# B Exercises 1
# Sets

1. Given $X = \{a, b\}$, show the following.

   (a) card($X$)
   *Ans:* 2. There are two elements, $a$ and $b$.

   (b) if $aa : X$, **one** possible value of $aa$
   *Ans:* Either of $a$ or $b$. **Not** $\{\}$ or $\{a, b\}$!

   (c) $\mathbb{P}(X)$, card($\mathbb{P}(X)$).
   *Ans:* $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$
   (The set of elements taken none at a time, one at a time, and two at a time.)
   *Ans:* 4

   (d) if $aaa : \mathbb{P}(X)$, **one** possible value of $aaa$
   *Ans:* Any of $\{\}$, $\{a\}$, $\{b\}$, $\{a, b\}$.

   (e) $\mathbb{P}(\mathbb{P}(X))$, card($\mathbb{P}(\mathbb{P}(X))$).
   *Ans:* $\{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \ldots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$
   (The set of things taken none at a time (empty set, shown), one at a time (set containing the empty set (shown), then a set of each of the remaining elements of part b one at a time (shown), then sets of the elements of part b taken two at a time (not shown), three at a time (not shown), and finally four at a time (shown).)
   *Ans:* 16

   (f) $\mathbb{P}(\mathbb{P}(\mathbb{P}(X)))$, card($\mathbb{P}(\mathbb{P}(\mathbb{P}(X)))$).
   *Ans:* $\{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\{a\}\}\}, \ldots,$
   $\{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \ldots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$
   *Ans:* 65536 ($2^{16}$)

   (g) $X \times \{0, 1\}$, $X \times \{\}$
   *Ans:* $\{(a, 0), (a, 1), (b, 0), (b, 1)\}$
   *Ans:* $\{\}$. (Note that $\text{card}(A \times B) = \text{card}(A) \times \text{card}(B)$)

2. Given $X = \{a, b\}$ and $Y = \{0, 1\}$, show as sets of maplets:

   (a) $X \nrightarrow Y$
   *Ans:* various answers are possible, such as $\{(a, 0)\}$ (ordered pair notation) or $\{a \mapsto 1, b \mapsto 0\}$ (maplet notation) or $\{b \mapsto 1\}$

   (b) $X \rightarrow Y$
   *Ans:* $\{f | f \in X \nrightarrow Y \wedge \text{dom}(f) = X\}$
   example: $\{a \mapsto 0, b \mapsto 1\}$

   What are the values of the following predicates?

   (c) $a \mapsto 0 \in X \nrightarrow Y$
   *Ans:* False (LHS should be a set)

   (d) $\{\{a \mapsto 0, a \mapsto 1\}\} \subseteq X \rightarrow Y$
   *Ans:* False (domain contains non-unique elements)

3. Students pass a subject if they gain at least 50 marks in the final examination. Given a function *results* : $\mathsf{STUDENTS} \nrightarrow \mathbb{N}$, that yields the examination result for a particular student, specify

(a) the set of students that pass;
*Ans:* $\{s | s \in STUDENTS \land s \in \text{dom}(results) \land results(s) \geq 50\}$
(Note: the $s \in STUDENTS$ clause can be omitted, since it is implied by $s \in \text{dom}(results)$)

(b) the set of students that fail.
*Ans:* $\{s | s \in STUDENTS \land (s \notin \text{dom}(results) \lor results(s) < 50)\}$
(Note: the $s \in STUDENTS$ clause cannot be omitted, since $s$ is otherwise unconstrained (untyped).

4. Memory on a modern computer can be considered to be an array of bytes.

(a) define sets to represent memory.
*Ans:* $BYTE = 0 \mathinner{\ldotp\ldotp} 255$ (8-bit natural numbers)
$MEMORY = 0 \mathinner{\ldotp\ldotp} (n-1) \rightarrow BYTE$ where $n$ is the size of the memory in bytes

(b) show a function application to represent memory lookup of location *loc*.
*Ans:* $MEMORY(loc)$

(c) show a function update (relational override) to represent assignment of value *val* to memory location *loc*
*Ans:* $MEMORY \oplus loc \mapsto val$

5. If we were modelling a taxi fleet company we might have three variables, *drivers*, *taxis* and *assigned* constrained by

$$\begin{aligned} drivers &: \mathbb{P}\,\mathsf{DRIVERS} \\ taxis &: \mathbb{P}\,\mathsf{TAXIS} \\ assigned &: drivers \rightarrowtail\!\!\!\!\!\!\!+\; taxis \end{aligned}$$

where *drivers* is the set of drivers working for the company, *taxis* is the set of taxis owned by the company, and *assigned* is a function recording the assignment of drivers to taxis.

The arrow $\rightarrowtail\!\!\!\!\!\!+$ denotes a *partial injective* function. An injective function is a one-to-one function.

(a) Why is *assigned* a partial function?
*Ans:* Because not every driver need be assigned to a taxi.

(b) Why is *assigned* an injective function?
*Ans:* Because each taxi can be assigned to only one driver.

(c) Specify the drivers who are currently assigned.
*Ans:* $\text{dom}(assigned)$

(d) Specify the drivers who are currently unassigned.
*Ans:* $drivers - \text{dom}(assigned)$

(e) Specify the taxis that are currently assigned.
*Ans:* $\text{ran}(assigned)$

(f) Specify the taxis that are currently unassigned.
*Ans:* $taxis - \text{ran}(assigned)$

6. (Schneider 6.1) The relation *eats* is defined as follows:

$$\begin{aligned} eats = \{ \quad & ian \mapsto eggs, ian \mapsto cheese, ian \mapsto pizza, \\ & jim \mapsto eggs, jim \mapsto salad, ken \mapsto pizza, \\ & lisa \mapsto cheese, lisa \mapsto salad, lisa \mapsto pizza\} \end{aligned}$$

   (a) Draw the relation *eats*
   (b) What is the relation $\{ian\} \lhd eats$?
   (c) What is the relation $\{jim\} \ntriangleleft eats$?
   (d) What is the relation $eats \rhd \{cheese, pizza\}$?
   (e) What is $\text{dom}(eats \rhd \{eggs\})$ ?
   (f) Using the notation on relations, express the set of people that eat either *eggs* or *pizza*.
   (g) Express the set of people that eat both *cheese* and *pizza*