Interior Point Methods

In this section we will give an (extremely) brief
Introduction to the concept of

interior point methods

- Logarithmic Barrier Method
- Method of Centers

We have previously seen methods that follow a path
On the boundary of the feasible region (Simplex).

As the name suggest, interior point methods instead
Follow a path through the interior of the feasible region.

The discussion follows Chapter 2 of the “Interior Point Methods - mini course” by Gjerrit Meinsma

Importance of Interior Point Methods

For linear problems, interior point methods that are based on primal-dual formulations are competitive to Simplex for large scale problems.

Interior point methods are of central importance for optimizing large *convex* sub-classes of non-linear problems, in particular, so-called geometric programs and semi-definite programs.

There is an enormous variety of interior point methods and we will only look at two very very basic methods to illustrate the general idea.

NLP

$$\text{minimize} \qquad f(x)$$

$$\text{subject to} \quad g_i(x) \geq 0$$

Assume *f* is convex and *gi* are concave,
And that the feasible region is bounded.

Then the above NLP is a convex program

Log Barrier

Phi defines a barrier function grows fast when we
Approach the boundary of the feasible set.

$$\Phi(x) = -\sum_i \log g_i(x)$$

Note that phi is convex on the feasible region!

Log Barrier Method

The idea is to use this as a penalty on the current position
to make sure that we cannot cross the boundary
of the feasible region.

The revised problem becomes

$$\text{minimize} \quad f(x) + \frac{1}{\alpha}\phi(x)$$

We solve iteratively for increasing values of alpha.

Note as alpha increases the barrier is more strongly localized
In the limit of very large alpha, the two problems coincide.

Sequential Unconstrained Minimization

Input: Start point $x0$, tolerance ε, initial barrier multiplier $\alpha 0$
barrier step factor β; m is number of constraints

begin
  $x := x^{(0)}, \alpha := \alpha^{(0)}$.
  repeat
    $v := -(\nabla^2 f(x) + \frac{1}{\alpha}\nabla^2\phi(x))^{-1}(\nabla f(x) + \frac{1}{\alpha}\nabla\phi(x))$; (Newton direction)
    $\delta^* := \text{argmin}_\delta f(x + \delta v) + \frac{1}{\alpha}\phi(x + \delta v)$; (a line search)
    $x := x + \delta^* v$;
  until $\|v\|$ very small
  return if $m/\alpha < \epsilon$
  $\alpha := \alpha\beta$.
end.

The stopping condition is based on the Theorem:

$$0 \leq f(x_\alpha^*) - f^* \leq \frac{m}{\alpha}.$$

For any alpha we can approximate the true
Optimum to at least m/a

Method of Centers

The method of centers is based on the idea of a center point of
The feasible set.

The defined center point is called the "analytic center"

$$x_{ac} := \underset{x \text{ feasible}}{\arg\max} \prod_{i=1}^{m} g_i(x)$$

It is obvious that the "analytic center" is always somewhere in
The feasible region.

Revised Problem for MoC

Again, we turn the original constraint problem

$$\text{minimize} \quad f(x)$$

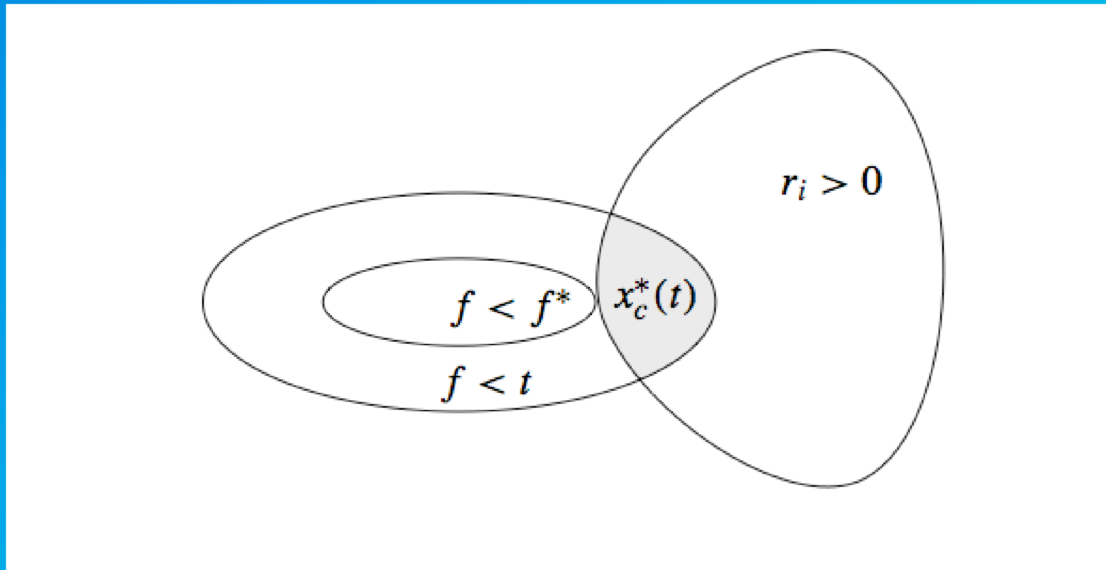
$$\text{subject to} \quad g_i(x) \geq 0$$

Into an unconstrained problem and perform a series of
Increasingly tight approximations. We look at the analytic center of
The original *gi* and the additional constraint $f(x) < t$.

$$x_{ac} := \underset{x \text{ feasible}}{\arg\max} \quad (t - f(x)) \prod_{i=1}^{m} g_i(x)$$

$$= \underset{x \text{ feasible}}{\arg\min} \quad -\log(t - f(x)) - \sum_{i=1}^{m} \log(g_i(x))$$

Geometric Interpretation of MoC



The analytic center is always in the intersection.
By reducing *t* we narrow the region of this intersection until
We are in an ε–neighborhood of the tangential point
of $f < f^*$ and $gi(x) > 0$.

We have

$$0 \leq f(x_c^*(t)) - f^* \leq m(t - f(x_c^*(t))).$$

(as t approaches f*, the optimum of the revised approaches the optimum of the original problem).

Method of Centers

Input: Start point x0, tolerance ε, initial upper bound *t*
update interpolation rate Θ

begin
$x := x^{(0)}$

repeat
$v := -[\nabla^2(-\log(t - f(x)) + \log \phi(x))]^{-1}$
$\times \nabla(-\log(t - f(x)) + \phi(x))$ (Newton direction)
$\delta^* := \text{argmin}_\delta - \log(t - f(x + \delta v)) + \phi(x + \delta v)$ (line search)
$x := x + \delta^* v$
until $\|v\|$ very small
return if $m(t - f(x)) < \epsilon$
$t := (1 - \theta) f(x) + \theta t$
end

The initial feasible point

Both methods discussed require an initial feasible interior point.

How can we find this?

- Think about how we did this for the Simplex method:
  - find a phase 0 problem with a trivial feasible point
  - Optimize this problem to obtain a starting point for the original problem