

B Exercises 1

Sets

1. Given $X = \{a, b\}$, show the following.
 - (a) $\text{card}(X)$
Ans: 2. There are two elements, a and b .
 - (b) if $aa : X$, **one** possible value of aa
Ans: Either of a or b . **Not** $\{\}$ or $\{a, b\}$!
 - (c) $\mathbb{P}(X)$, $\text{card}(\mathbb{P}(X))$.
Ans: $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$
 (The set of elements taken none at a time, one at a time, and two at a time.)
Ans: 4
 - (d) if $aaa : \mathbb{P}(X)$, **one** possible value of aaa
Ans: Any of $\{\}, \{a\}, \{b\}, \{a, b\}$.
 - (e) $\mathbb{P}(\mathbb{P}(X))$, $\text{card}(\mathbb{P}(\mathbb{P}(X)))$.
Ans: $\{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \dots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$
 (The set of things taken none at a time (empty set, shown), one at a time (set containing the empty set (shown), then a set of each of the remaining elements of part b one at a time (shown), then sets of the elements of part b taken two at a time (not shown), three at a time (not shown), and finally four at a time (shown).)
Ans: 16
 - (f) $\mathbb{P}(\mathbb{P}(\mathbb{P}(X)))$, $\text{card}(\mathbb{P}(\mathbb{P}(\mathbb{P}(X))))$.
Ans: $\{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\{a\}\}\}, \dots, \{\{\}, \{\{\}\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \dots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$
Ans: 65536 (2^{16})
 - (g) $X \times \{0, 1\}$, $X \times \{\}$
Ans: $\{(a, 0), (a, 1), (b, 0), (b, 1)\}$
Ans: $\{\}$. (Note that $\text{card}(A \times B) = \text{card}(A) \times \text{card}(B)$)
2. Given $X = \{a, b\}$ and $Y = \{0, 1\}$, show as sets of maplets:
 - (a) $X \leftrightarrow Y$
Ans: various answers are possible, such as $\{(a, 0)\}$ (ordered pair notation) or $\{a \mapsto 1, b \mapsto 0\}$ (maplet notation) or $\{b \mapsto 1\}$
 - (b) $X \rightarrow Y$
Ans: $\{f \mid f \in X \leftrightarrow Y \wedge \text{dom}(f) = X\}$
 example: $\{a \mapsto 0, b \mapsto 1\}$

What are the values of the following predicates?

- (c) $a \mapsto 0 \in X \leftrightarrow Y$
Ans: False (LHS should be a set)
- (d) $\{\{a \mapsto 0, a \mapsto 1\}\} \subseteq X \rightarrow Y$
Ans: False (domain contains non-unique elements)

3. Students pass a subject if they gain at least 50 marks in the final examination. Given a function $results : STUDENTS \rightarrow \mathbb{N}$, that yields the examination result for a particular student, specify

- (a) the set of students that pass;

$$Ans: \{s \mid s \in STUDENTS \wedge s \in \text{dom}(results) \wedge results(s) \geq 50\}$$

(Note: the $s \in STUDENTS$ clause can be omitted, since it is implied by $s \in \text{dom}(results)$)

- (b) the set of students that fail.

$$Ans: \{s \mid s \in STUDENTS \wedge (s \notin \text{dom}(results) \vee results(s) < 50)\}$$

(Note: the $s \in STUDENTS$ clause cannot be omitted, since s is otherwise unconstrained (untyped).

4. Memory on a modern computer can be considered to be an array of bytes.

- (a) define sets to represent memory.

$$Ans: BYTE = 0..255 \text{ (8-bit natural numbers)}$$

$$MEMORY = 0..(n-1) \rightarrow BYTE \text{ where } n \text{ is the size of the memory in bytes}$$

Alternatively, to represent $BYTE$ as an (ordered) collection of BIT s, where $BIT = \{0, 1\}$, we could use $BYTE = 0..7 \rightarrow BIT$

In this case, the function $MEMORY$ becomes a function returning a function.

- (b) show a function application to represent memory lookup of location loc .

$$Ans: MEMORY(loc)$$

- (c) show a function update (relational override) to represent assignment of value val to memory location loc

$$Ans: MEMORY \leftarrow loc \mapsto val$$

5. If we were modelling a taxi fleet company we might have three variables, $drivers$, $taxis$ and $assigned$ constrained by

$$drivers : \mathbb{P} DRIVERS$$

$$taxis : \mathbb{P} TAXIS$$

$$assigned : drivers \rightsquigarrow taxis$$

where $drivers$ is the set of drivers working for the company, $taxis$ is the set of taxis owned by the company, and $assigned$ is a function recording the assignment of drivers to taxis.

The arrow \rightsquigarrow denotes a *partial injective* function. An injective function is a one-to-one function.

- Why is $assigned$ a partial function?
- Why is $assigned$ an injective function?
- Specify the drivers who are currently assigned.
- Specify the drivers who are currently unassigned.
- Specify the taxis that are currently assigned.
- Specify the taxis that are currently unassigned.

6. (Schneider 6.1) The relation *eats* is defined as follows:

$$\begin{aligned} \textit{eats} = \{ & \textit{ian} \mapsto \textit{eggs}, \textit{ian} \mapsto \textit{cheese}, \textit{ian} \mapsto \textit{pizza}, \\ & \textit{jim} \mapsto \textit{eggs}, \textit{jim} \mapsto \textit{salad}, \textit{ken} \mapsto \textit{pizza}, \\ & \textit{lisa} \mapsto \textit{cheese}, \textit{lisa} \mapsto \textit{salad}, \textit{lisa} \mapsto \textit{pizza} \} \end{aligned}$$

- (a) Draw the relation *eats*
- (b) What is the relation $\{\textit{ian}\} \triangleleft \textit{eats}$?
- (c) What is the relation $\{\textit{jim}\} \triangleleft \textit{eats}$?
- (d) What is the relation $\textit{eats} \triangleright \{\textit{cheese}, \textit{pizza}\}$?
- (e) What is $\text{dom}(\textit{eats} \triangleright \{\textit{eggs}\})$?
- (f) Using the notation on relations, express the set of people that eat either *eggs* or *pizza*.
- (g) Express the set of people that eat both *cheese* and *pizza*