# Probability

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# Overview

- Terminology
- Interpretations of Probability
- Probability Theory
- Conditional Probability
- Independence
- Bayes' Theorem

# Terms in Probability Theory

- *Sample space $\Omega$*
  - A set whose elements represent outcomes of a hypothetical experiment.
- *Event*
  - A subset of the sample space.
- *Probability P*
  - A function that assigns to each event a number.
- *Random Variable*
  - A function that assigns to each outcome a value.

# Interpretations of Probability

- Principle of Indifference.
- Relative Frequency.
- Subjective Probability.
- Axiomatic

# Principle of Indifference

*Outcomes are considered to be equiprobable if we have no reason to expect or prefer one over another.*

1. Works well for simple cases.
2. Some problems where it is difficult to use this principle:
   1. Occupancy problem.
   2. Random Chord.
   3. Thumbtack problem.

# Occupancy Problem

*What is the probability that 3 objects occupy 3 particular cells out of 5 cells?*

- Maxwell-Boltzman model.
  - The objects are distinguishable.
- Bose-Einstein model.
  - The objects are indistinguishable.
- Fermi-Dirac model.
  - The objects are indistinguishable and no more than one can occupy a cell.

## The random chord problem

*Given a circle of radius R. Choose a chord at random. What is the probability that the length of this chord is greater than R?*

- Centre of the chord is uniform.
- Angle subtended by the chord is uniform.
- Endpoints are uniform on the circumference and independent.

## Thumbtack Problem

*Toss a thumbtack. What is the probability that it lands with edge of the head and the end of the point touching the ground?*

## Relative Frequency

*If an experiment is repeated many times the probability of an event is:*

$$\frac{\text{number of times the event occurs}}{\text{number of experiments}}$$

*as the number of experiments approach infinite.*

## Relative Frequency (cont.)

- Probably used by gamblers before formal theory of probability was developed.
- Not possible to do an infinite number of trials.
- Some experiments are not repeatable.

## Subjective Probability

- Bernoulli 1713, Laplace 1825, De Morgan 1847, Keynes 1921, Jeffreys 1952, etc.
  - *Degree of belief or confidence*.
- Ramsey 1931 and De Finetti 1937
  - *Related probabilities to betting odds*
- Raiffa 1968
  - *Judgemental probability*

## Urn Problem

- Consider the urn problem in the first lecture, but now suppose you do not know how many urns are Type 1 and Type 2.
- Suppose you had a quick look at the urns before their labels were removed.
- Can you use your impression of the number of each type of urn in your decision problem?

## Calibration

Find a **p** such that you are indifferent to the following two options:

**<u>Option 1:</u>** You receive reward W if the random urn is of Type 1, otherwise you receive L.

**<u>Option 2:</u>** You receive a **p-BRLT**.

## Judgemental Probability

*Consider a lottery which gives prize W if the event E occurs, and L otherwise. Then if a decision maker is indifferent to the lottery and a p-BRLT we say that the judgemental probability of E is p.*

- Can be shown to satisfy the Axioms of Probability using the principles of substitutability and transitivity.

## Axioms of Probability

- The probability of an event is between 0 and 1.
  - $0 \leq P(E) \leq 1$.
- The probability of the sample space is 1.
  - $P(\Omega) = 1$.
- For disjoint events, A and B,
  - $P(A \cup B) = P(A) + P(B)$
- For pairwise disjoint events, $A_1$, $A_2$,…
  - $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

## Conditional Probability

- Let A and B be events and $P(B) > 0$.
- The *Conditional Probability of A given B* is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

## Independence

- Let A and B be events.
- A and B are independent iff:

$$P(A \cap B) = P(A)P(B)$$

- If $P(B) > 0$ then A is independent of B, iff:

$$P(A \mid B) = P(A)$$

## Law of Total Probability

- Let A be an event and
- Let $H_1$, .., $H_n$, be a partition of of the sample space.
- Then

$$P(A) = \sum_{k=1}^{n} P(A \mid H_k) P(H_k)$$

## Bayes' Theorem

- Let A and B be events and P(A) > 0 and P(B) > 0. Then

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

- Let $H_1$, .., $H_n$, be a partition of of the sample space. Then

$$P(H_m \mid A) = \frac{P(A \mid H_m)P(H_m)}{\sum_{k=1}^{n} P(A \mid H_k)P(H_k)}$$

## Evidence Example

The following problem dates back to 1685:

*Two witness separately report that a particular event took place. The probability that the witnesses are telling the truth is $p_1$ and $p_2$ respectively.*

*What is the probability the event took place?*

## AIDS Example

- Suppose 0.6% of the population have AIDS.
- Suppose we have a test for AIDS such that:
  - For a person with AIDS the test would give a positive result with probability 0.977
  - For a person without AIDS the test would give a negative result with probability 0.926

*What would be the probability that if the result of the test was positive for a person, then that person did **not** have AIDS?*

## Urn Example

- Two urns. First one has 70 green balls and 30 red balls. The second has 70 red balls and 30 green balls.
- A urn is chosen at random and you draw 12 balls with replacement, 8 green balls and 4 red balls.

*What is the probability that the urn chosen is the one with* 70 green balls and 30 red balls*?*