LP and SDP upper bounds for Permutation Arrays

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(joint work with Peter Dukes UVIC)

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Definitions

A **Permutation array** $\Gamma(n, d; s)$ of order n is an $s \times n$ array such that:

- Each cell contains one element of $\{1, \ldots, n\}$
- Each symbol occurs exactly once in each row
- Any two rows disagree in at least d columns

. . . each row is a permutation of the n symbols.

1	2	3	4	5
2	1	5	4	3
2	5	1	3	4
3	2	5	1	4
3	4	1	5	2
3	5	4	2	1
4	3	2	1	5
5	1	4	3	2

A $\Gamma(5, 4, 8)$

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- 1974 Blake: Application to PLC.
- 1977-1986 subject studied by Vanstone, Deza, Mathon, Cameron, . . .
- 1999 - now: Colbourn, Chu, Dukes, Cameron, Bailey, Kløve, Han Vinck,. . .

Distance

Let d_H be **the Hamming distance** on $Sym(n)$:

$$d_H(\phi, \psi) = |\{i : \phi(i) \neq \psi(i)\}| = n - |fix(\phi\psi^{-1})|$$

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Property

The Hamming distance on $Sym(n)$ is bi-invariant:
$d_H(\phi, \psi) = d_H(\alpha\phi, \alpha\psi) = d_H(\phi\alpha, \psi\alpha) \qquad \forall \alpha, \phi, \psi \in Sym(n)$

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Any bi-invariant distance is invertible: $d_H(\phi,\psi) = d_H(\phi^{-1},\psi^{-1})$.

Theorem Farahat 1960

The isometry group of $(Sym(n), d_H)$ is isomorphic to $Sym(n) \wr 2$

Maximal PA

A $\Gamma(n, d; s)$ is **maximal** if it is not included in a $\Gamma(n, d; s+1)$ (can't add a row).

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A maximal $\Gamma(5, 4; 7)$

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$\mu(n,d)$ =Maximal size s such that a $\Gamma(n,d;s)$ exists.

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- $\mu(n,d) \leq \frac{n!}{(d-1)!}$
- $\frac{n!}{V_n(d-1)} \leq \mu(n,d) \leq \frac{n!}{V_n(\frac{d}{2})}$ $\qquad V_n(t) = \sum_{k=0}^{t} \binom{n}{k} \lfloor \frac{k!}{e} \rceil$

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  (card. of a ball of radius t)
- A $PG(2,q)$ gives a $\Gamma(q, q-1; q(q-1))$.

UB and LB for $\mu(n, d)$

$\mu(n,d)$	3	4	5	6	7	8	9
5	**60**	**20**	**5**				
6	**360**	**120**	**18**	**6**			
7	**2520**	[349-*543*]	[77-*140*]	**42**	**7**		
8	**20160**	[2688-*4135*]	[560- *926*]	**336**	**56**	**8**	
9	**181440**	[18144-*32989*]	[1944- *7128*]	[1512-*1962*]	**504**	**72**	**9**
10	$\frac{10!}{2}$	[86400-*302400*]	[13680-*64800*]	[4320-*16941*]	[?? -*4699*]	**720**	[35-90

Cayley graphs

G a group, Ω a subset s.t. $\Omega^{-1} = \Omega$ and $Id \notin \Omega$

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A maximal permutation array Γ corresponds to a **maximal clique** in $Cay(Sym(n), \Omega_d)$,
with $\Omega_d :=$ {all perms with $\leq n - d$ fixed points }.
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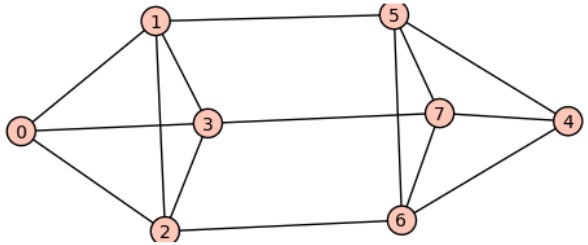
The action of $Aut(Cay(G, \Omega))$ on the vertices is **sharply transitive**

Lovasz θ number



$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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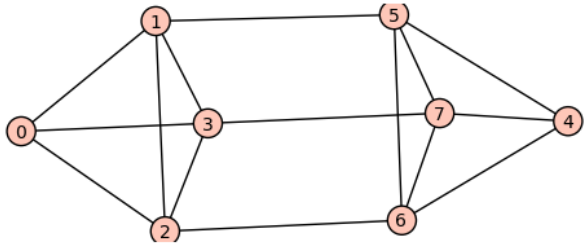

$$I + A_G = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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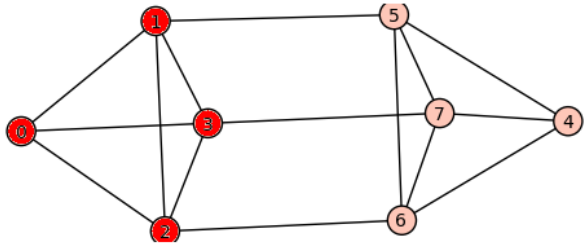

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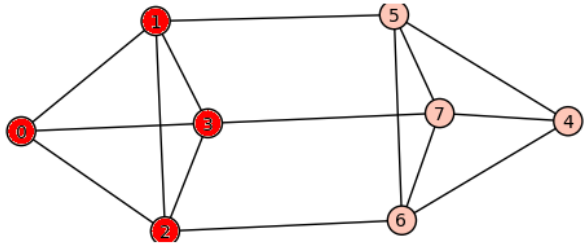

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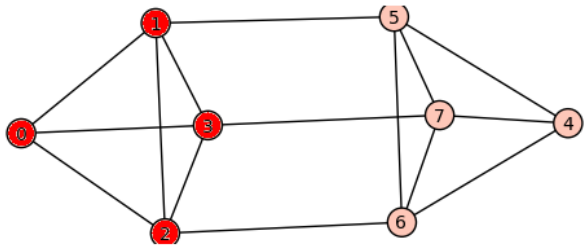

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Properties of $\tilde{A}_K = \frac{A_K}{|K|}$

$$|K| = \operatorname{tr} J\tilde{A}_K$$

$$[(i,j) \notin E \ \& \ i \neq j] \qquad \Rightarrow (\tilde{A}_K)_{ij} = 0$$

$$\operatorname{tr} \tilde{A}_K = 1 \qquad \tilde{A}_K \succeq 0 \qquad \tilde{A}_K \text{ symmetric}$$

Lovàsz θ function

Lovàsz $\bar{\theta}(G) = \theta(\bar{G})$ function

$$\bar{\theta}(G) = \max \operatorname{tr} JX$$

subject to:

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- Sandwich theorem $\omega(G) \leq \bar{\theta}(G) \leq \chi(G)$

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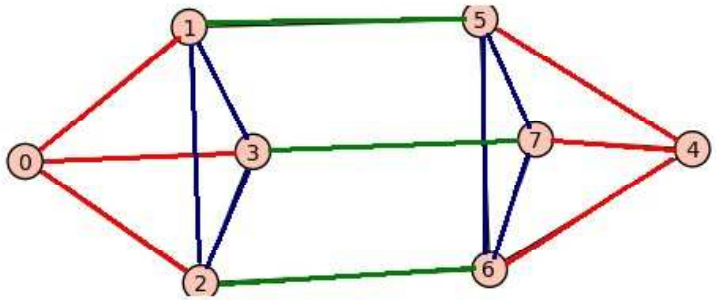
$$A_K^{sym} = \begin{pmatrix} x & a & a & a & 0 & 0 & 0 & 0 \\ a & y & b & b & 0 & c & 0 & 0 \\ a & b & y & b & 0 & 0 & c & 0 \\ a & b & b & y & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & x & b & b & a \\ 0 & c & 0 & 0 & b & y & b & a \\ 0 & 0 & c & 0 & b & b & y & a \\ 0 & 0 & 0 & c & a & a & a & y \end{pmatrix}$$

see E. De Klerk



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For $\Gamma(n, d; s)$, $R_\Gamma := \frac{1}{|\Gamma|} 1_\Gamma 1_\Gamma^\top \qquad \tilde{R}_\Gamma := \frac{1}{|Iso(n)|} \sum_{t \in Iso(n)} R_{t(\Gamma)}$

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Then: $\mathrm{tr}(J\tilde{R}_\Gamma) = s$, $\tilde{R}_\Gamma = \sum a_i A_i$, $\mathrm{tr}(\tilde{R}_\Gamma) = 1$, $(\tilde{R}_\Gamma)_{i,j} = 0$ if $0 < d_H(\phi_i, \phi_j) < d$

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SDP in $\mathcal{M}$

$$\mu_{\mathcal{M}}(n, d) := \max \operatorname{tr}(JX) \text{ s.t. } \begin{cases} X = \sum a_i A_i \\ (X)_{i,j} = 0 \text{ if } 0 < d_H(\phi_i, \phi_j) < d \\ \operatorname{tr}(X) = 1 \end{cases}$$

$$\mu_{\mathcal{M}}(n, d) \geq \mu(n, d)$$

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$$\mu_{\mathcal{M}}(n, d) := \max \operatorname{tr}(JX) \text{ s.t. } \begin{cases} X = \sum a_i A_i \\ (X)_{i,j} = 0 \text{ if } 0 < d_H(\phi_i, \phi_j) < d \\ \operatorname{tr}(X) = 1 \end{cases}$$

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The $n! \times n!$ symmetric matrices (A_i) generate the commutative Bose-Mesner algebra $\mathcal{M}$.

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See L. Schrijver for bin.

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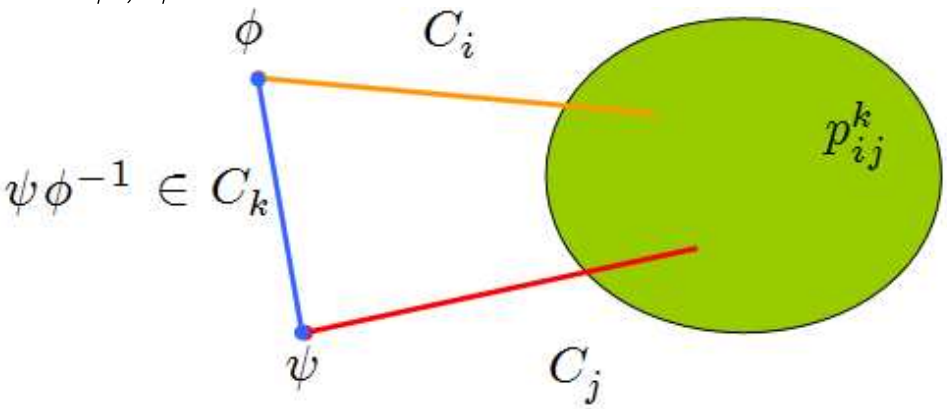
$$X \succeq 0 \Leftrightarrow QXQ^\top \geq 0$$

Conjugacy scheme

Suppose $\phi\psi^{-1} \in C_k$.
$p_{ij}^k := |\{\theta \in Sym(n) : \phi\theta^{-1} \in C_i, \psi\theta^{-1} \in C_j\}|$ does not depend on ϕ, ψ.



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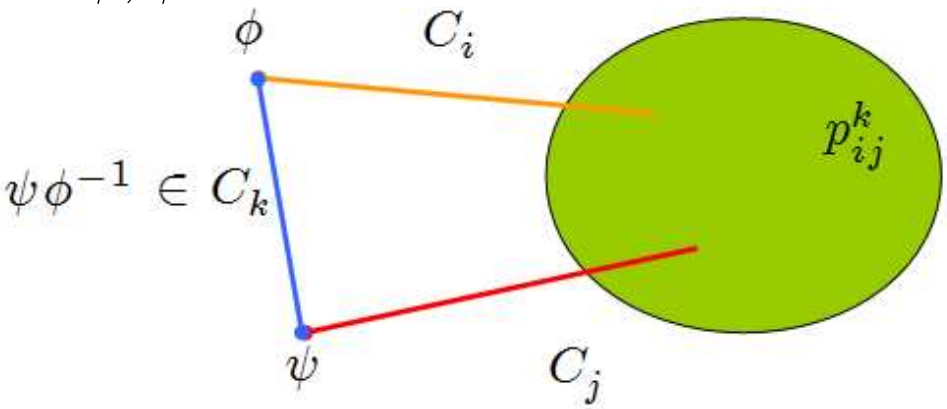

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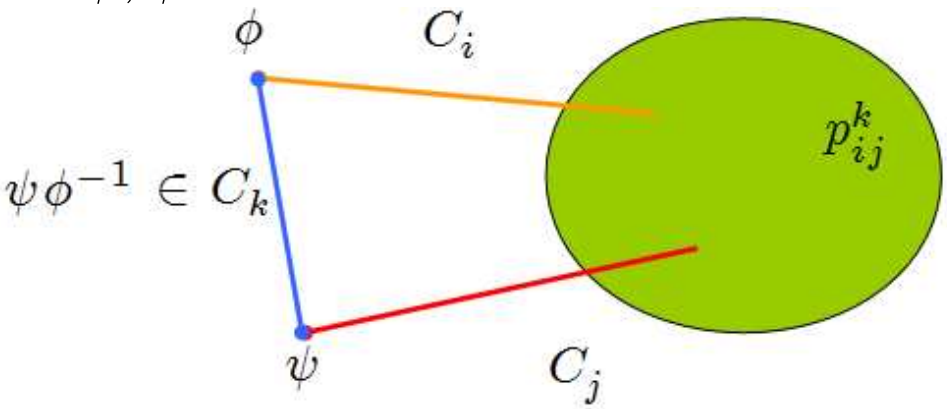

$(Sym(n); C_0, \ldots, C_{p(n)})$ is an association scheme (**conjugacy scheme**).
$\mathcal{M}$ has an orthogonal basis, with transition matrices P and Q. In the conjugacy scheme, the transition matrix Q is obtained by evaluation of the irreducible characters χ_i of $Sym(n)$.

Conjugacy scheme (2)

$C_0, \ldots C_{p(n)}$: conjugacy classes of $Sym(n)$.

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For any $\Gamma(n, d; s)$, the distribution vector $\bar{a} = (a_0, \dots, a_{p(n)})$ satisfies:

$$\begin{cases} \sum_j Q_{ij} a_j \geq 0 & \forall i \\ a_0 = 1 & \\ a_i = 0 & \text{if } \phi\psi^{-1} \notin C_i \quad \forall \phi, \psi \in \Gamma \\ \sum_i a_i = s & \end{cases}$$

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Theorem (LP bound for permutation codes (Tarnanen,1999))

Let $D \subset \{1, \ldots, p(n)\}$
$E \subset S_n$ *s. t. for any distinct permutations* $\phi, \psi \in E$,
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Tarnanen gives upper bounds for $n \leq 10$.
Irreducible characters are available on GAP, Magma,...

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- are known up to $Sym(19)$ (NP-Hard)

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$\mu(n,d)$	LP bound	Older bound
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Define $\chi|_{C_l}(t) :=$ number of perms fixed by $t \in Iso(n)_{Id}$

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n	m_k
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5	$1, 1, 3, 3, 5, 5, 6, 7$
6	$1, 1, 1, 1, 1, 1, 3, 3, 4, 6, 7, 8, 8, 9, 9, 11, 15$
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Improvements via SDP in $\mathcal{T}$

	$\mu_{\mathcal{T}}$	$\mu_{\mathcal{M}}$
$(6,5)$	24	30
$(7,5)$	134	140
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For $n \geq 8$

- Hard to compute ϵ_i's
- Too many variables...

More constraints

$$a_i = \frac{|\{(\phi, \psi) : \phi\psi^{-1} \in C_i, \phi, \psi \in \Gamma(n, d; s)\}|}{s}$$

$=$ average number of $\psi \in \Gamma$ with $\psi\phi^{-1} \in C_i$ for a fixed $\phi \in \Gamma$.

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Constraint (1b)

$max(|E_i(d)|)$ can be upper bounded by Lovasz θ number.

More constraints

Consider the conjugacy classes C_i, $i \in D_w$ such that $\bigcup_{i \in D_w} C_i$ contains all ϕ with $|Supp(\phi)| = w$.

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Theorem:

- $\mu(n, n, n) = n - 1$
- $\mu(n, n, n-1) = n$
- $\mu(n, d, w) \leq \binom{n}{k} \mu(k, d, w)$ for $w \leq k \leq n$.
- $\mu(n, d, w) \leq \mu(n-1, d, w) + (n-1)(\mu(n-1, d, w-1) + \mu(n-2, d, w-2))$ for $w < n$.
- $\mu(n, d, n) \leq (n-1)(\mu(n-1, d, n-1) + \mu(n-2, d, n-2))$
- $\mu(n, n-2, n) \leq (n-1)(\mu(n-1, n-2) - 1)$

Consequences of the new constraints

For $n = 6, d = 5$, the new constraints change $\mu(6,5) \leq 30$ into $\mu(6,5) \leq 21$.

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For $\nu(n,d)$:

$\nu(n,d)$	3	4	5	6
4	**6**	**24**		
5	**6**	**24**	**120**	
6	**6**	**24**	**120**	**720**
7	**7**	**24**	120	720
8	**8**	**29**	120	720
9	**9**	**37**	120	720
10	**10**	46	120	720
11	**11**	56	146	720

Lower bounds for $\nu(n,d)$

Thank you