Minimum Opaque Covers for Polygonal Regions

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The Opaque Cover Problem

Given a set S in the plane, an *opaque cover (OC)* for S is any set F having the property that any line in the plane intersecting S also intersects F.

The problem of finding an opaque cover of minimum length for any given planar set S is known as the *Opaque Cover Problem (OPC)*.

Intuitively, an opaque cover forms a barrier that makes it impossible to see through S from any vantage point

Formal Definitions

How do we define "length"?

Defining Length

For set $F \in \Re^2$, the *1-dimensional Hausdorff measure* of S is defined by

$$\lambda_1(F) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(E_i) \mid \bigcup_{i=1}^{\infty} E_i = F, \mathrm{diam}(E_i) \leq \delta \right\}$$

where $\mathrm{diam}(E)$ is the supremum of the distance between any two points of E.

Note that this matches the normal definition of Euclidean length when it is defined, but exists for any set in $\Re^2$.

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Formal Definitions

The Opaque Cover Problem (OCP)

Given: a compact connected set in S in $\Re^2$.
Find: a set F of minimum 1-dimensional Hausdorff measure, such that F is an OC for S.

Some versions of the OCP :
interior OCs: F is required to lie entirely inside S.
graphical OCs: F is required to be composed of a finite number of straight-line segments.
connected OCs: F is required to be graphical and connected.
single-path OCs: F is required to be graphical and a single path.

Some Basic Theorems

Notation. For any set S, $\bar{S}$ represents the convex hull of S.

3 OCP Theorems

1) A set F is an OC for S if and only if it is an OC for $\bar{S}$.

2) If F is an OC for S then $S \subseteq \bar{F}$.

3) Any OC F for S has length $\lambda_1(F) \geq diam(S)$.

The OCP for polygonal regions

We now assume that the boundary of S, ∂S, is a *convex polygon*, with vertices denoted by V_S.

The Graphical Conjecture

A minimum OC is always graphical.

Let $\rho(S)$ denote the length of ∂S, and let $\mathrm{st}(S)$ be the length of a minimum Steiner Tree (MST) on V_S.

Basic Polygonal OCP Theorems

Let F be a minimum graphical OC for S in $\Re^2$. Then
1) each component C of F is an MST on $V_{\bar{C}}$, and
2) $\rho(S)/2 \leq \lambda_1(F) \leq \mathrm{st}(V_S)$.

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Interior connected OCs

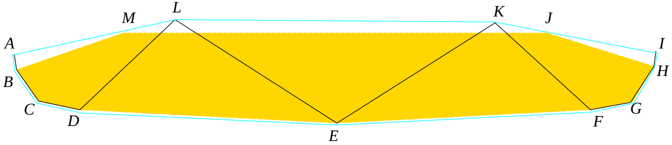
The interior connected OCP is the only known OC problem for which the solution can be fully characterised and computed.

Interior Connected OCP Theorems

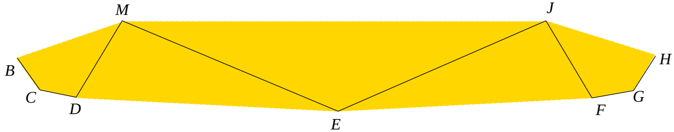
1) A minimum interior connected OC for S is an MST on V_S.

2) The interior connected OCP is NP-hard, but has a fully polynomial approximation scheme.

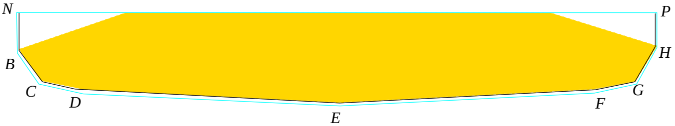
General Connected OCs



(a) Conjectured minimum connected OC: length = 4091.17



(b) MST on V_S: length = 4100.58



(c) "Tent" path: length = 4093.04

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General Connected OCs

2 Structural Lemmas

1) The vertices of $\partial\bar{F}$ are exactly the degree 1 or 2 vertices of F.
2) There are at most $2|V_S|$ vertices in $\partial\bar{F}$.

Definitions - Critical Points and Lines

1) A *critical point* of a graphical OC F for S is a vertex v of F that is not in V_S but which can be perturbed, along with its adjacent edges in F, in such a way that the length of F decreases.
2) A *critical line* L of F is the limit of violating lines obtained by length-decreasing perturbations of critical points.

General Connected OCs

Critical Points Lemmas

Let F be a minimum connected OC for S. The critical points of F are precisely the vertices of $\partial\bar{F}$ that are not vertices of ∂S.

Definition - Free Critical Points

A critical point v is a *free critical point (FCP)* if it lies on only one critical line.

It appears likely that the only "difficult" critical points to locate are FCPs. By considering perturbations we can develop a list of possible locations of FCPs on critical lines.

Connected OCs for Triangular Regions

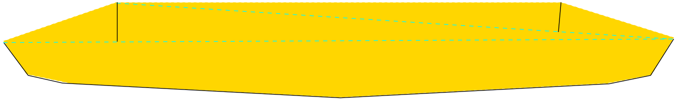
Connected OCs for Triangular Regions Theorem

For any triangular region $S = \triangle abc$, the minimum connected OC for S is the MST on $\{a, b, c\}$.

This does not generalise for polygonal regions with more than three edges of their boundary.

General Graphical OCP

The Conjectured minimum OC (with no constraints) for the previous example is shown below and has length $= 4089.27$.



Note: Each component C of a graphical minimum OC is an MST on the vertices of its convex hull $\bar{C}$. Further, the union of these $\bar{C}$'s must block all lines passing through S.

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General Graphical OCP

For a graphical OC F made up of multiple components, let $\mathcal{F}$ be the set of extreme points of the convex hulls of the components of F.

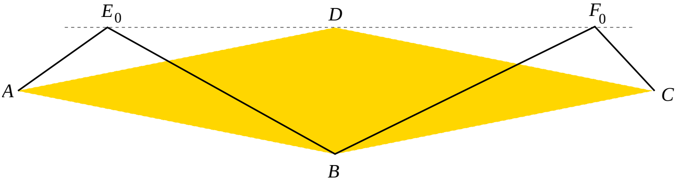
Lemma

The critical points for F are precisely the elements of $\mathcal{F} \setminus V_S$.

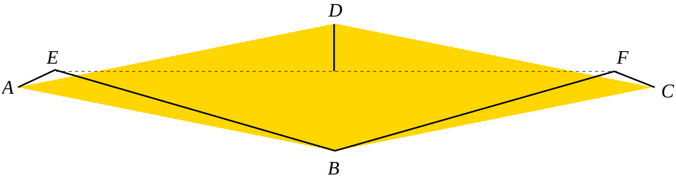
As before, we can develop a list of constraints on locations of FCPs on critical lines and the angles the incident edges of F make with a critical line.

We can also show there is a restriction on the number of Steiner points in any component of F.

General Graphical OCP



(a) minimum connected OC



(b) conjectured minimum OC

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Open Questions and Future Directions

The long-term aim is to find a polynomial time (or even finite) algorithm for solving OCP for a polygonal region. Some key open questions are the Graphical Conjecture and the following:

1. *Is the minimum graphical OC for a triangular region S the MST on V_S?* We conjecture that this is the case, and have proved it for OCs containing at most two connected components.
2. *In a minimum graphical OC, under what conditions can there exist critical points that are not free?* All of the conjectured minimum OCs that the authors have seen have the property that any critical point that is not free is determined by two critical lines that are extensions of edges of S.