

B Exercises 1 Solutions to Sets

1. Given $X = \{a, b\}$, show the following.

(a) $\text{card}(X)$

Ans: 2. There are two elements, a and b .

(b) if $aa : X$, **one** possible value of aa

Ans: Either of a or b . **Not** $\{\}$ or $\{a, b\}$!

(c) $\mathbb{P}(X)$, $\text{card}(\mathbb{P}(X))$.

Ans: $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$

(The set of elements taken none at a time, one at a time, and two at a time.)

Ans: 4

(d) if $aaa : \mathbb{P}(X)$, **one** possible value of aaa

Ans: Any of $\{\}, \{a\}, \{b\}, \{a, b\}$.

(e) $\mathbb{P}(\mathbb{P}(X))$, $\text{card}(\mathbb{P}(\mathbb{P}(X)))$.

Ans: $\{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \dots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$

(The set of things taken none at a time (empty set, shown), one at a time (set containing the empty set (shown), then a set of each of the remaining elements of part b one at a time (shown), then sets of the elements of part b taken two at a time (not shown), three at a time (not shown), and finally four at a time (shown).)

Ans: 16

(f) $\mathbb{P}(\mathbb{P}(\mathbb{P}(X)))$, $\text{card}(\mathbb{P}(\mathbb{P}(\mathbb{P}(X))))$.

Ans: $\{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\{a\}\}\}, \dots, \{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \dots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$

Ans: 65568 (2^{16})

(g) $X \times \{0, 1\}$, $X \times \{\}$

Ans: $\{(a, 0), (a, 1), (b, 0), (b, 1)\}$

Ans: $\{\}$. (Note that $\text{card}(A \times B) = \text{card}(A) \times \text{card}(B)$)

2. Given $X = \{a, b\}$ and $Y = \{0, 1\}$, show as sets of maplets:

(a) $X \leftrightarrow Y$

Ans: various answers are possible, such as $\{(a, 0)\}$ (ordered pair notation) or $\{a \mapsto 1, b \mapsto 0\}$ (maplet notation) or $\{b \mapsto 1\}$

(b) $X \rightarrow Y$

Ans: $\{f \mid f \in X \rightarrow Y \wedge \text{dom}(f) = X\}$

example: $\{a \mapsto 0, b \mapsto 1\}$

What are the values of the following predicates?

(c) $a \mapsto 0 \in X \leftrightarrow Y$

Ans: False (LHS should be a set)

(d) $\{\{a \mapsto 0, a \mapsto 1\}\} \subseteq X \rightarrow Y$

Ans: False (domain contains non-unique elements)

3. Students pass a subject if they gain at least 50 marks in the final examination. Given a function $results : STUDENTS \rightarrow \mathbb{N}$, that yields the examination result for a particular student, specify

(a) the set of students that pass;

Ans: $\{s \mid s \in STUDENTS \wedge s \in \text{dom}(results) \wedge results(s) \geq 50\}$

(Note: the $s \in STUDENTS$ clause can be omitted, since it is implied by $s \in \text{dom}(results)$)

(b) the set of students that fail.

Ans: $\{s \mid s \in STUDENTS \wedge (s \notin \text{dom}(results) \vee results(s) < 50)\}$

(Note: the $s \in STUDENTS$ clause cannot be omitted, since s is otherwise unconstrained (untyped).

4. Memory on a modern computer can be considered to be an array of bytes.

(a) define sets to represent memory.

Ans: $BYTE = 0..255$ (8-bit natural numbers)

$MEMORY = 0..(n-1) \rightarrow BYTE$ where n is the size of the memory in bytes

(b) show a function application to represent memory lookup of location loc .

Ans: $MEMORY(loc)$

(c) show a function update (relational override) to represent assignment of value val to memory location loc

Ans: $MEMORY \triangleleft loc \mapsto val$

5. Assuming the following C declarations

```
int a[4] = {5,2,8,9};
char b[] = "the";
struct {int ca; char cb;} c;
```

give a set representation of the data structures **a**, **b**, **c**

Answer: All of these can be considered to be functions from VAR to \mathbb{X} , where VAR is the set of all variables, and \mathbb{X} is the set of all result types (sets). Call this the set of variables $VARIABLES \in VAR \rightarrow \mathbb{X}$

Answer: **a:** an example of a is $\{0 \mapsto 5, 1 \mapsto 2, 2 \mapsto 8, 3 \mapsto 9\}$

We have that $(a, x) \in VARIABLES \Rightarrow (a \in VAR) \wedge (x \in 0..3 \rightarrow \mathbb{N}) \wedge (0..3 \rightarrow \mathbb{N} \in \mathbb{X})$

Answer: **b:** an example of b is $\{0 \mapsto 't', 1 \mapsto 'h', 2 \mapsto 'e'\}$ (Note: the values $'t'$, $'h'$, $'e'$ are assumed to stand for themselves (like the natural numbers)

We have that $(b, y) \in VARIABLES \Rightarrow (b \in VAR) \wedge (y \in 0..2 \rightarrow \mathbb{C}) \wedge (0..2 \rightarrow \mathbb{C} \in \mathbb{X})$. \mathbb{C} is the set of all characters.

Answer: c : Structures are *variable composers*. They make variables out of composite names. There are two ways of viewing this. One is to think of ca , cb as functions that take an argument of type VAR , and then $c.ca$ is really $ca(c)$. Alternatively, you can think of c as a function from $FIELDS$ to a new variable, and then regard $c.ca$ as the function application $c(ca)$

First approach: an example of ca is $\{(c, 23)\}$ and cb is $\{(c, 'k')\}$

We have that $\{(ca, x), (cb, y)\} \subset VARIABLES \Rightarrow (((ca \in VAR) \wedge (x \in VAR \mapsto \mathbb{X})) \Rightarrow (\exists p, q.(p, q) \in x \wedge q \in VAR \mapsto \mathbb{N} \wedge c \in \text{dom}(q) \wedge VAR \mapsto \mathbb{N} \in \mathbb{X})) \wedge \text{similar expression for } cb$

Second approach: an example of c is $\{(ca, 23), (cb, 'k')\}$

We have that $(c, x) \in VARIABLES \Rightarrow (((c \in VAR) \wedge (x \in FIELDS \mapsto \mathbb{X})) \Rightarrow (\exists p, q.(p, q) \in x \Rightarrow (p = ca \Rightarrow q \in \mathbb{N} \wedge \mathbb{N} \in \mathbb{X}) \vee (p = cb \Rightarrow q \in \mathbb{C} \wedge \mathbb{C} \in \mathbb{X})))$

Other approaches are just as valid!

6. If we were modelling a taxi fleet company we might have three variables, *drivers*, *taxis* and *assigned* constrained by

$$\begin{aligned} drivers & : \mathbb{P} \text{DRIVERS} \\ taxis & : \mathbb{P} \text{TAXIS} \\ assigned & : drivers \mapsto taxis \end{aligned}$$

where *drivers* is the set of drivers working for the company, *taxis* is the set of taxis owned by the company, and *assigned* is a function recording the assignment of drivers to taxis.

The arrow \mapsto denotes a *partial injective* function. An injective function is a one-to-one function.

- Why is *assigned* a partial function?
- Why is *assigned* an injective function?
- Specify the drivers who are currently assigned.
- Specify the drivers who are currently unassigned.
- Specify the taxis that are currently assigned.
- Specify the taxis that are currently unassigned.

7. In the **SimpleLibrary** machine, we have three variables

$$\begin{aligned} books_in_library & \subseteq BOOK \\ books_on_shelf & \subseteq books_in_library \\ books_on_loan & \in books_in_library \mapsto users \end{aligned}$$

We want *books_on_shelf* and $\text{dom}(books_on_loan)$ to *partition* the set *books_in_library*. Normally, we would specify this by:

$$\begin{aligned} \text{dom}(books_on_loan) \cup books_on_shelf & = books_in_library \wedge \\ \text{dom}(books_on_loan) \cap books_on_shelf & = \{\} \end{aligned}$$

but it was specified by

$$\text{dom}(books_on_loan) = books_in_library - books_on_shelf$$

- Show that the former follows from the latter.

- (b) Would $\text{dom}(\text{books_on_loan}) \cup \text{books_on_shelf} = \text{books_in_library}$ have been equally effective.
8. For the $\text{Borrow}(\text{user}, \text{book})$ operation of the **SimpleLibrary** machine we get a proof obligation to show that

$$\text{dom}(\text{books_on_loan} \cup \{\text{book} \mapsto \text{user}\}) = \text{books_in_library} - (\text{books_on_shelf} - \{\text{book}\})$$

Reason why this is true.