B Exercises 1
Sets

1. Given $X = \{a, b\}$, show the following.

   (a) card(X)
   Ans: 2. There are two elements, a and b.

   (b) if $aa : X$, **one** possible value of aa
   Ans: Either of a or b. **Not** $\{\}$ or $\{a, b\}$!

   (c) $\mathbb{P}(X)$, card($\mathbb{P}(X)$).
   Ans: $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$
   (The set of elements taken none at a time, one at a time, and two at a time.)
   Ans: 4

   (d) if $aaa : \mathbb{P}(X)$, **one** possible value of aaa
   Ans: Any of $\{\}$, $\{a\}$, $\{b\}$, $\{a, b\}$.

   (e) $\mathbb{P}(\mathbb{P}(X))$, card($\mathbb{P}(\mathbb{P}(X))$).
   Ans: $\{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \ldots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$
   (The set of things taken none at a time (empty set, shown), one at a time (set containing the empty set (shown), then a set of each of the remaining elements of part b one at a time (shown), then sets of the elements of part b taken two at a time (not shown), three at a time (not shown), and finally four at a time (shown).)
   Ans: 16

   (f) $\mathbb{P}(\mathbb{P}(\mathbb{P}(X)))$, card($\mathbb{P}(\mathbb{P}(\mathbb{P}(X)))$).
   Ans: $\{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\{a\}\}\}, \ldots,$
   $\{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \ldots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$
   Ans: 65536 (2^{16})

   (g) $X \times \{0, 1\}$, $X \times \{\}$
   Ans: $\{(a, 0), (a, 1), (b, 0), (b, 1)\}$
   Ans: $\{\}$. (Note that $\text{card}(A \times B) = \text{card}(A) \times \text{card}(B)$)

2. Given $X = \{a, b\}$ and $Y = \{0, 1\}$, show as sets of maplets:

   (a) $X \nrightarrow Y$
   Ans: various answers are possible, such as $\{(a, 0)\}$ (ordered pair notation) or $\{a \mapsto 1, b \mapsto 0\}$ (maplet notation) or $\{b \mapsto 1\}$

   (b) $X \rightarrow Y$
   Ans: $\{f | f \in X \nrightarrow Y \land \text{dom}(f) = X\}$
   example: $\{a \mapsto 0, b \mapsto 1\}$

   What are the values of the following predicates?

   (c) $a \mapsto 0 \in X \nrightarrow Y$
   Ans: False (LHS should be a set)

   (d) $\{\{a \mapsto 0, a \mapsto 1\}\} \subseteq X \rightarrow Y$
   Ans: False (domain contains non-unique elements)

3. Students pass a subject if they gain at least 50 marks in the final examination. Given a function *results* : $\mathsf{STUDENTS} \nrightarrow \mathbb{N}$, that yields the examination result for a particular student, specify

(a) the set of students that pass;
Ans: $\{s | s \in STUDENTS \land s \in \mathrm{dom}(results) \land results(s) \geq 50\}$
(Note: the $s \in STUDENTS$ clause can be omitted, since it is implied by $s \in \mathrm{dom}(results)$)

(b) the set of students that fail.
Ans: $\{s | s \in STUDENTS \land (s \notin \mathrm{dom}(results) \lor results(s) < 50)\}$
(Note: the $s \in STUDENTS$ clause cannot be omitted, since s is otherwise unconstrained (untyped).

4. Memory on a modern computer can be considered to be an array of bytes.

   (a) define sets to represent memory.
   (b) show a function application to represent memory lookup of location *loc*.
   (c) show a function update (relational override) to represent assignment of value *val* to memory location *loc*

5. If we were modelling a taxi fleet company we might have three variables, *drivers*, *taxis* and *assigned* constrained by

$$\begin{aligned} drivers &: \mathbb{P}\,\mathsf{DRIVERS} \\ taxis &: \mathbb{P}\,\mathsf{TAXIS} \\ assigned &: drivers \rightarrowtail\!\!\!\!\!\!+\;\; taxis \end{aligned}$$

   where *drivers* is the set of drivers working for the company, *taxis* is the set of taxis owned by the company, and *assigned* is a function recording the assignment of drivers to taxis.

   The arrow ⤔ denotes a *partial injective* function. An injective function is a one-to-one function.

   (a) Why is *assigned* a partial function?
   (b) Why is *assigned* an injective function?
   (c) Specify the drivers who are currently assigned.
   (d) Specify the drivers who are currently unassigned.
   (e) Specify the taxis that are currently assigned.
   (f) Specify the taxis that are currently unassigned.

6. (Schneider 6.1) The relation *eats* is defined as follows:

$$\begin{aligned} eats = \{ \quad & ian \mapsto eggs, ian \mapsto cheese, ian \mapsto pizza, \\ & jim \mapsto eggs, jim \mapsto salad, ken \mapsto pizza, \\ & lisa \mapsto cheese, lisa \mapsto salad, lisa \mapsto pizza\} \end{aligned}$$

   (a) Draw the relation *eats*
   (b) What is the relation $\{ian\} \lhd eats$?
   (c) What is the relation $\{jim\} \ntriangleleft eats$?
   (d) What is the relation $eats \rhd \{cheese, pizza\}$?
   (e) What is $\mathrm{dom}(eats \rhd \{eggs\})$?
   (f) Using the notation on relations, express the set of people that eat either *eggs* or *pizza*.
   (g) Express the set of people that eat both *cheese* and *pizza*