Quadratic Programming

$$\max_{\vec{x}}/\min_{\vec{x}} \quad f(\vec{x}) \quad \text{subject to} \quad C(\vec{x})$$

where $f(\vec{x}) = f(x_1, \ldots x_n)$

is a quadratic function $f : R^n \rightarrow R$ and

$C(\vec{x}) = c_1(x_1, \ldots x_n) \wedge \ldots \wedge c_k(x_1, \ldots, x_n)$

is a conjunction of linear constraints c_i of forms

$f_i(x_1, \ldots, x_n) \leq k_i$ and

$f_i(x_1, \ldots, x_n) = k_i$ and

$f_i(x_1, \ldots, x_n) \geq k_i$

Wolfe's Method

Wolfe's method is an extended simplex procedure, which can be applied to QPP in which all problem variables are non-negative.

The basis are the complimentary slackness conditions for QPP

at the optimum point we have:

- slack (excess) for the *i*-th constraint and λi cannot both be positive.
  (follows from the geometric K-T interpretation)

- *xi* and μi in the "main" K-T condition cannot both be positive.

Slackness vs. K-T Conditions

The first complementary slack condition secures the second K-T condition

- slack (excess) for the *i*-th constraint and λ*i* cannot both be positive.

$$\forall i: \quad \lambda_i (b_i - g_i(\vec{x})) = 0$$

The second complementary slack condition secures the third K-T condition

- *xi* and μi in the “main” K-T condition cannot both be positive.

$$\forall j: \quad \left[\frac{\partial f(\vec{x})}{\partial x_j} + \sum_i \lambda_i \cdot \frac{\partial g_i(\vec{x})}{\partial x_j} \right] \cdot x_j = 0$$

Wolfe's QPP Idea

To solve a QPP we can rewrite the original constraints with slack and use the complementary slack conditions to ensure the second and third K-T conditions.

For a QPP, the main K-T condition is obviously linear.

We can therefore calculate the main K-T condition (1st condition) for the given QPP and use an extended Simplex that ensures the complementary slackness conditions (Wolfe's QPP Simplex) to find the optimum.

Review: (linear) Simplex

1. identify best substitute x[j] from highest opportunity cost
2. identify limiting resource s[i]
3. select pivot row according to (2)
4. solve pivot row for variable x[j]
5. substitute (4) in all other equations

6a. terminate if all opportunity costs negative or zero

6b otherwise goto step 1

Wolfe's Simplex Extensions

1. Construct the main K-T condition for non-negative variables for the problem

2. Add slack and excess variables for the inequalities.

3. Add artificial variables for any equation that does not have an obvious basic variable.

4. Apply phase 0 simplex with the following modifications:

    - Never pivot such that the excess *ei* from the *i*-th constraint and *xi* both become basic (this could make both of them positive)

    - Never pivot such that the slack/excess for the *i*-th constraint and λi both become basic (this could make both of them positive)

QPP Example

$$\text{minimize } z = -x_1 - x_2 + \tfrac{1}{2}x_1^2 + x_2^2 - x_1x_2$$

$$\text{subject to } \begin{cases} x_1 + x_2 \le 3 \\ -2x_1 - 3x_2 \le -6 \\ x_1, x_2 \ge 0 \end{cases}$$

rewrite to

$x_1 - 1 - x_2 + \lambda_1 - 2\lambda_2 - e_1 = 0$ (main K - T cond. for x_1)

$2x_2 - 1 - x_1 + \lambda_1 - 3\lambda_2 - e_2 = 0$ (main K - T cond. for x_2)

$x_1 + x_2 + s_1' = 3$ (original g_1 with slack)

$2x_1 + 3x_2 - e_2' = 6$ (original g_2 with excess)

$\lambda_2 e_2' = 0 \quad \lambda_1 s_1' = 0 \quad e_1 x_1 = 0 \quad e_2 x_2 = 0$ (complementary slackness)

All problem variables non - negative

QPP Tableau (I)

minimize $a_1 + a_2 + a_2'$

subject to
$$\begin{cases} x_1 - x_2 + \lambda_1 - 2\lambda_2 - e_2 + a_1 = 1 \\ -x_1 + 2x_2 + \lambda_1 - 3\lambda_2 - e_2 + a_2 = 1 \\ x_1 + x_2 + s_1' = 3 \\ 2x_1 + 3x_2 - e_2' + a_2' = 6 \end{cases}$$

solve for artificial variables and substitute into objective

objective	x_1	x_2	λ_1	λ_2	e_1	e_2	s_1'	e_2'	a_1	a_2	a_2'	*rhs*
1	2	4	2	–5	–1	–1	0	–1	0	0	0	8
0	1	–1	1	–2	–1	0	0	0	1	0	0	1
0	–1	2	1	–3	0	–1	0	0	0	1	0	1
0	1	1	0	0	0	0	1	0	0	0	0	3
0	2	3	0	0	0	0	0	–1	0	0	1	6

QPP Tableau (II)

The first pivot (*x*2 enters the basis) is a standard pivot

obj.	x_1	x_2	λ_1	λ_2	e_1	e_2	s_1'	e_2'	a_1	a_2	a_2'	*rhs*
1	4	0	0	1	−1	1	0	−1	0	−2	0	6
0	1/2	0	3/2	−7/2	−1	−1/2	0	0	1	1/2	0	3/2
0	−1/2	1	1/2	−3/2	0	−1/2	0	0	0	1/2	0	1/2
0	3/2	0	−1/2	3/2	0	1/2	1	0	0	−1/2	0	5/2
0	7/2	0	−3/2	9/2	0	3/2	0	−1	0	−3/2	1	9/2

QPP Tableau (III)

The second pivot ($x1$ enters the basis) is also a standard pivot

$obj.$	x_1	x_2	λ_1	λ_2	e_1	e_2	s_1'	e_2'	a_1	a_2	a_2'	rhs
1	0	0	12/7	−29/7	−1	−5/7	0	1/7	0	−2/7	−8/7	6/7
0	0	0	12/7	−29/7	−1	−5/7	0	1/7	1	5/7	−1/7	6/7
0	0	1	2/7	−6/7	0	−2/7	0	−1/7	0	2/7	1/7	8/7
0	0	0	1/7	−3/7	0	−1/7	1	3/7	0	1/7	−3/7	4/7
0	1	0	−3/7	9/7	0	3/7	0	−2/7	0	−3/7	2/7	9/7

Now $\lambda 1$ should enter the basis.
But since s1’ is already in the basis Wolfe’s algorithm does not allow this.

e2’ is the next best choice...

QPP Tableau (IV)

$obj.$	x_1	x_2	λ_1	λ_2	e_1	e_2	s_1'	e_2'	a_1	a_2	a_2'	rhs
1	0	0	5/3	−4	−1	−2/3	−1/3	0	0	−1/3	−1	2/3
0	0	0	5/3	−4	−1	−2/3	−1/3	0	1	2/3	0	2/3
0	0	1	1/3	−1	0	−1/3	1/3	0	0	1/3	0	4/3
0	0	0	1/3	−1	0	−1/3	7/3	1	0	1/3	−1	4/3
0	1	0	−1/3	1	0	1/3	2/3	0	0	−1/3	0	5/3

Now λ1 can enter the basis since s1' is no longer basic.

the final tableau is...

Final QPP Tableau

$obj.$	x_1	x_2	λ_1	λ_2	e_1	e_2	s_1'	e_2'	a_1	a_2	a_2'	rhs
1	0	0	0	0	0	0	0	0	-1	-1	-1	0
0	0	0	1	$-12/5$	$-3/5$	$-2/5$	$-1/5$	0	$3/5$	$2/5$	0	$2/5$
0	0	1	0	$-1/5$	$1/5$	$-1/5$	$2/5$	0	$-1/5$	$1/5$	0	$6/5$
0	0	0	0	$-1/5$	$1/5$	$-1/5$	$12/5$	1	$-1/5$	$1/5$	-1	$6/5$
0	1	0	0	$1/5$	$-1/5$	$1/5$	$3/5$	0	$1/5$	$-1/5$	0	$9/5$

Summary

In this section we have looked at

- Review of nonlinear single variable optimization
- Golden Section Search

- Unconstrained multi variable NLP
    - Steepest Ascent Method

- Constrained multi variable NLP
    - Euler-Lagrange Multipliers
    - Kuhn-Tucker Conditions

- Quadratic Programming
    - Wolfe’s extended Simplex Method