

A Universal Bijection for Catalan Families

R. Brak

School of Mathematics and Statistics
University of Melbourne

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Catalan Numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$1, 2, 5, 14, 42, 132, 429, \dots$$

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$$

$$G(z) = \sum_{n \geq 0} C_n z^n, \quad G = 1 + zG^2$$

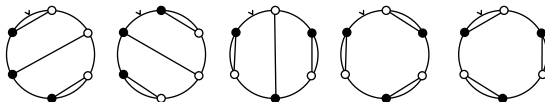
A few Catalan families

Examples of C_3 objects

\mathbb{F}_1 – Matching brackets and Dyck words

$\{\{\{\}\}\}$ $\{\{\}\{\}\}$ $\{\{\{\}\}\}$ $\{\{\}\}\{\}$ $\{\{\}\}\{\}$

\mathbb{F}_2 – Non-crossing chords the circular form of nested matchings



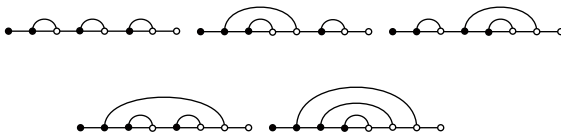
\mathbb{F}_3 – Complete Binary trees and Binary trees



\mathbb{F}_4 – Planar Trees



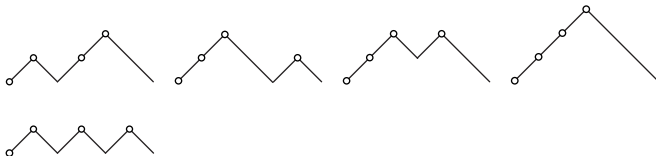
\mathbb{F}_5 – Nested matchings or Link Diagrams



\mathbb{F}_6 – Non-crossing partitions



\mathbb{F}_7 – Dyck paths



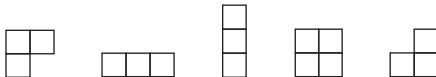
\mathbb{F}_8 – Polygon triangulations



\mathbb{F}_9 – 321-avoiding permutations

123, 213, 132, 312, 231.

\mathbb{F}_{10} – Staircase polygons



\mathbb{F}_{11} – Pyramid of heaps of segments



\mathbb{F}_{12} – Two row standard tableau

1	3	5	1	2	5	1	2	3	1	2	4	1	3	4
2	4	6	3	4	6	4	5	6	3	5	6	2	5	6

\mathbb{F}_{13} – Non-nested matchings



\mathbb{F}_{14} – Frieze Patterns: $n - 1$ row periodic repeating rhombus

$$\begin{array}{cccccccccccccccc}
 \cdots & 1 & & 1 & & \cdots & & 1 & & \cdots & & & & & & & \\
 & \cdots & a_1 & & a_2 & & \cdots & & a_n & & \cdots & & & & & & \\
 & & \cdots & b_1 & & b_2 & & \cdots & & b_n & & \cdots & & & & & \\
 & & & & \ddots & & & & & & \ddots & & & & & & \\
 & & & \cdots & r_1 & & r_2 & & \cdots & & r_n & & \cdots & & & & \\
 & & & & \cdots & 1 & & 1 & & \cdots & & 1 & & \cdots & & &
 \end{array}$$

with

$$\begin{array}{c} r \\ s \quad t \\ u \end{array} \quad \text{and} \quad st - ru = 1$$

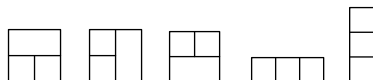
12213, 22131, 21312, 13122, 31221.

The Catalan Problem

- Over 200 families of Catalan objects:
Richard Stanley: "Catalan Numbers" (2015)
- Regular trickle of new families ...
- Alternative Tableau (2015) – related to Weyl algebra

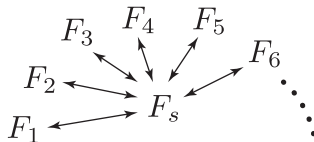


- Floor plans (2018)

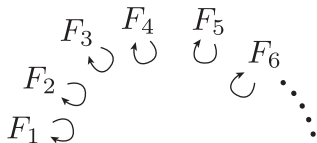


- How to prove Catalan: Focus on bijections
- **Problem I:** Too many bijections.
- Assume 200 families: F_1, F_2, F_3, \dots
- $\Rightarrow \binom{200}{2} = 19\,900$ possible bijections

- Better: Biject to a common family



- Which family F_s ?
- Even better:



■ **Problem II:** Proofs can be lengthy

Dyck words \leftrightarrow Staircase polygons (Delest & Viennot 1984)



■ **Problem III:** Uniqueness:

If $|A| = |B| = n$ then $n!$ possible bijections.

■ Why choose any one?

The Magma

- Solution to all three problems:
- Replace “bijection” by “isomorphism”
- What algebra?
- Magma

■ Definition (Magma – Bourbaki 1970)

A **magma** defined on \mathcal{M} is a pair (\mathcal{M}, \star) where \star is a map

$$\star : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$$

called the **product map** and \mathcal{M} a non-empty set, called the **base set**.

- No conditions on map.

Additional definitions

- **Unique factorisation magma:** if product map \star is injective.
- **Magma morphism:** Two magmas, (\mathcal{M}, \star) and (\mathcal{N}, \bullet) and a map

$$\theta : \mathcal{M} \rightarrow \mathcal{N}$$

satisfying

$$\theta(m \star m') = \theta(m) \bullet \theta(m').$$

- **Irreducible elements:** elements not in the image (range) of the product map.

Example magma

\star	1	2	3	4	5	...
1	5	7	10	3	16	22
2	6	9	4	15	21	\ddots
3	8	4	14	20	27	\ddots
4	11	13	19	26		\ddots
5	12	18	25			\ddots
\vdots	17	24	\ddots	\ddots	\ddots	\ddots

- ii) Not a unique factorisation magma: $4 = 2 \star 3 = 3 \star 2$.
- iii) Two “irreducible” elements: 1, 2 absent.

Standard Free magma

Definition

Let X be a non-empty finite set. Define the sequence $W_n(X)$ of sets of nested 2-tuples recursively by:

$$W_1(X) = X$$

$$W_n(X) = \bigcup_{p=1}^{n-1} W_p(X) \times W_{n-p}(X), \quad n > 1,$$

$$\mathcal{W}_X = \bigcup_{n \geq 1} W_n(X).$$

Let $\mathcal{W}_X = \bigcup_{n \geq 1} W_n(X)$.

Define the product map $\diamond : \mathcal{W}_X \times \mathcal{W}_X \rightarrow \mathcal{W}_X$ by

$$m_1 \diamond m_2 \mapsto (m_1, m_2)$$

The pair $(\mathcal{W}_X, \diamond)$ is called the **standard free magma** generated by X .

- Elements of \mathcal{W}_X for $X = \{\epsilon\}$:

$$\begin{aligned} \epsilon, \quad (\epsilon, \epsilon), \quad (\epsilon, (\epsilon, \epsilon)), \quad ((\epsilon, \epsilon), \epsilon), \\ (\epsilon, (\epsilon, (\epsilon, \epsilon))), \quad ((\epsilon, (\epsilon, \epsilon)), \epsilon), \quad (\epsilon, ((\epsilon, \epsilon), \epsilon)), \\ (((\epsilon, \epsilon), \epsilon), \epsilon), \quad ((\epsilon, \epsilon), (\epsilon, \epsilon)) \dots \end{aligned}$$

- Three ways to write products:

$$\epsilon\epsilon\epsilon\blacklozenge\blacklozenge, \quad \blacklozenge\epsilon\blacklozenge\epsilon\epsilon \quad \text{and} \quad (\epsilon\blacklozenge(\epsilon\blacklozenge\epsilon))$$

all give

$$(\epsilon, (\epsilon, \epsilon)).$$

We need one additional ingredient to make connection with Catalan numbers.

■ Definition (Norm)

Let (\mathcal{M}, \star) be a magma. A **norm** is a super-additive map

$$\|\cdot\| : \mathcal{M} \rightarrow \mathbb{N}$$

.

- Super-additive: For all $m_1, m_2 \in \mathcal{M}$
 $\|m_1 \star m_2\| \geq \|m_1\| + \|m_2\|.$
- If (\mathcal{M}, \star) has a norm it will be called a **normed magma**.
- Standard Free magma norm: if $m \in W_n$ then $\|m\| = n.$
- eg. $\|(\epsilon, (\epsilon, \epsilon))\| = 3.$

With a norm we now get:

Proposition (Segner 1761)

Let $\mathcal{W}(X)$ be the standard free magma generated by the finite set X . If

$$W_\ell = \{m \in \mathcal{W}_\epsilon : \|m\| = \ell\}, \quad \ell \geq 1,$$

then

$$|W_\ell| = |X|^\ell C_{\ell-1} = |X|^\ell \frac{1}{\ell} \binom{2\ell-2}{\ell-1}, \quad (2)$$

and for a single generator, $X = \{\epsilon\}$, we get the Catalan numbers:

$$|W_\ell| = C_{\ell-1} = \frac{1}{\ell} \binom{2\ell-2}{\ell-1}. \quad (3)$$

Main theorem

■ Theorem (RB)

Let (\mathcal{M}, \star) be a unique factorisation normed magma. Then (\mathcal{M}, \star) is isomorphic to the standard free magma $\mathcal{W}(X)$ generated by the irreducible elements of \mathcal{M} .

■ Proof

- Use norm to prove reducible elements have finite recursive factorisation.
- Use injectivity to get bijective map to set of reducible elements.
- Morphism straightforward.

Definition (Catalan Magma)

A unique factorisation normed magma with only one irreducible element is called a **Catalan magma**.

Consequences...

- If we can define a product

$$\star_i : \mathbb{F}_i \times \mathbb{F}_i \rightarrow \mathbb{F}_i$$

on a set \mathbb{F}_i and:

- show \star_i is injective,
 - has one irreducible element
 - and define a norm, then
- \mathbb{F}_i is a Catalan magma and \mathbb{F}_i isomorphic to $W(\varepsilon)$:

$$\Gamma_i : \mathbb{F}_i \rightarrow W(\varepsilon)$$

- and thus
 - Γ_i is in bijection,
 - norm partitions \mathbb{F}_i into Catalan number sized subsets,
 - the bijection is recursive,
 - and embedded bijections, Narayana statistic correspondence, ...

Universal Bijection

- The proof is constructive and thus gives

$$\Gamma_i : \mathbb{F}_i \rightarrow W(\varepsilon)$$

explicitly.

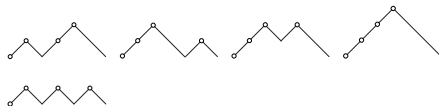
- Furthermore, the bijection is “universal” – same (meta) algorithm for all pairs of families.
-

$$\begin{array}{ccc} \mathbb{F}_i & \xrightarrow{\pi} & W_{\varepsilon_i} \\ \vdots \Gamma_{i,j} & & \downarrow \theta_{i,j} \\ \mathbb{F}_j & \xleftarrow{\mu} & W_{\varepsilon_j} \end{array} \quad (4)$$

- Morphism implies recursive: $\Gamma(m_1 \star m_2) = \Gamma(m_1) \bullet \Gamma(m_2)$.

Example: Dyck path Magma

■ Dyck Paths

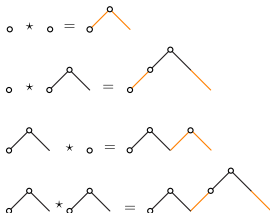


■ Product



■ Generator: $\varepsilon = \circ$ (a vertex).

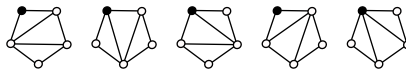
■ Examples



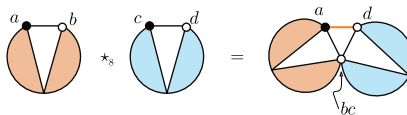
■ Norm = Number of up steps + 1

Example: Triangulation Magma

■ Polygon Triangulation's

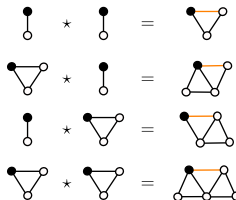


■ Product:



■ Generator $\epsilon =$

■ Examples:



■ Norm = (Number of triangles) + 1

Example: Frieze pattern Magma (Conway and Coxeter 1973)

\mathbb{F}_{14} – Frieze Patterns: $n - 1$ row periodic repeating rhombus

$$\begin{array}{cccccccccccccccc}
\dots & 1 & & 1 & & \dots & & 1 & & \dots & & & & & & & \\
& \dots & a_1 & & a_2 & & \dots & & a_n & & \dots & & & & & & \\
& & \dots & b_1 & & b_2 & & \dots & & b_n & & \dots & & & & & \\
& & & & \ddots & & & & & & \ddots & & & & & & \\
& & & \dots & r_1 & & r_2 & & \dots & & r_n & & \dots & & & & \\
& & & & \dots & 1 & & 1 & & \dots & & 1 & & \dots & & &
\end{array}$$

with $\begin{matrix} r \\ s \end{matrix} \begin{matrix} t \\ u \end{matrix}$ and $st - ru = 1$

12213, 22131, 21312, 13122, 31221.

- Product: $a_1, a_2, \dots, a_n \star b_1, b_2, \dots, b_m = c_1, c_2, \dots, c_{n+m-1}$
where

$$c_i = \begin{cases} a_1 + 1 & i = 1 \\ a_i & 1 < i < n \\ a_n + b_1 + 1 & i = n \\ b_i & n < i < n + m - 1 \\ b_m + 1 & i = n + m - 1 \end{cases} \quad (5)$$

- Generator: $\varepsilon = 00$.
- Examples:

$$00 \star 00 = 111$$

$$00 \star 111 = 1212$$

$$111 \star 00 = 2121$$

$$111 \star 111 = 21312$$

- Norm = (Length of sequence) - 1

Bijections

- First, factorise path to its generators

$$\begin{aligned}
 & \text{Diagram of a path with 4 vertices and 3 edges} = \text{Diagram of a path with 4 vertices and 3 edges, with the middle edge highlighted in orange} \\
 & = \text{Diagram of a path with 3 vertices and 2 edges} \star_7 \text{Diagram of a path with 2 vertices and 1 edge} \\
 & = (\circ \star_7 \circ) \star_7 (\circ \star_7 \circ)
 \end{aligned}$$

- then change generators and product rules: $\star_7 \rightarrow \star_8$:

$$(\circ \star_7 \circ) \star_7 (\circ \star_7 \circ) \mapsto (\bullet \star_8 \bullet) \star_8 (\bullet \star_8 \bullet)$$

- then re-multiply:

$$\begin{aligned}
 (\bullet \star_8 \bullet) \star_8 (\bullet \star_8 \bullet) & = \text{Diagram of a triangle with one black vertex} \star_8 \text{Diagram of a triangle with one black vertex} \\
 & = \text{Diagram of a pentagon with one black vertex and all internal diagonals}
 \end{aligned}$$

- which gives the bijection

$$\text{Diagram of a path with 4 vertices and 3 edges} \mapsto \text{Diagram of a pentagon with one black vertex and all internal diagonals}$$

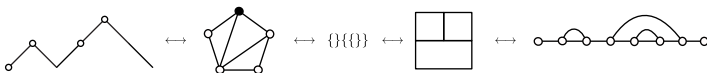
- Similarly, if we perform the same multiplications for matching brackets:

$$(\emptyset \star_1 \emptyset) \star_1 (\emptyset \star_1 \emptyset) = \{\} \star_1 \{\} = \{\}\{\}\{\}$$

- or for nested matchings,

$$(\bullet \circ \star_{27} \bullet \circ) \star_{27} (\bullet \circ \star_{27} \bullet \circ) = \bullet \overset{\curvearrowright}{\bullet} \circ \star_{27} \bullet \overset{\curvearrowright}{\bullet} \circ = \bullet \overset{\curvearrowright}{\bullet} \circ \bullet \overset{\curvearrowright}{\bullet} \circ$$

- Thus we have the bijections:



Conclusion

- Magmatisation of Catalan families gives “universal” recursive bijection.
- Also, embedded bijections, Narayanaya statistic etc.
- Adding a unary map gives Fibonacci, with binary map gives Motzkin, Schröder paths etc.
- Current projects:
 - Extending to coupled algebraic equations eg. pairs of ternary trees
 - Reformulating the “symbolic” method.
- Reference: [arXiv:1808.09078](https://arxiv.org/abs/1808.09078) [math.CO]

– Thank You –