2-Factorisations of the Complete Graph

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The Oberwolfach problem

The Oberwolfach problem was posed by Ringel in the 1960s.
At the Conference center in Oberwolfach, Germany

The Oberwolfach problem was originally motivated as a seating problem:

The Oberwolfach problem

Given n attendees at a conference with t circular tables each of which which seat a_i, $i = 1, \ldots, t$ people $\left(\sum_{i=1}^{t} a_i = n\right)$.
Find a seating arrangement so that every person sits next to each other person around a table exactly once over the r days of the conference.

Factors

Definition

A *k-factor* of a graph G is a k-regular spanning subgraph of G.

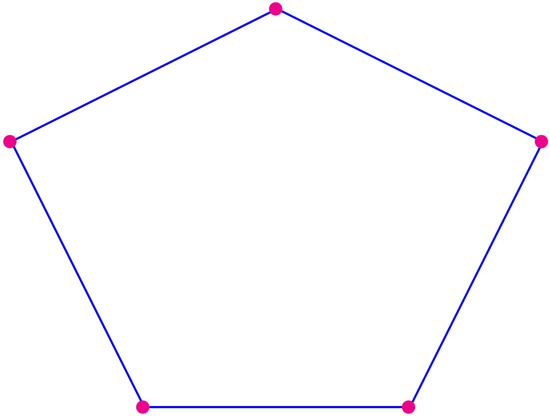
Definition

Given a factor F, an $F-$Factorisation of a graph G is a decomposition of the edges of G into copies of F.

Definition

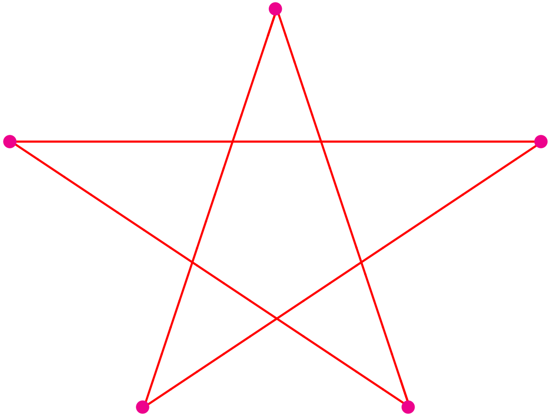
Given a set of factors $\mathcal{F}$, an $\mathcal{F}-$Factorisation of a graph G is a decomposition of the edges of G into copies of factors $F \in \mathcal{F}$.

Example $n = 5$



A 2−Factor of K_5

Example $n = 5$



A 2−Factor of K_5

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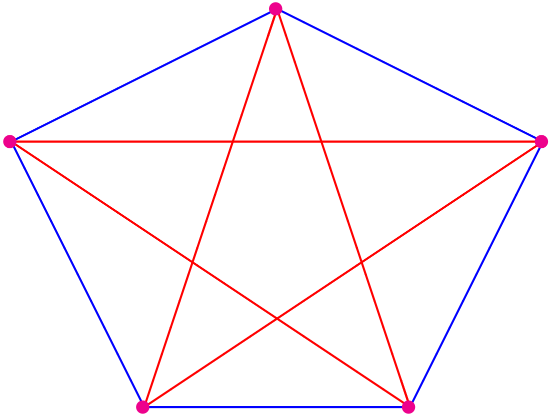


A $2-$Factorisation of K_5

The Oberwolfach problem

When n is odd, the Oberwolfach problem $\mathrm{OP}(F)$ asks for a factorisation of K_n into a specified $2-$factor F of order n.

$$\left(r = \frac{n-1}{2}\right)$$

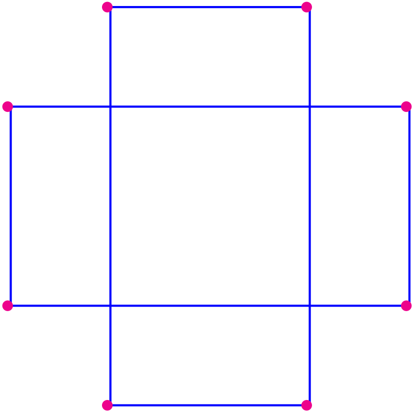
When n is even, the Oberwolfach problem $\mathrm{OP}(F)$ asks for a factorisation of $K_n - I$ into a specified $2-$factor F of order n.
Where $K_n - I$ is the complete graph on n vertices with the edges of a 1-factor removed.

$$\left(r = \frac{n-2}{2}\right)$$

We will use the notation $[m_1, m_2, \ldots, m_t]$ to denote the 2-regular graph consisting of t (vertex-disjoint) cycles of lengths $m_1, m_2, \ldots, m_t$.

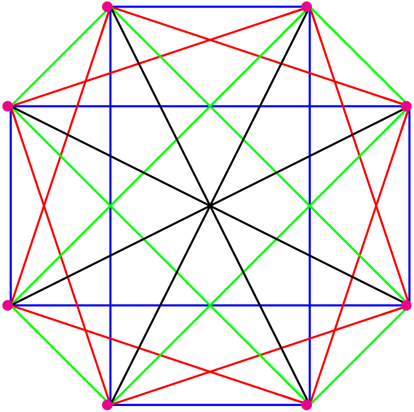
The Oberwolfach problem can be thought of as a generalisation of Kirkman Triple Systems, which are the case $F = [3, 3, \ldots, 3]$.

Example $n = 8, \quad F = [4, 4]$



An $F-$Factor of K_8

Example $n = 8, \quad F = [4, 4]$



A $F-$Factorisation of K_8 with a $1-$factor remaining

Hamilton - Waterloo: Include the Pub

Hamilton-Waterloo

In the Hamilton-Waterloo variant of the problem the conference has two venues

The first venue (Hamilton) has circular tables corresponding to a 2−factor F_1 of order n.

The second venue (Waterloo) circular tables each corresponding to a 2−factor F_2 of order n.

Hamilton-Waterloo

The Hamilton-Waterloo problem thus requires a factorisation of K_n or $K_n - I$ if n is even into two 2-factors, with α_1 classes of the form F_1 and α_2 classes of the form F_2. Here the number of days is

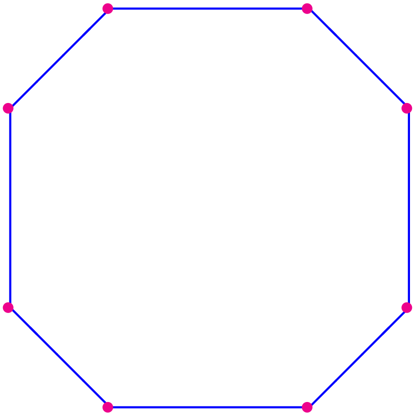
$$r = \alpha_1 + \alpha_2 = \begin{cases} \frac{n-1}{2} & n \text{ odd} \\ \frac{n-2}{2} & n \text{ even} \end{cases} .$$

When n is even

If $n \equiv 2 \bmod 4$ then $\frac{n-2}{2}$ is even and α_1 and α_2 have the same parity. i.e. Either both α_1, α_2 are even or both are odd.

If $n \equiv 0 \bmod 4$ thenthen $\frac{n-2}{2}$ is odd and so α_1 and α_2 have opposite parity. i.e. one of α_1, α_2 is even and the other is odd.

Example $n = 8$, $F_1 = [8], \alpha_1 = 2$, $F_2 = [4, 4]$, $\alpha_2 = 1$



An F_1 –Factor of K_8

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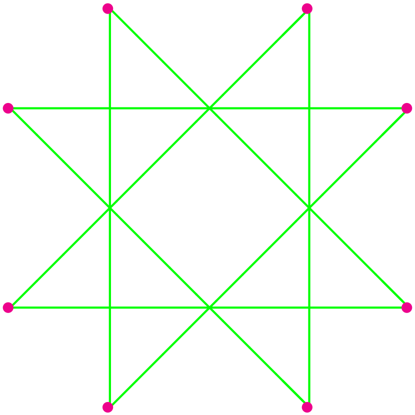

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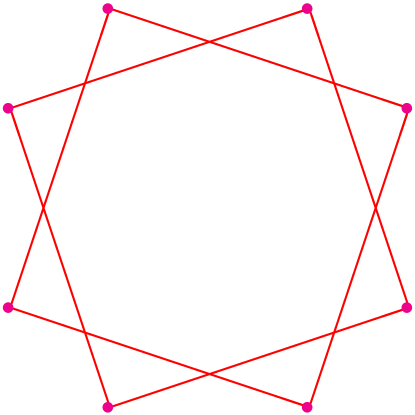
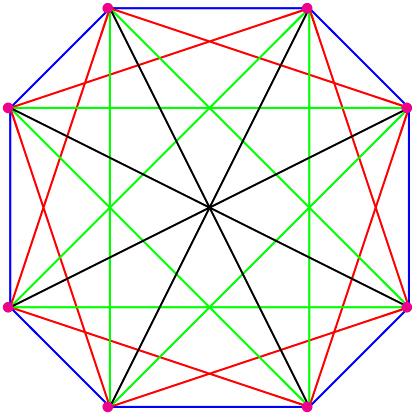


An F_2−Factor of K_8

Example $n = 8$, $F_1 = [8], \alpha_1 = 2$, $F_2 = [4, 4]$, $\alpha_2 = 1$



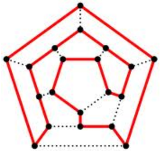
A Solution to the given Hamilton-Waterloo Problem

Hamilton? - Waterloo?

Hamilton?

Hamilton? - Waterloo?

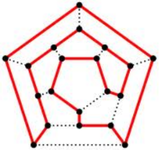
Hamilton?



Hamiltonian?

Hamilton? - Waterloo?

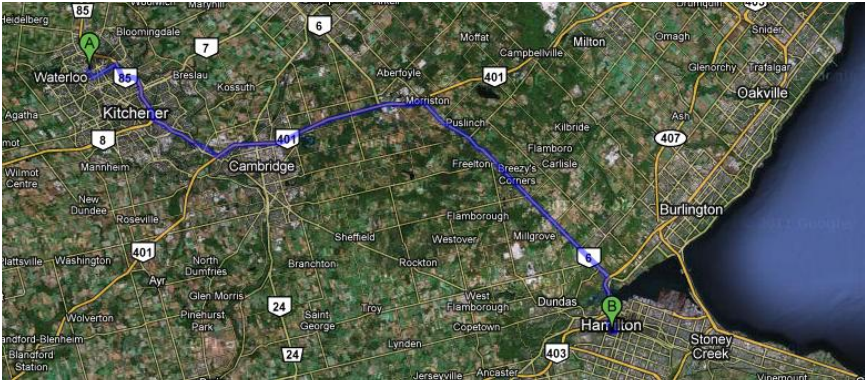
Hamilton?



Hamiltonian?

Waterloo?

Hamilton - Waterloo



3rd Ontario Combinatorics Workshop
McMaster University, **Hamilton**, Ontario, Feb. 1988.
University of Waterloo, **Waterloo**, Ontario, Oct. 1987;

Hamilton - Waterloo?

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Organizers

Alex Rosa
McMaster University
Hamilton, Ontario

Charlie Colbourn
University of Waterloo
Waterloo, Ontario

Generalise to $\mathrm{OP}(F_1, \ldots, F_t)$

Given 2−factors $F_1, F_2, \ldots, F_t$ order n and non-negative integers $\alpha_1, \alpha_2, \ldots, \alpha_t$ such that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_t = \begin{cases} \frac{n-1}{2} & n \text{ odd} \\ \frac{n-2}{2} & n \text{ even} \end{cases}$$

Find a 2-factorisation of K_n, or $K_n - I$ if un is even, in which there are exactly α_i 2-factors isomorphic to F_i for $i = 1, 2, \ldots, t$.

Generalise to other Graphs G, $\mathrm{OP}(G; F_1, \ldots, F_t)$

We can also consider Factorisations of other graphs G.

Of particular interest is the case when $G = K_{n^r}$, the multipartite complete graph with r parts of size n.

In order for $G = K_{n^r}$ to have a factorisation into 2−factors $F_1, \ldots, F_t$ is that every vertex is of even degree ($n(r-1)$ is even).

What is known about $\mathrm{OP}(F)$

It is known that there is no solution to $\mathrm{OP}(F)$ for

$$F \in \{[3,3],[4,5],[3,3,5],[3,3,3,3]\},$$

A solution is known for all other instances with $n \leq 40$.
Deza, Franek, Hua, Meszka, Rosa (2010), Adams & Bryant (2006), Franek & Rosa (2000), Bolstad (1990), Huang, Kotzig & A. Rosa (1979).

The case where all the cycles in F are of the same length has been solved.
Govzdjak (1997), Alspach & Häggkvist (1985), Alspach, Schellenberg, Stinson & D. Wagner (1989), Hoffman & Schellenberg (1991), Huang, Kotzig & A. Rosa (1979), Ray-Chaudhuri & Wilson (1971).

What is known about Hamilton - Waterloo

It is known that the following instances of the Hamilton-Waterloo Problem have no solution.

$$\mathrm{HW}([3,4],[7];2,1) \quad \mathrm{HW}([3,5],[4^2];2,1) \quad \mathrm{HW}([3,5],[4^2];1,2)$$
$$\mathrm{HW}([3^3],[4,5];2,2)$$

$$\mathrm{HW}([3^3],F;3,1) \text{ for } F \in \{[4,5],[3,6],[9]\} \quad \text{and}$$
$$\mathrm{HW}([3^5],F;6,1) \text{ for } F \in \{[3^2,4,5],[3,5,7],[5^3],[4^2,7],[7,8]\}.$$

Every other instance of the Hamilton-Waterloo Problem has a solution when $n \leq 17$ and odd and when $n \leq 10$ and even Adams, Bryant (2006), Franek, Rosa (2000, 2004).

What is known about Hamilton - Waterloo

Theorem (Danziger, Quottrocchi, Stevens (2009))

If F_1 is a collection of $3-$cycles and F_2 is a collection of $4-$cycles then $\mathrm{OP}(F_1, F_2)$ exists if and only if $n \equiv 0 \bmod 12$, with 14 possible exceptions.

Theorem (Horak, Nedela, Rosa (2004), Dinitz, Ling (2009))

If F_1 is a collection of $3-$cycles and F_2 is a Hamiltonan cycle and $n \not\equiv 0 \bmod 6$ then $\mathrm{OP}(F_1, F_2)$ exists, except when $n = 9, \alpha_2 = 1$, with 13 possible exceptions. (The case $n \equiv 0 \bmod 6$ is still open.)

Generalised OP and Häggkvist

Theorem (Häggkvist (1985))

Let $n \equiv 2 \mod 4$, and $F_1, \ldots, F_t$ be bipartite $2-$factors of order n then $\mathrm{OP}(F_1, \ldots, F_t)$ has solution, with an even number of factors isomorphic to each F_i.

Corollary ($t = 1$)

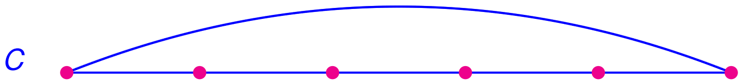
Let $n \equiv 2 \mod 4$, and F be a bipartite $2-$factor of order n then $\mathrm{OP}(F)$ has solution.

Corollary ($t = 2$)

Let $n \equiv 2 \mod 4$, and F_1, F_2 be bipartite $2-$factors of order n then $\mathrm{OP}(F_1, F_2)$ (Hamilton-Waterloo) has solution where there are an even number of each of the factors. (Both α_1 and α_2 are even)

Doubling

For any given graph G, the graph $G^{(2)}$ is defined by
$V(G^{(2)}) = V(G) \times \mathbb{Z}_2$,
$E(G^{(2)}) = \{\{(x,a),(y,b)\} : \{x,y\} \in E(G),\ a,b \in \mathbb{Z}_2\}$.



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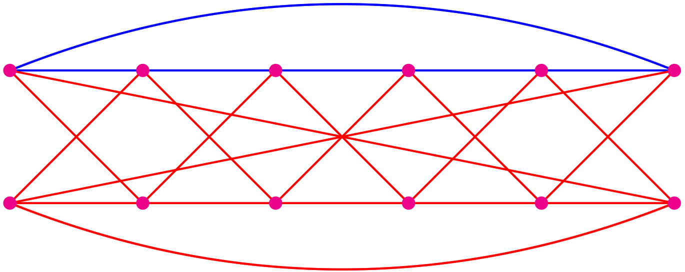
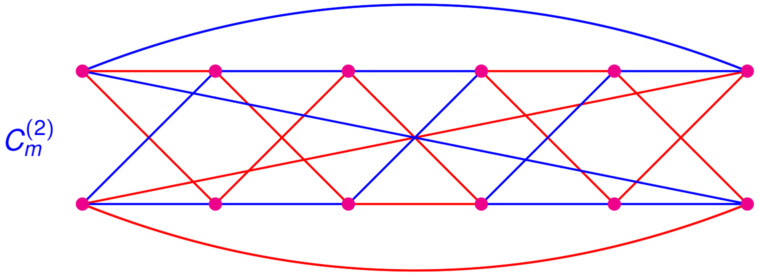
$C^{(2)}$



Factoring $C_n^{(2)}$

Lemma (Häggkvist (1985))

For any $m > 1$ and for each bipartite 2-regular graph F of order $2m$, there exists a 2-factorisation of $C_m^{(2)}$ in which each 2-factor is isomorphic to F.



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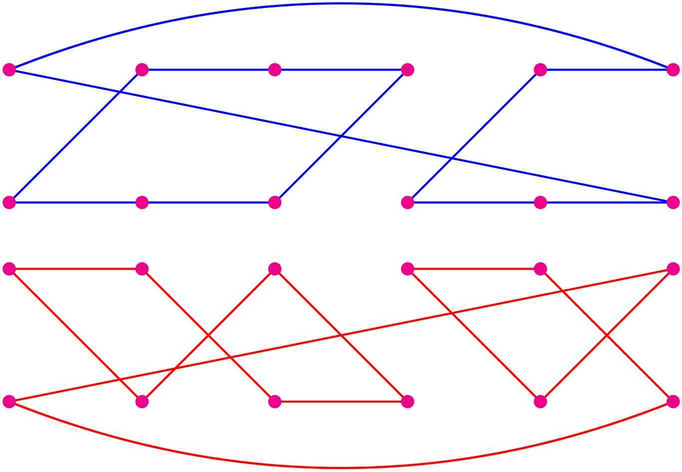


Häggkvist and Doubling

Let m be odd, and $n = 2m \equiv 2 \bmod 4$.

Given bipartite 2−factors $F_1, \ldots, F_{\frac{m-1}{2}}$, of order $2m$ (not necessarily distinct).

Since m is odd, K_m has a factorisation into Hamiltonian cycles H_i, $1 \leq i \leq \frac{m-1}{2}$.

Now doubling, we have a $H^{(2)}$ factorisation of $K_m^{(2)}$

We can factor the square of the i^{th} Hamiltonian cycle $H_i^{(2)}$ into 2 copies of F_i by Häggkvist doubling as above.

Result is a factorisation of $K_m^{(2)} \cong K_{2m}$ into pairs of factors each isomorphic to F_i, $i = 1, \ldots \frac{m-1}{2}$.

Results

Theorem (Bryant, Danziger (2011))

If $n \equiv 0 \bmod 4$ and $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order n and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are non-negative integers such that

- $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$,
- α_i *is even for* $i = 2, 3, \ldots, t$,
- $\alpha_1 \geq 3$ *is odd*,

then $\mathrm{OP}(F_1, \ldots, F_t)$ *has a solution with* α_i *2-factors isomorphic to* F_i *for* $i = 1, 2, \ldots, t$.

Results ...and so...

Corollary ($t = 1$)

Let n be even and F be a bipartite $2-$factor of order n then $\mathrm{OP}(F)$ has solution.

Corollary ($t = 2$)

Let n be even and F_1, F_2 be bipartite $2-$factors of order n then $\mathrm{OP}(F_1, F_2)$ (Hamilton-Waterloo) has solution, except possibly in the case where all but one of the 2-factors are isomorphic ($\alpha_1 = 1$ or $\alpha_2 = 1$).

To Show:

Theorem (Bryant, Danziger (2011))

If $n \equiv 0 \bmod 4$ *and* $F_1, F_2, \ldots, F_t$ *are bipartite 2-regular graphs of order* n *and* $\alpha_1, \alpha_2, \ldots, \alpha_t$ *are non-negative integers such that*

- $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$,
- α_i *is even for* $i = 2, 3, \ldots, t$,
- $\alpha_1 \geq 3$ *is odd*,

then $\mathrm{OP}(F_1, \ldots, F_t)$ *has a solution with* α_i *2-factors isomorphic to* F_i *for* $i = 1, 2, \ldots, t$.

The Plan

When $n \equiv 0 \bmod 4$ we don't have the Hamiltonian Factorisation of $K_{\frac{n}{2}}$ for Häggkvist doubling.

Idea is to decompose $K_{\frac{n}{2}}$ into Hamiltonian Cycles, $H_1, \ldots, H_{\frac{n-4}{2}}$, and a known 3-regular Graph G.

Use the factorisation above to factor $K_{\frac{n}{2}}^{(2)}$ as follows:

Factor the Doubled Hamiltonian Cycles, $H_1^{(2)}, \ldots, H_{\frac{n-4}{2}}^{(2)}$, into pairs isomorphic to $F_2, \ldots, F_t$ by Häggkvist doubling.

Factor the 7-regular graph $G^{(2)} \cup \{(x_0, x_1)\}$ into copies of F_1 and a 1-factor I.

Notation

A Cayley graph on a cyclic group is called a *circulant graph*.
We will always use vertex set $\mathbb{Z}_n$.

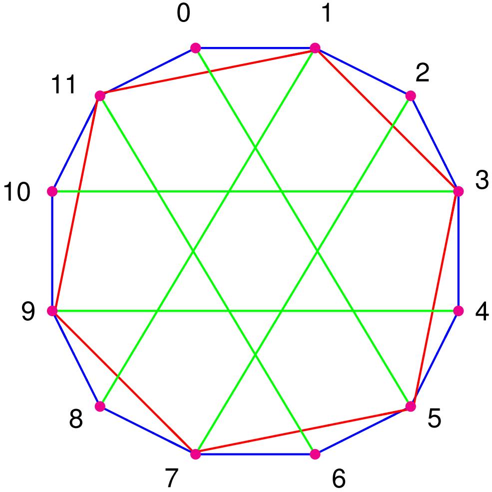
The length of an edge $\{x, y\}$ in a graph is defined to be either $x - y$ or $y - x$, whichever is in $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$

We denote by $\langle S \rangle_n$ the graph with vertex set $\mathbb{Z}_n$ and edge set the edges of length s for each $s \in S$.

We call $\{\{x, x+s\} : x = 0, 2, \ldots, n-2\}$ the even edges of length s.
We call $\{\{x, x+s\} : x = 1, 3, \ldots, n-1\}$ the odd edges of length s.

If we wish to include in our graph only the even edges of length s then we give s the superscript e.
If we wish to include only the odd edges of length s then we give s the superscript o.

Example $\langle\{1, 2^{o}, 5^{e}\}\rangle_{12}$ on $\mathbb{Z}_{12}$



Factoring Circulants

Lemma

For each even $m \geq 8$ there is a factorisation of K_m into $\frac{m-4}{2}$ Hamilton cycles and a copy of $G = \langle\{1, 3^{\mathrm{e}}\}\rangle_m$.

We can create factors from the Hamiltonian cycles using the Häggkvist doubling construction.

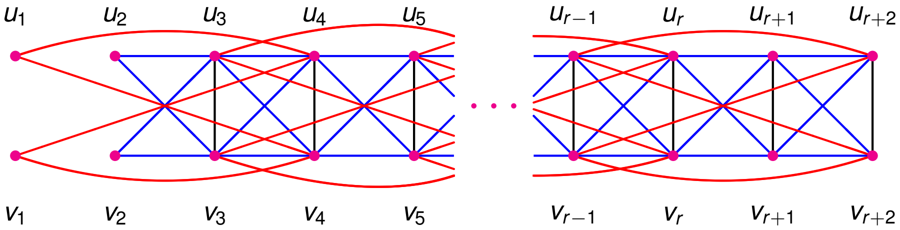
It remains to factor $G_{2m} = G^{(2)} \cup I = (\langle\{1, 3^{\mathrm{e}}\}\rangle_m)^{(2)} \cup I$.

The Graph J_{2r}

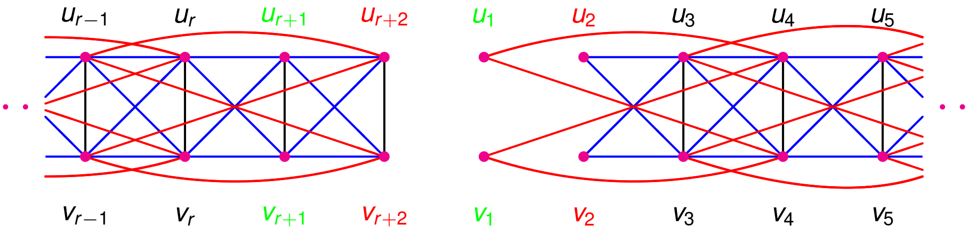
For each even $r \geq 2$ we define the graph J_{2r} by

$$V(J_{2r}) = \{u_1, u_2, \ldots, u_{r+2}\} \cup \{v_1, v_2, \ldots, v_{r+2}\}$$

$$\begin{aligned} E(J_{2r}) = \; & \{\{u_i, v_i\} : i = 3, 4, \ldots, r+2\} \cup \\ & \{\{u_i, u_{i+1}\}, \{v_i, v_{i+1}\}, \{u_i, v_{i+1}\}, \{v_i, u_{i+1}\} : i = 2, 3, \ldots, r+1\} \\ & \{\{u_i, u_{i+3}\}, \{v_i, v_{i+3}\}\{u_i, v_{i+3}\}, \{v_i, u_{i+3}\} : i = 1, 3, \ldots, r-1\}. \end{aligned}$$



If we identify vertices u_1 with u_{r+1}, u_2 with u_{r+2}, v_1 with v_{r+1}, and v_2 with v_{r+2}, then the resulting graph is isomorphic to G_{2r}.



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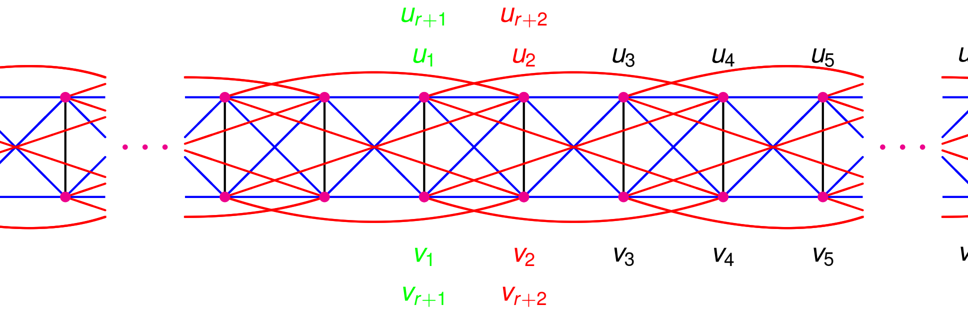
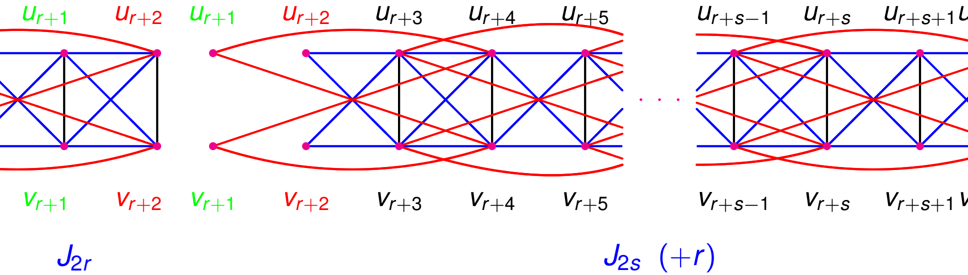


The $\oplus$ Operation on J

Similarly, by $J_{2r} \oplus J_{2s}$ we mean adjoining J_{2r} to a copy of J_{2s} which has been shifted by r, to obtain $J_{2(r+s)}$.



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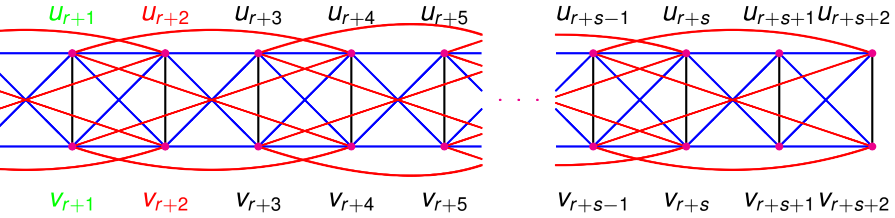


$J_{2(r+s)}$

Decompositions $J \mapsto F$

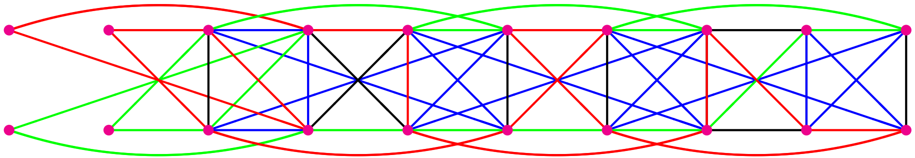
Given a factor F the idea is to divide it into smaller factors F_i of order $2m_i$ and to divide G_{2n} into edge disjoint J_{m_i}, so that $\bigoplus_i J_{m_i} \cong G_n$, with the identification above at the ends.

We then factor each of the J_{m_i} into F_i and join up the results.

Since G_{2m} is $7-$regular we require a factorisation of each $J_{m_i}(k)$ into three partial cycle factors, H_j, $j \in \{0, 1, 2\}$ and a $1-$factor, H_3.

Index j	Missed points in $J_m(k)$
0	$u_i, v_i, i \in \{1, 2\}$
1	$\{v_1, v_2, u_{r+1}, u_{r+2}\}$
2	$\{u_1, u_2, v_{r+1}, v_{r+2}\}$
$1-$factor	$u_i, v_i, i \in \{1, 2\}$

Decomposition of $J \mapsto [16] + [16] + [16]$



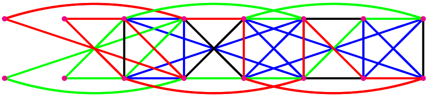
Index j	Missed points in $J_m(k)$
0	$u_i, v_i, i \in \{1, 2\}$
1	$\{v_1, v_2, u_{r+1}, u_{r+2}\}$
2	$\{u_1, u_2, v_{r+1}, v_{r+2}\}$
1−factor	$u_i, v_i, i \in \{1, 2\}$

Joining Decompositions

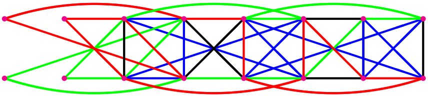
Lemma

If F and F' are 2-regular graphs such that $J \mapsto F$ and $J \mapsto F'$, then $J \mapsto F''$ where F'' is the union of vertex-disjoint copies of F and F'.

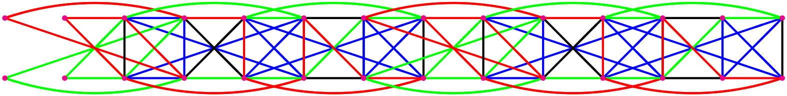
$$[12] + [12] + [12] \quad \bigoplus \quad [12] + [12] + [12]$$



$\bigoplus$



$$= [12, 12] + [12, 12] + [12, 12]$$



Lemma 10 - Ingredient Decompositions $J \mapsto F$

Lemma (10)

For each graph F in the following list we have $J \mapsto F$.

- $[m]$ *for each* $m \in \{8, 12, 16, \ldots\}$
- $[4, m]$ *for each* $m \in \{4, 8, 12, \ldots\}$
- $[m, m']$ *for each* $m, m' \in \{6, 10, 14, \ldots\}$
- $[4, m, m']$ *for each* $m, m' \in \{6, 10, 14, \ldots\}$
- $[4, 4, 4]$

Factoring G_{2m}

Lemma

If F is a bipartite 2-regular graph of order $2m$ where $m \geq 8$ is even, then there is a factorisation of G_{2m} into three 2-factors each isomorphic to F, and a 1-factor.

Proof We show that there is a decomposition of F into bipartite 2-regular subgraphs $F_1, F_2, \ldots, F_s$ such that Lemma 10 covers $J \mapsto F_i$ for $i = 1, 2, \ldots, s$.

We then use $\bigoplus$ to join these Factorisations into $J \mapsto F$.

Finally we identify endpoints to obtain the required factorisation of G_{2m}.

Main Theorem

Theorem

If $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order n and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are non-negative integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$, $\alpha_1 \geq 3$ is odd, and α_i is even for $i = 2, 3, \ldots, t$, then there exists a 2-factorisation of $K_n - I$ in which there are exactly α_i 2-factors isomorphic to F_i for $i = 1, 2, \ldots, t$.

Proof The conditions guarantee that $n \equiv 0 \bmod 4$.

Factor $K_{\frac{n}{2}}$ into Hamiltonian factors and $\langle\{1, 3^{\mathrm{e}}\}\rangle_{\frac{n}{2}}$.

Factor $C_{\frac{n}{2}}^{(2)}$ into pairs of F_i, $i = 1, \ldots, t$.

Remaining edges of $K_{\frac{n}{2}}^{(2)}$ are isomorphic to G_n, which we can factor into 3 copies of F_1.

Lemma 10 - Ingredient Decompositions $J \mapsto F$

We now want to prove

Lemma (10)

For each graph F in the following list we have $J \mapsto F$.

- $[m]$ *for each* $m \in \{8, 12, 16, \ldots\}$
- $[4, m]$ *for each* $m \in \{4, 8, 12, \ldots\}$
- $[m, m']$ *for each* $m, m' \in \{6, 10, 14, \ldots\}$
- $[4, m, m']$ *for each* $m, m' \in \{6, 10, 14, \ldots\}$
- $[4, 4, 4]$

Cycle Length $m \equiv 0 \bmod 4$, $m > 4$

We describe the three cycle factors in three parts.

A left hand end $[\ell$, consisting of an $\ell-$path

A continuing part c that consist of two paths whose total length is c.

The continuing part is designed so that it can be repeated.

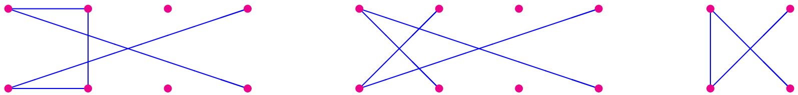
And a right hand end $r]$, consisting of an $r-$path.

We use $\oplus$ to adjoin these parts:

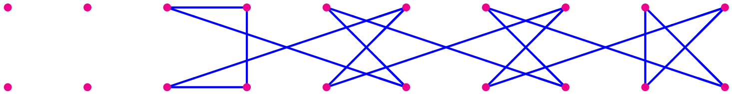
$$[\ell \oplus c \oplus c \oplus r] = [(\ell + c + c + r)]$$

Blue Factor

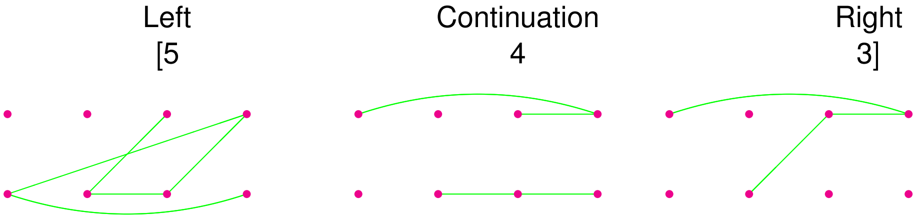
Left [5 | Continuation 4 | Right 3]

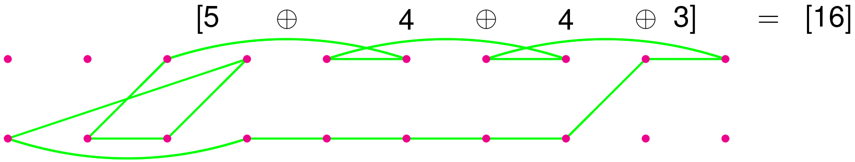


[5 ⊕ 4 ⊕ 4 ⊕ 3] = [16]

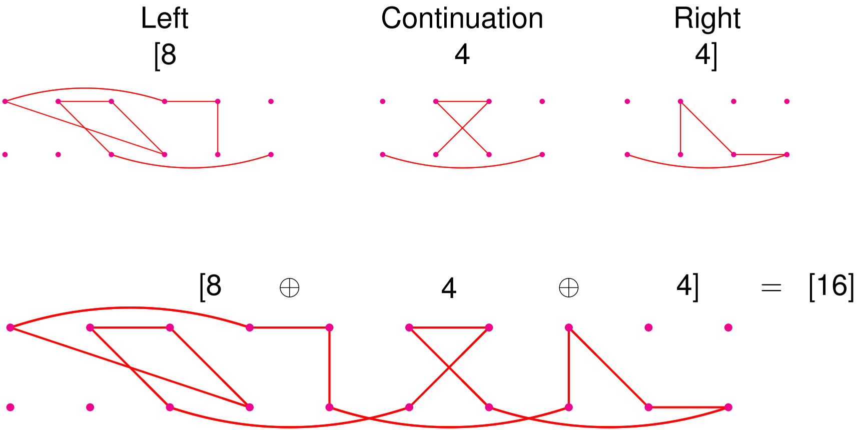


Green Factor



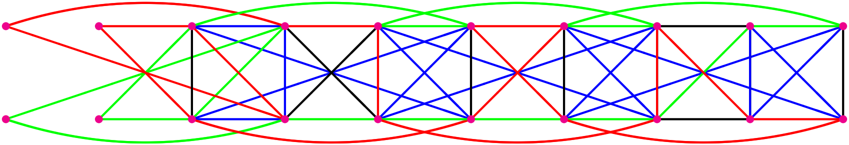


Red Factor



Putting it Together

$$([5+[5+[8)\oplus((4\oplus 4)+(4\oplus 4)+4)\oplus(3]+3]+4]) = [16]+[16]+[16]$$

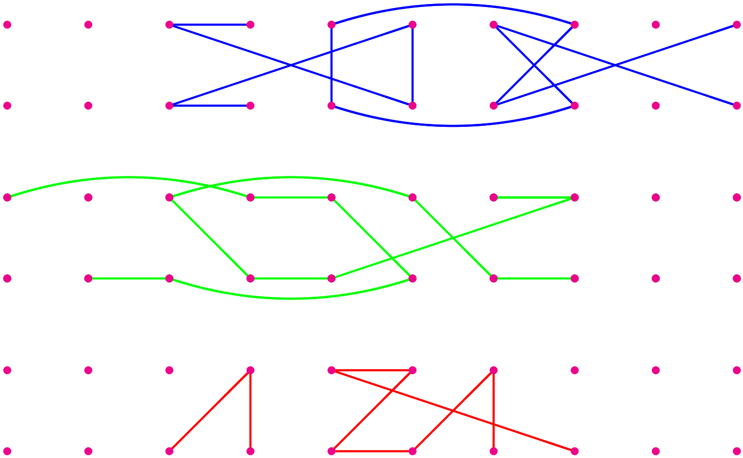


$[m, m'], \quad m \equiv m' \equiv 2 \bmod 4$

5, 7

5, 7

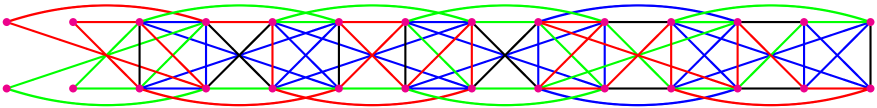
2, 6



The Join

$$([5+[5+[8)\oplus(4+4+4)\oplus(5,7+5,7+2,6)\oplus(0+0+0)\oplus(3]+3]+4])$$

$$=[14,10]+[14,10]+[14,10]$$



Remaining Cases

The cases $[4, m]$ are dealt with using a special left end.

The following are dealt with as special cases:
$[8], [12], [4, 8], [4, 12], [4, 4], [4, 4, 4]$

Hamilton-Waterloo (Two Factors F_1 and F_2)

When n is even

If $n \equiv 2 \bmod 4$ then $\frac{n-2}{2}$ is even and α_1 and α_2 have the same parity. i.e. Either both α_1, α_2 are even or both are odd.

If $n \equiv 0 \bmod 4$ thenthen $\frac{n-2}{2}$ is odd and so α_1 and α_2 have opposite parity. i.e. one of α_1, α_2 is even and the other is odd.

Corollary ($t = 2$)

Let n be even and F_1, F_2 be bipartite $2-$factors of order n then $\mathrm{OP}(F_1, F_2)$ (Hamilton-Waterloo) has solution, except possibly in the case where all but one of the 2-factors are isomorphic ($\alpha_1 = 1$ or $\alpha_2 = 1$).

Refinement

Definition

Given two 2-regular graphs F_1 and F_2,
F_1 is called a refinement of F_2 if F_1 can be obtained from by replacing each cycle of F_2 with a 2-regular graph on the same vertex set

Example

- $[4, 4]$ is a refinement of $[8]$
- $[4, 8^3, 10^2, 12]$ is a refinement of $[4, 16, 18, 22]$,
- $[4, 18^2, 20]$ is not a refinement of $[4, 16, 18, 22]$.
- Every 2-regular graph of order n is a refinement of an n-cycle.

General result, $n \equiv 2 \bmod 4$

Theorem (Bryant, Danziger, Dean (2012))

If $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order $n \equiv 2 \bmod 4$, and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are positive integers such that

- F_1 *is a refinement of* F_2;
- α_i *even for* $i = 3, 4 \ldots, t$;
- $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$;

then K_n has a factorisation into α_i copies of F_i for $i = 1, 2, \ldots, t$ and a 1-factor.

General result, $n \equiv 0 \bmod 4$

Theorem (Bryant, Danziger, Dean (2012))

If $t \geq 3$, $F_1, F_2, \ldots, F_t$ are bipartite 2-regular graphs of order n, and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are positive integers such that

- *F_1 is a refinement of F_2;*
- *$\alpha_1, \alpha_2, \alpha_3$ are odd with $\alpha_3 \geq 3$;*
- *α_i is even for $i = 4, 5, \ldots, t$;*
- *$\alpha_1 + \alpha_2 + \cdots + \alpha_t = \frac{n-2}{2}$;*
- *$F_2 \notin \{[4,4,4], [4,8], [12], [4,6,6], [6,10]\}$; and*
- *$F_3 \notin \{[6^r], [4,6^r] : r \equiv 2 \bmod 4\}$;*

then K_n has a factorisation into α_i copies of F_i for $i = 1, 2, \ldots, t$ and a 1-factor.

Hamilton Waterloo

Theorem (Bryant, Danziger, Dean (2012))

If F_2 is a bipartite 2-regular graph of order n and F_1 is a bipartite refinement of F_2, then for all non-negative α_1, α_2 satisfying $\alpha_1 + \alpha_2 = \frac{n-2}{2}$ there is a factorisation of K_n into α_1 copies of F_1, α_2 copies of F_2, and a 1-factor.

Corollary

Let F_1, F_2 be bipartite 2−factors of order n such that F_1 a refinement of F_2 then $\mathrm{OP}(F_1, F_2)$ (Hamilton-Waterloo) has solution.

Multipartite Graphs - One Factor F

We wish to consider factorisations of the complete multipartite graph K_{n^r} into a single biparite $2-$factor F.

Ths is the multipartite case of the original Oberwolfach problem.

Necessary Conditions

Recall In order for the complete multipartite graph K_{n^r} to have a factorisation into $2-$factors $F_1, \ldots, F_t$ we require that every vertex is of even degree, i.e. $n(r-1)$ is even.

Now we only have one bipartite factor F, of even order nr,

But $n(r-1)$ is also even,so:

Theorem

In order for the complete multipartite graph K_{n^r}, $r \geq 2$, to have a factorisation into a single bipartite $2-$factor, n must be even.

What is known

Theorem (Auerbach and Laskar (1976))

A complete multipartite graph has a Hamilton decomposition if and only if it is regular of even degree.

Theorem (Piotrowski (1991))

If F is a bipartite $2-$regular graph of order $2n$, then the complete bipartite graph, $K_{n,n}$ has a $2-$factorisation into F except when $n = 6$ and $F = [6, 6]$.

Theorem (Liu (2003))

The complete multipartite graph K_{n^r}, $r \geq 2$, has a $2-$factorisation into $2-$factors composed of $k-$cycles if and only if $k \mid rn$, $(r-1)n$ is even, further k is even when $r = 2$, and $(k, r, n) \notin \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}$.

What is known

It is known that there is no 2-factorisation of $K_{6,6}$ into $[6,6]$.

Corollary ((Bryant, Danziger (2010), $t = 1$)

If F is a bipartite 2−regular graph of order $2r$, then the complete multipartite graph K_{2^r} has a 2−factorisation into F.

Corollary (Bryant, Danziger, Dean (2012))

Let $n \equiv 0 \bmod 4$ with $n \geq 12$. For each bipartite 2-regular graph F of order n, there is a factorisation of $\langle\{1, 3^e\}\rangle^{(2)}_{n/2}$ into three copies of F; except possibly when $F \in \{[6^r], [4, 6^r] : r \equiv 2 \bmod 4\}$.

Setting up

Note that when $n = 2m$

$$K_{n^r} \cong K_{m^r}^{(2)}$$

Also

$$K_{m^r} \cong \langle \{1, 2, \ldots, \frac{rm}{2}\} \setminus \{r, 2r, \ldots, \frac{m-1}{2} r\} \rangle_{rm}.$$

Lemma

For each even $r \geq 4$ and each odd $m \geq 1$, except $(r, m) = (4, 1)$, there is a factorisation of K_{m^r} into $\frac{(r-1)m-3}{2}$ Hamilton cycles and a copy of $\langle \{1, 3^e\} \rangle_{rm}$.

Necessary Conditions are Sufficient

Theorem (Bryant, Danziger, Pettersson (2013))

If F is a bipartite 2-regular graph of order rn, then there exists a 2-factorisation of K_{n^r}, $r \geq 2$, into F if and only if n is even; except that there is no 2-factorisation of $K_{6,6}$ into $[6,6]$.

If m is even or r is odd, then K_{m^r} has even degree, and hence has a Hamilton decomposition by Auerbach and Laskar's result. If $n = 2m$ then $K_{n^r} \cong K_{m^r}^{(2)}$ and we can complete the proof using Häggkvist's doubling.

$n = 2$ and $r = 2$ are done above.

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$n = 2$ and $r = 2$ are done above.

Proof - Sketch

Assume $m \geq 3$ is odd and $r \geq 4$ is even.

By the Lemma there is a factorisation of K_{m^r} into $\frac{(r-1)m-3}{2}$ Hamilton cycles and a copy of $\langle\{1, 3^e\}\rangle_{rm}$.

Double, use Häggkvist doubling on the Hamiltonian cycles $C_{rm}^{(2)}$ and the second corollary on $\langle\{1, 3^e\}\rangle_{rm}^{(2)}$.

This leaves the case $r \geq 4$ is even, $m = \frac{n}{2} \geq 3$ is odd, and $F = [4, 6^{4x+2}]$ for some $x \geq 1$, which is done as a special case.

□

The End

Thank You