B Exercises 1
Solutions to Sets

1. Given $X = \{a, b\}$, show the following.

   (a) $\mathsf{card}(X)$
   Ans: 2. There are two elements, a and b.

   (b) if $aa : X$, **one** possible value of aa
   Ans: Either of a or b. **Not** $\{\}$ or $\{a, b\}$!

   (c) $\mathbb{P}(X)$, $\mathsf{card}(\mathbb{P}(X))$.
   Ans: $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$
   (The set of elements taken none at a time, one at a time, and two at a time.)
   Ans: 4

   (d) if $aaa : \mathbb{P}(X)$, **one** possible value of aaa
   Ans: Any of $\{\}$, $\{a\}$, $\{b\}$, $\{a, b\}$.

   (e) $\mathbb{P}(\mathbb{P}(X))$, $\mathsf{card}(\mathbb{P}(\mathbb{P}(X)))$.
   Ans: $\{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \ldots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$
   (The set of things taken none at a time (empty set, shown), one at a time (set containing the empty set (shown), then a set of each of the remaining elements of part b one at a time (shown), then sets of the elements of part b taken two at a time (not shown), three at a time (not shown), and finally four at a time (shown).)
   Ans: 16

   (f) $\mathbb{P}(\mathbb{P}(\mathbb{P}(X)))$, $\mathsf{card}(\mathbb{P}(\mathbb{P}(\mathbb{P}(X))))$.
   Ans: $\{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\{a\}\}\}, \ldots,$
   $\{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \ldots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$

   Ans: 65568 (2^{16})

   (g) $X \times \{0, 1\}$, $X \times \{\}$
   Ans: $\{(a, 0), (a, 1), (b, 0), (b, 1)\}$

   Ans: $\{\}$. (Note that $\mathsf{card}(A \times B) = \mathsf{card}(A) \times \mathsf{card}(B)$)

2. Given $X = \{a, b\}$ and $Y = \{0, 1\}$, show as sets of maplets:

   (a) $X \rightarrowtail\!\!\!\!+ Y$
   Ans: various answers are possible, such as $\{(a, 0)\}$ (ordered pair notation) or $\{a \mapsto 1, b \mapsto 0\}$ (maplet notation) or $\{b \mapsto 1\}$

   (b) $X \rightarrow Y$
   Ans: $\{f \mid f \in X \nrightarrow Y \wedge \mathsf{dom}(f) = X\}$
   example: $\{a \mapsto 0, b \mapsto 1\}$

   What are the values of the following predicates?

   (c) $a \mapsto 0 \in X \nrightarrow Y$
   Ans: False (LHS should be a set)

   (d) $\{\{a \mapsto 0, a \mapsto 1\}\} \subseteq X \rightarrow Y$
   Ans: False (domain contains non-unique elements)

3. Students pass a subject if they gain at least 50 marks in the final examination. Given a function $results : \mathsf{STUDENTS} \nrightarrow \mathbb{N}$, that yields the examination result for a particular student, specify
   (a) the set of students that pass;
   Ans: $\{s \mid s \in STUDENTS \wedge s \in \mathsf{dom}(results) \wedge results(s) \geq 50\}$
   (Note: the $s \in STUDENTS$ clause can be omitted, since it is implied by $s \in \mathsf{dom}(results)$)
   (b) the set of students that fail.
   Ans: $\{s \mid s \in STUDENTS \wedge (s \notin \mathsf{dom}(results) \vee results(s) < 50)\}$
   (Note: the $s \in STUDENTS$ clause cannot be omitted, since s is otherwise unconstrained (untyped).

4. Memory on a modern computer can be considered to be an array of bytes.
   (a) define sets to represent memory.
   (b) show a function application to represent memory lookup of location *loc*.
   (c) show a function update (relational override) to represent assignment of value *val* to memory location *loc*

5. Assuming the following C declarations

```
int a[4] = {5,2,8,9};
char b[] = "the";
struct {int ca; char cb;} c;
```

   give a set representation of the data structures `a`, `b`, `c`

6. If we were modelling a taxi fleet company we might have three variables, *drivers*, *taxis* and *assigned* constrained by

$$\begin{aligned} drivers &: \mathbb{P}\,\mathsf{DRIVERS} \\ taxis &: \mathbb{P}\,\mathsf{TAXIS} \\ assigned &: drivers \rightarrowtail\!\!\!\!\!\nrightarrow taxis \end{aligned}$$

   where *drivers* is the set of drivers working for the company, *taxis* is the set of taxis owned by the company, and *assigned* is a function recording the assignment of drivers to taxis.

   The arrow $\rightarrowtail\!\!\!\!\!\nrightarrow$ denotes a *partial injective* function. An injective function is a one-to-one function.

   (a) Why is *assigned* a partial function?
   (b) Why is *assigned* an injective function?
   (c) Specify the drivers who are currently assigned.
   (d) Specify the drivers who are currently unassigned.
   (e) Specify the taxis that are currently assigned.

(f) Specify the taxis that are currently unassigned.

7. In the **SimpleLibrary** machine, we have three variables

$$\begin{aligned} books\_in\_library &\subseteq BOOK \\ books\_on\_shelf &\subseteq books\_in\_library \\ books\_on\_loan &\in books\_in\_library \nrightarrow users \end{aligned}$$

We want $books\_on\_shelf$ and $\mathsf{dom}(books\_on\_loan)$ to *partition* the set $books\_in\_library$. Normally, we would specify this by:

$$\mathsf{dom}(books\_on\_loan) \cup books\_on\_shelf = books\_in\_library \wedge$$
$$\mathsf{dom}(books\_on\_loan) \cap books\_on\_shelf = \{\}$$

but it was specified by

$$\mathsf{dom}(books\_on\_loan) = books\_in\_library - books\_on\_shelf$$

(a) Show that the former follows from the latter.

(b) Would $\mathsf{dom}(books\_on\_loan) \cup books\_on\_shelf = books\_in\_library$ have been equally effective.

8. For the Borrow(user,book) operation of the **SimpleLibrary** machine we get a proof obligation to show that

$$\mathsf{dom}(books\_on\_loan \cup \{book \mapsto user\}) = books\_in\_library - (books\_on\_shelf - \{book\})$$

Reason why this is true.