Conjugacy growth in groups, geometry and combinatorics

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Counting elements and conjugacy classes

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- Growth of G: number of elements of length n in G, for all $n \geq 0$.
- Standard growth functions:

$$\texttt{sphere} \longrightarrow a(n) = a_{G,X}(n) := \sharp\{g \in G \mid |g|_X = n\}$$

$$\texttt{ball} \longrightarrow A(n) = A_{G,X}(n) := \sharp\{g \in G \mid |g|_X \leq n\}.$$

A finite example

Let $G = S_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle$, $X = \{a, b\}$.

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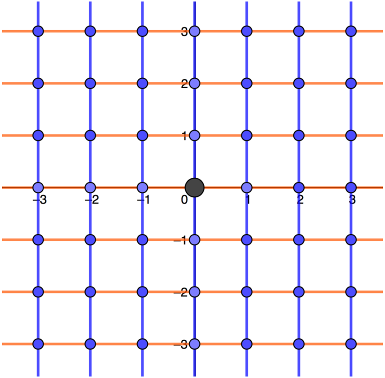
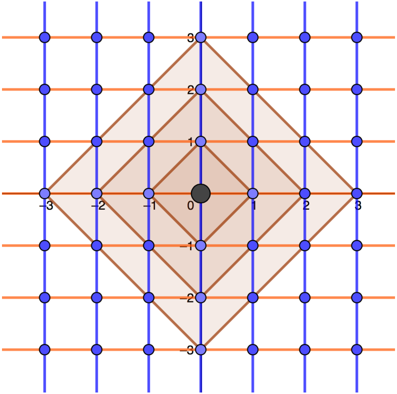
Cayley graph of $\mathbb{Z}^2$ with standard generators a and b



$\mathbb{Z}^2$ with standard generators a and b



$$a(k) = 4k, \quad A(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1$$

Counting in groups

- Conjugacy growth of G: number of conjugacy classes containing an element of length n in G, for all $n \geq 0$.

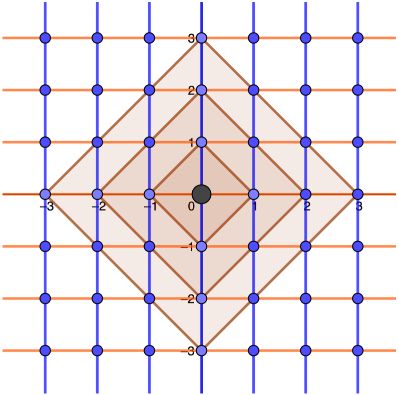
- Conjugacy growth functions:

$$c(n) = c_{G,X}(n) := \sharp\{[g] \in G \mid |g|_c = n\}$$

$$C(n) = C_{G,X}(n) := \sharp\{[g] \in G \mid |g|_c \leq n\},$$

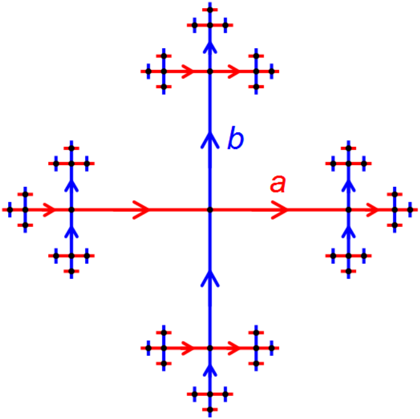
where $|g|_c$ is the length of a shortest element in the conjugacy class $[g]$, with respect to X.

$\mathbb{Z}^2$ with standard generators a and b



$$a(k) = c(k) = 4k, \quad A(n) = C(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1$$

Examples: F_2 with free generators a and b



$$a(n) = 4 \cdot 3^{n-1}$$

Asymptotics of conjugacy growth in the free group F_r

Idea: take all cyclically reduced words of length n, whose number

is $(2r-1)^n + 1 + (r-1)[1+(-1)^n]$, and divide by n.

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Coornaert (2005): For the free group F_k, the primitive (non-powers) conjugacy growth function is given by

$$c_p(n) \sim \frac{(2r-1)^{n+1}}{2(r-1)n} = K\frac{(2r-1)^n}{n},$$

where $K = \frac{2r-1}{2(r-1)}$.

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In general, when powers are included, one cannot divide by n.

Conjugacy growth: history and motivation

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- 1960s (Sinai, Margulis): $M=$ complete Riemannian manifolds or compact manifolds of pinched negative curvature;
- 1990s - 2000s (Knieper, Coornaert, Link): some classes of (rel) hyperbolic or CAT(0) groups.

Conjugacy growth asymptotics

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- Rivin (2000), Coornaert (2005): asymptotics for the free groups.
- Guba-Sapir (2010): asymptotics for various groups.
- **Conjecture (Guba-Sapir)**: groups* of standard exponential growth have exponential conjugacy growth.

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- Breuillard-Cornulier-Lubotzky-Meiri (2011): exponential conjugacy growth for linear (non virt. nilpotent) groups.
- Hull-Osin (2014): all acylindrically hyperbolic groups have exponential conjugacy growth.

What is the relation between $A(n)$ and $C(n)$?

C(n) vs A(n)??

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  $\star$ Exclude the Osin or Ivanov type 'monsters'!

- Easy/Hard: Compare standard and conjugacy growth rates.

Growth rates

The standard growth rate of G wrt X always exists and is

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Hull: There are groups for which

$$\liminf_{n \to \infty} \sqrt[n]{c(n)} < \limsup_{n \to \infty} \sqrt[n]{c(n)},$$

that is, the limit does not exist.

Conjugacy vs. standard growth

	Standard growth	Conjugacy growth
Type	pol., int., exp.	pol., int.*, exp.
Quasi-isometry invariant	yes	no**, but group invariant
Rate of growth	exists	exists (not always)

* Bartholdi, Bondarenko, Fink.

** Hull-Osin (2013): conjugacy growth not quasi-isometry invariant.

The conjugacy growth series

The conjugacy growth series

Let G be a group with finite generating set X.

- The conjugacy growth series of G with respect to X records the number of conjugacy classes of every length. It is

$$\sigma_{(G,X)}(z) := \sum_{n=0}^{\infty} c(n) z^n,$$

where $c(n)$ is the number of conjugacy classes of length n.

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- Element representatives: $\{1, a, b, ab, ba, aba\}$ $\quad 1 + 2z + 2z^2 + z^3$
- Conjugacy representatives: $\{1, a, ab\}$ $\quad 1 + z + z^2$

Conjugacy growth series in $\mathbb{Z}$, $\mathbb{Z}_2 * \mathbb{Z}_2$

In $\mathbb{Z}$ the conjugacy growth series is the same as the standard one:

$$\sigma_{(\mathbb{Z},\{1,-1\})}(z) = 1 + 2z + 2z^2 + \cdots = \frac{1+z}{1-z}.$$

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In $\mathbb{Z}_2 * \mathbb{Z}_2$ a set of conjugacy representatives is $1, a, b, ab, abab, \ldots$, so

$$\sigma_{(\mathbb{Z}_2*\mathbb{Z}_2,\{a,b\})}(z) = 1 + 2z + z^2 + z^4 + z^6 \cdots = \frac{1+2z-2z^3}{1-z^2}.$$

Growth rates from power series

For any complex power series

$$f(z) = \sum_{i=0}^{\infty} a_i z^i$$

with radius of convergence $RC(f)$ we have

$$RC(f) = \frac{1}{\limsup_{i\to\infty} \sqrt[i]{a_i}} = \frac{1}{\alpha}.$$

Radius of convergence for rational series

For any rational function $f(z) = \frac{P(z)}{Q(z)}$ the radius of convergence $RC(f)$ of f is

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Rational conjugacy growth series give conjugacy asymptotics.

Question: For which groups are conjugacy growth series rational?

Rational, algebraic, transcendental

A generating function $f(z)$ is

- rational if there exist polynomials $P(z)$, $Q(z)$ with integer coefficients such that $f(z) = \frac{P(z)}{Q(z)}$;
- algebraic if there exists a polynomial $P(x, y)$ with integer coefficients such that $P(z, f(z)) = 0$;
- transcendental otherwise.

Conjugacy growth series: results

Rationality

Being rational/algebraic/transcendental is not a group invariant!

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Theorem [Stoll, 1996]

The higher Heisenberg groups H_r have rational growth with respect to one choice of generating set and transcendental with respect to another.

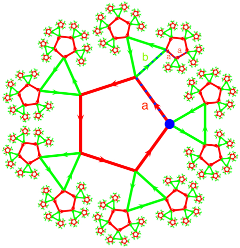
$$H_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a, b, c, d, e \in \mathbb{Z} \right\}$$

Conjugacy in hyperbolic groups

Hyperbolic groups

Motivation: Most (finitely presented, i.e. X, R finite) groups are hyperbolic.

Definition: Groups whose 'picture' looks like the hyperbolic plane.



Examples: free groups, free products of finite groups, $SL(2, \mathbb{Z})$, virtually free groups *, surface groups, small cancellation groups, and many more.

* Virtually free = groups with a free subgroup of finite index.

Virtually cyclic groups: $\mathbb{Z}$, $\mathbb{Z}_2 * \mathbb{Z}_2$

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In $\mathbb{Z}_2 * \mathbb{Z}_2$ a set of conjugacy representatives is $1, a, b, ab, abab, \ldots$, so

$$\sigma_{(\mathbb{Z}_2 * \mathbb{Z}_2,\{a,b\})}(z) = 1 + 2z + z^2 + z^4 + z^6 \cdots = \frac{1+2z-2z^3}{1-z^2}.$$

The conjugacy growth series in free groups

- Rivin (2000, 2010): the conjugacy growth series of F_k is not rational:

$$\sigma(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left(\frac{1}{1-(2k-1)x^d} - 1 \right).$$

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If G hyperbolic, then the conjugacy growth series of G is rational if and only if G is virtually cyclic.

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Theorem (Antolín - C., 2017)

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$\Leftarrow$

Theorem (C. - Hermiller - Holt - Rees, 2016)

Let G be a virtually cyclic group. Then the conjugacy growth series of G is rational.

NB: Both results hold for all symmetric generating sets of G.

Idea of proof: Asymptotics of conjugacy growth in hyperbolic groups

Theorem. (Coornaert - Knieper 2007, Antolín - C. 2017)

Let G be a non-elementary word hyperbolic group. Then there are positive constants A, B and n_0 such that

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

for all $n \geq n_0$, where α is the growth rate of G.

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MESSAGE:

The number of conjugacy classes in the ball of radius n is asymptotically the number of elements in the ball of radius n divided by n.

End of the proof: Analytic combinatorics at work

The transcendence of the conjugacy growth series for non-elementary hyperbolic groups follows from the bounds

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End of the proof: Analytic combinatorics at work

The transcendence of the conjugacy growth series for non-elementary hyperbolic groups follows from the bounds

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

together with

Lemma (Flajolet: Trancendence of series based on bounds).

Suppose there are positive constants $A, B, \mathbf{h}$ and an integer $n_0 \geq 0$ s.t.

$$A\frac{e^{\mathbf{h}n}}{n} \leq a_n \leq B\frac{e^{\mathbf{h}n}}{n}$$

for all $n \geq n_0$. Then the power series $\sum_{i=0}^{\infty} a_n z^n$ is not algebraic.

Consequence of Rivin's Conjecture

Corollary (Antolín - C.)

For any hyperbolic group G with generating set X the standard and conjugacy growth rates are the same:

$$\lim_{n\to\infty} \sqrt[n]{c(n)} = \gamma_{G,X} = \alpha_{G,X}.$$

Consequence of Rivin's Conjecture

Corollary (Antolín - C.)

Let G be a hyperbolic group, X any finite generating set, and $\mathcal{L}_c$ be a set containing one minimal length representative of each conjugacy class.

Then $\mathcal{L}_c$ is not unambiguous context-free, so not regular.

2nd Consequence: Formal languages and the Chomsky hierarchy

Let X be a finite alphabet. A formal language over X is a set $L \subset X^*$ of words.

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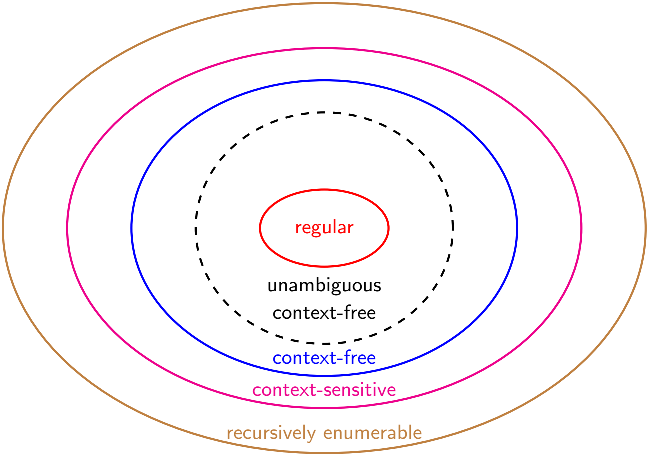

Formal languages and their algebraic complexity

Let $L \subset X^*$ be a language.

- The growth function $f_L : \mathbb{N} \to \mathbb{N}$ of L is:

$$f_L(n) = \sharp\{w \in L \mid w \text{ of length } n\}.$$

- The growth series of L is

$$\mathcal{S}_L(z) = \sum_{n=0}^{\infty} f_L(n) z^n.$$

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Theorem

- Regular languages have RATIONAL growth series.
- Unambiguous context-free languages have ALGEBRAIC growth series.

  (Chomsky-Schützenberger)

Acylindrically hyperbolic groups

Main Theorem (Antolín - C., 2017)

Let G be an acylindrically hyperbolic group, X any finite generating set, and $\mathcal{L}_c$ be a set containing one minimal length representative of each conjugacy class.

Then $\mathcal{L}_c$ is not unambiguous context-free, so not regular.

Rivin's conjecture for other groups

Theorem (Gekhtman and Yang, 2018)

Let G be a non-elementary group with a finite generating set S. If G has a contracting element with respect to the action on the corresponding Cayley graph, then the conjugacy growth series is transcendental.

Examples: relatively hyperbolic groups, (non-abelian) RAAGs, RAACs, graph products of finite groups, graphical small cancellation groups.

Rationality of standard and conjugacy growth series

	Standard Growth Series	Conjugacy Growth Series
Hyperbolic	Rational (Cannon, Gromov, Thurston)	Transcendental (C.- Antolín' 17*)
Virtually abelian		
Heisenberg H_1		

FOR ALL GENERATING SETS!

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Virtually abelian	Rational (Benson '83)	Rational (Evetts '18)
Heisenberg H_1	Rational (Duchin-Shapiro '19)	Transcendental

FOR ALL GENERATING SETS!

Standard generating set ... for now

	Conjugacy Growth Series	Formula
Wreath products	Transcendental (Mercier '17)	✓
Graph products	Transcendental[1] (C.- Hermiler - Mercier '19)	✓
BS(1,m)	Transcendental (C.- Evetts - Ho, '19)	✓

[1] in most cases

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- How do the conjugacy growth series behave when we change generators?

  Stoll: The rationality of the standard growth series depends on generators.

Thank you!