CSE4213 Lecture Notes

Revision of Logic, Substitutions

Predicate Logic and Substitutions

Schneider, chapter 2

20050326 / Lecture 9

1 Predicate Calculus

Predicates

- *Predicates* are statements which are either true or false
- Predicates can be used to make statements about sets, and hence are very useful in B
- $x \in PRICE$ is a predicate that states something about x, a potential member of the set $PRICE$. If true, then x is an element of $PRICE$
- Hence use predicates to define sets with *set comprehension*
- $\{x \mid x \in \mathbb{N} \wedge x \leq 10\}$

Operations on Predicates

- *negation* (not): $\neg P$
- *disjunction* (or): $P \vee Q$
- *conjunction* (and): $P \wedge Q$
- *implication* (implies): $P \Rightarrow Q$
- *equivalence* (if and only if): $P \Leftrightarrow Q$

Quantification

- We often want to state predicates across a set
- *universal quantification* defines a predicate across all members of a set: $\forall x.(x \in S \Rightarrow P)$
- *existential quantification* defines a predicate for at least one member of a set: $\exists x.(x \in S \wedge P)$
- note the difference in connectives (why?)

Duality of Quantification

- If universal quantification is false, then there must be at least one element that makes it false
- If existential quantification is false, then all elements must not satisfy the predicate.
- Hence we have:

$$\neg \forall x.(x \in S \Rightarrow P) \Leftrightarrow \exists x.(x \in S \wedge \neg P)$$
$$\neg \exists x.(x \in S \wedge P) \Leftrightarrow \forall x.(x \in S \Rightarrow \neg P)$$

Constraining Predicates

- There are some contexts where it is stated that a predicate P must *constrain* some list of variables z
- To constrain the variable x, the predicate P must contain predicates of the form: $x \in S$, $x \subseteq S$, $x \subset S$, or $x = E$, where $x \backslash S$, $x \backslash E$ and S is a set, and E is an expression.
- $x \backslash E$ means that x is not free in E
- We say that x is a *bound variable* in P.

Free Variables

- A variable x is free in an expression if it is not bound by a quantifier.
- All free instances of z in P and Q are bound in $\forall z.(P \Rightarrow Q)$ and $\exists z.(P \wedge Q)$
- Hence, $z \backslash \forall z.(P \Rightarrow Q)$
- Note that z is *free* in P and Q, but *bound* in $\forall z.(P \Rightarrow Q)$

2 Substitutions

Substitutions

- *Substitutions* are crucial to the B-Method
- Substitutions are the way in which we effect change of state
- A substitution (declarative) is the formal equivalent of assignment (imperative)
- $P[E/x]$ means: *P, with every occurrence of x replaced by E*
- multiple substitutions are possible: $P[E, F, \ldots/x, y, \ldots]$