On choosability of graphs with limited number of colours

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Outline

- A starting example
- Choosability: definitions and first examples
- Some classes of choosable graphs
- Complexity of choosability
- Choosability with few colors

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- **A starting example**
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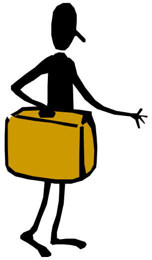
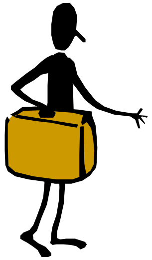
Booking system: coloring interval graphs

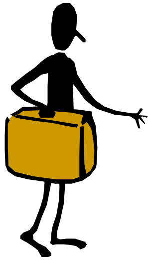
3 rooms 5 requirements



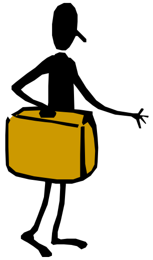
1 to 6



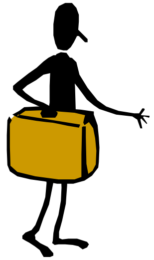
2 to 4



3 to 7



5 to 8



7 to 9

?

Blue

Pink

Green

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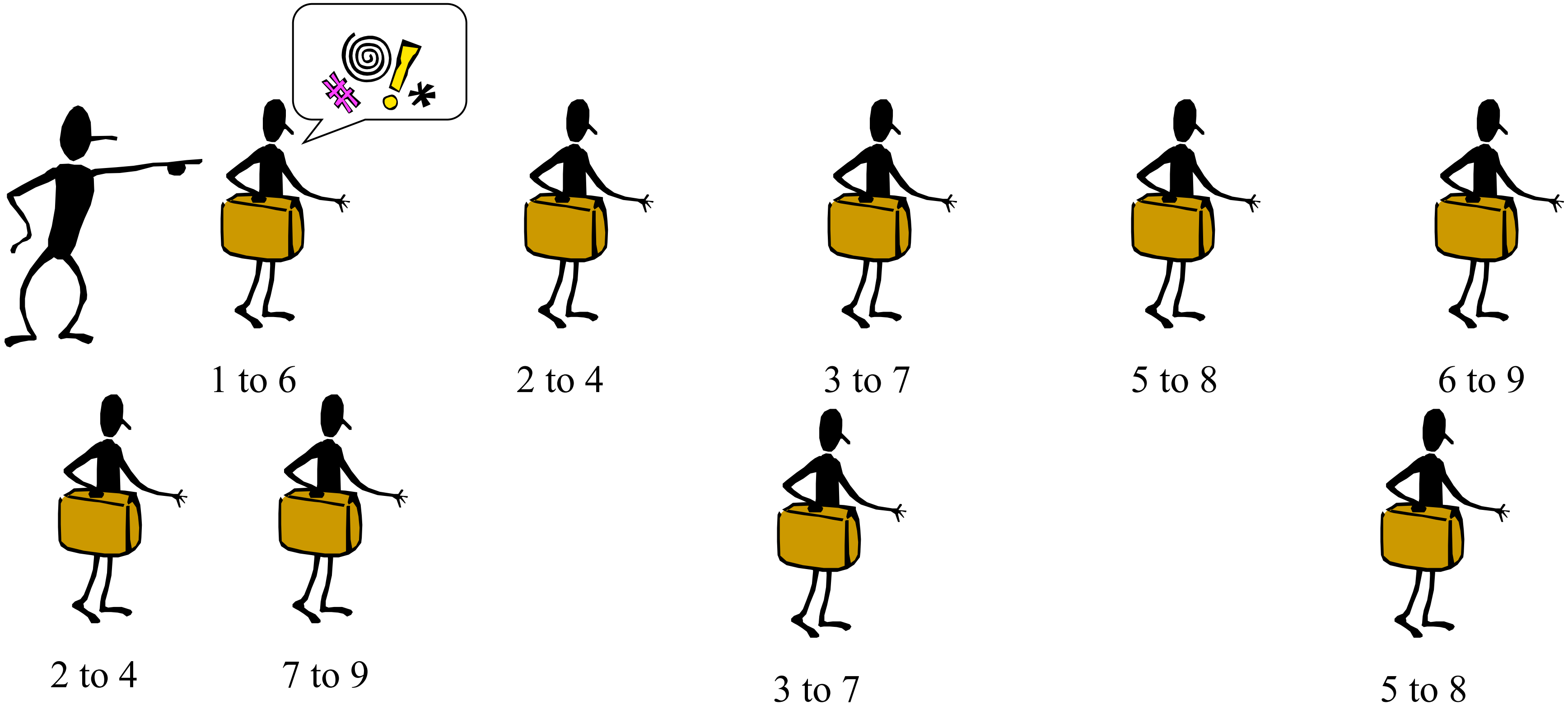
Booking system



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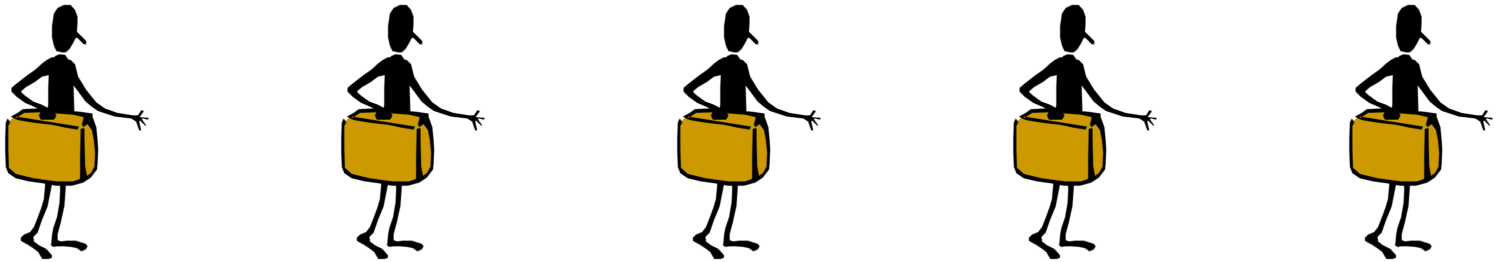
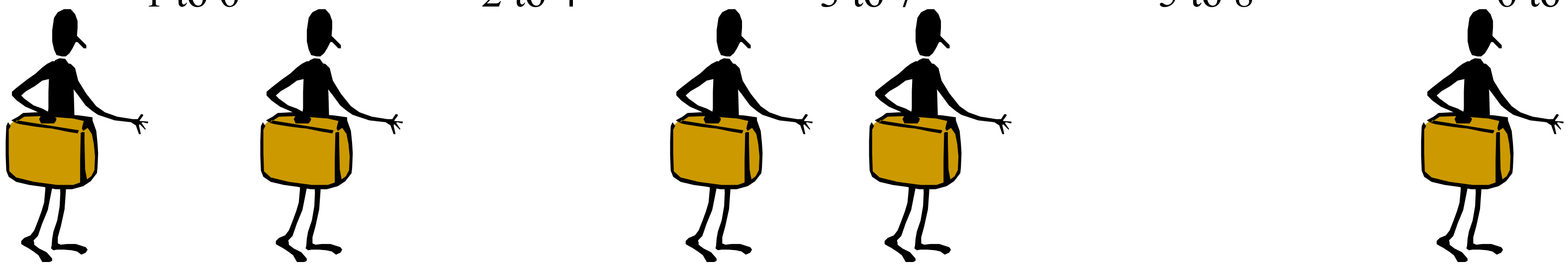


1 to 6 2 to 4 3 to 7 5 to 8 6 to 9



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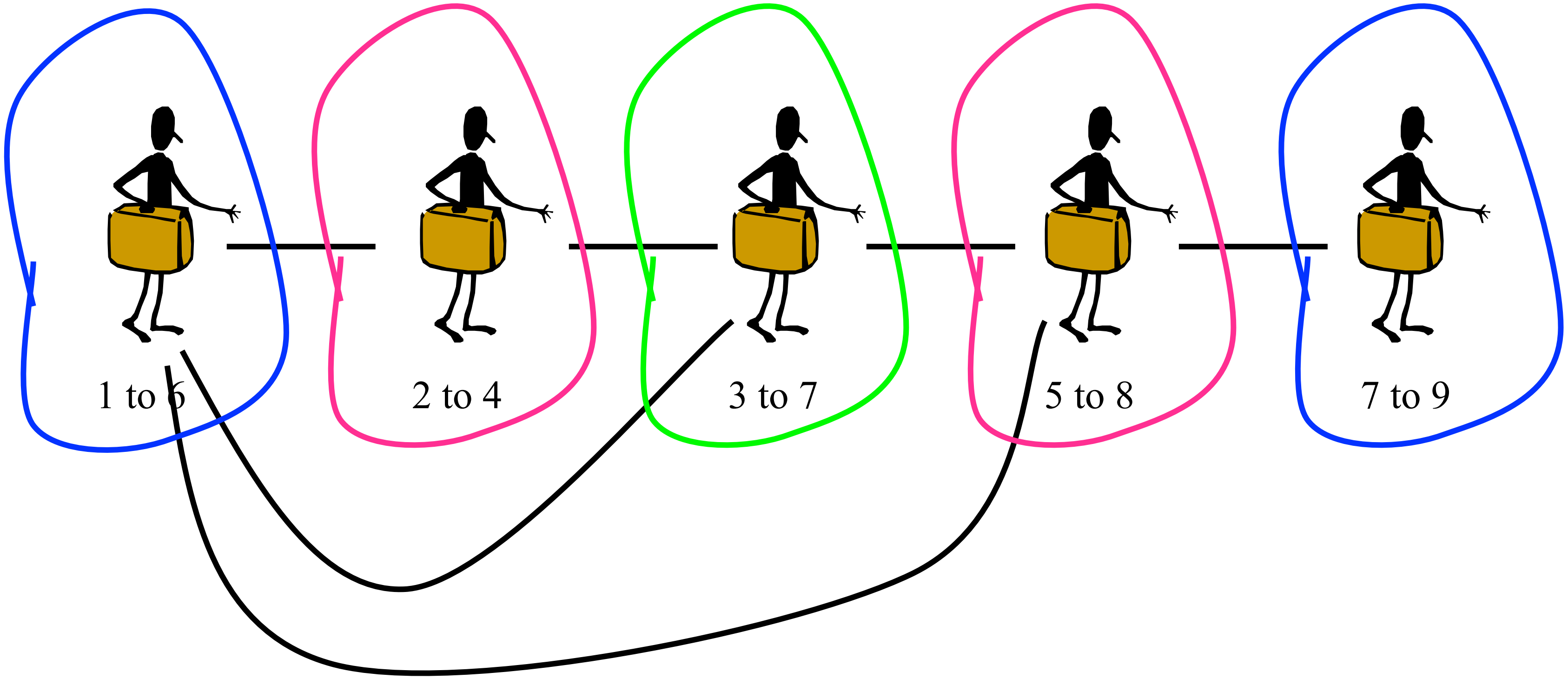
Pink

Green

Booking system



Blue

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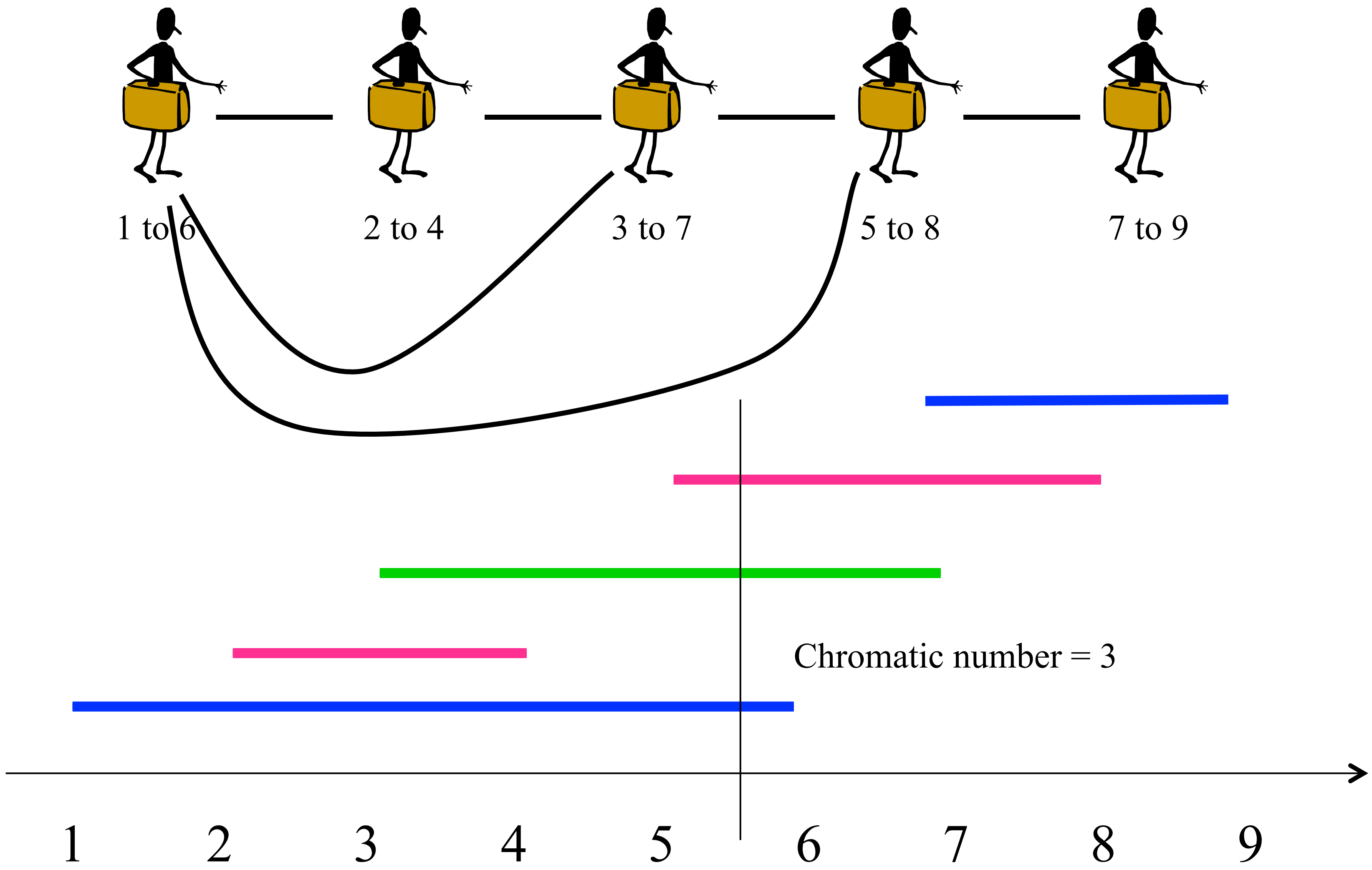
Green

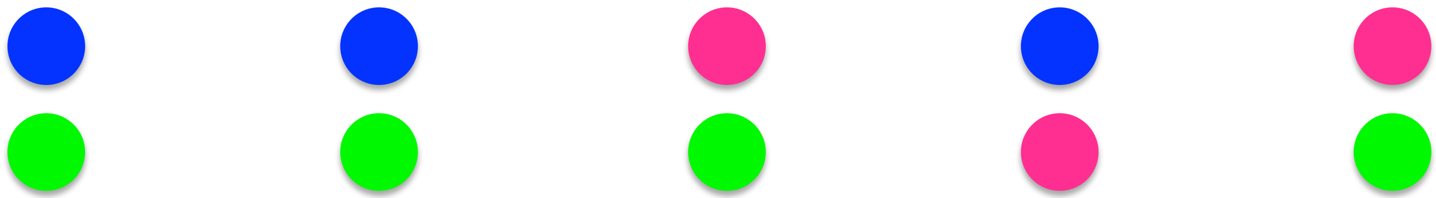
Interval graph → Polynomial



Each traveler can propose choices: list-coloring

3 rooms 5 requirements

1 to 6 | 2 to 4 | 3 to 7 | 5 to 8 | 7 to 9



Blue

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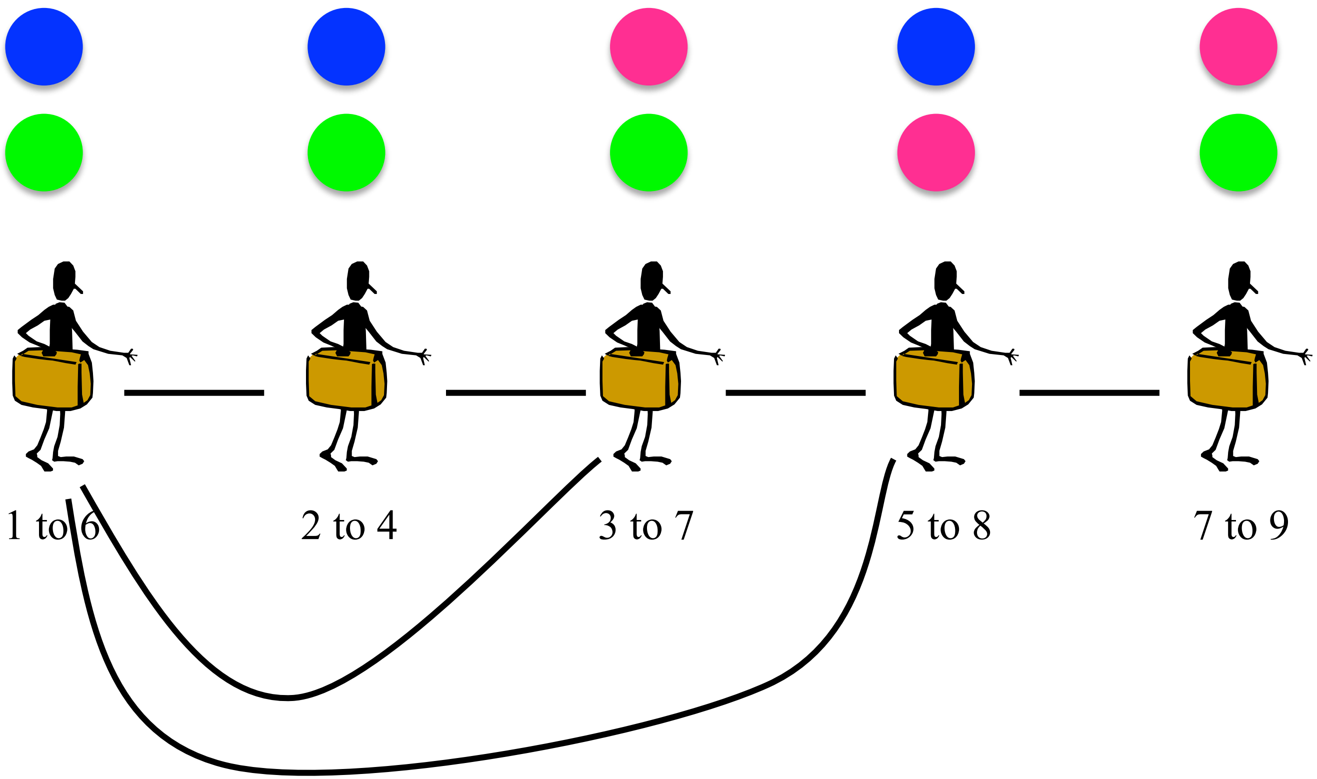
Green


1 to 6
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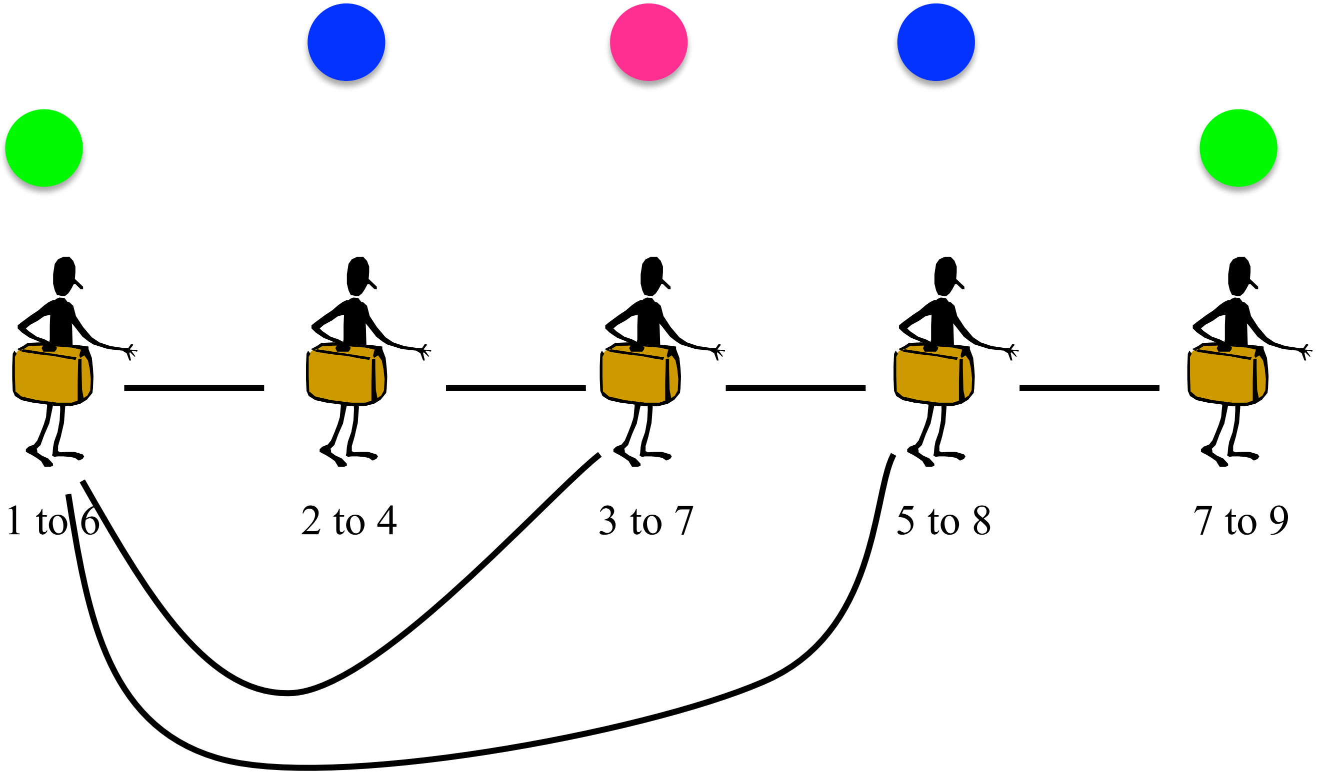

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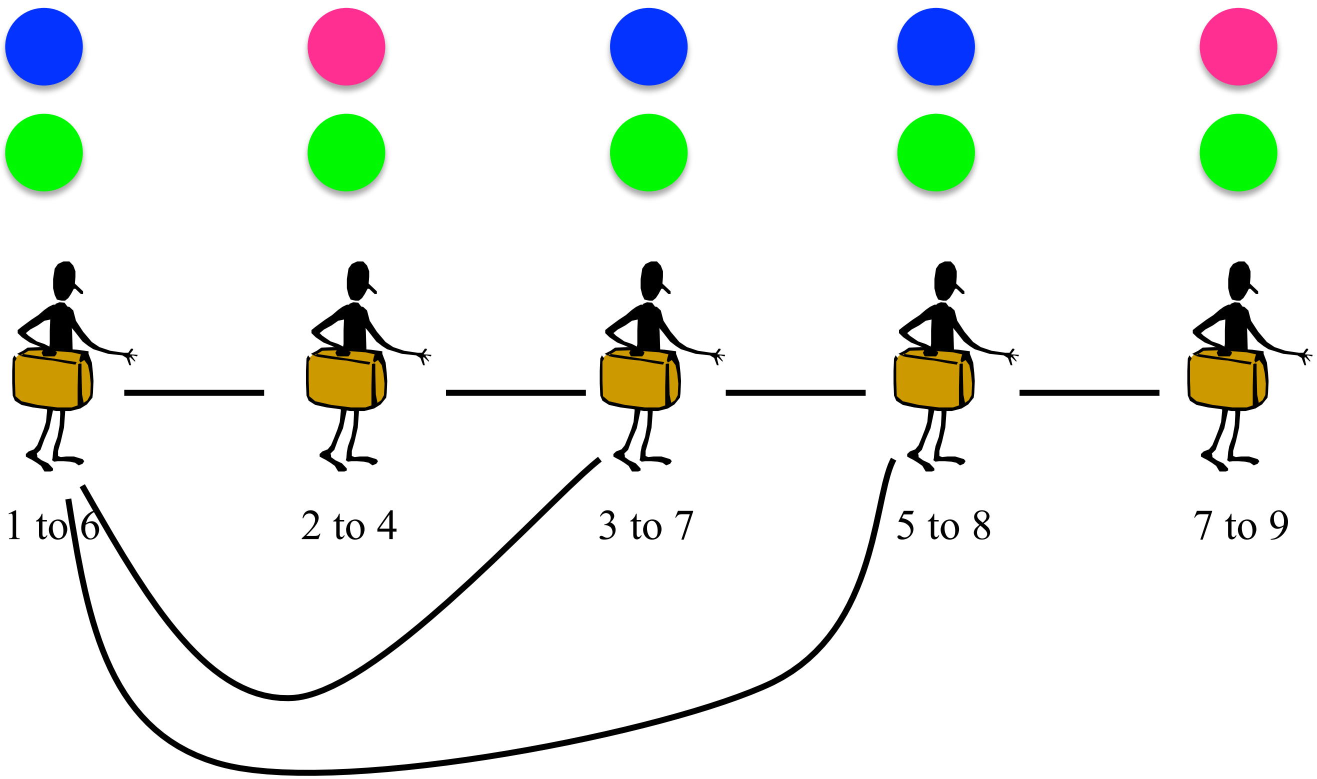
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Considered problems: list coloring

Graph $G = (V, E)$. Set $K = \{1, \ldots, k\}$ of colors.

k-coloring: each vertex v is associated with a color $c(v) \in K$.
If $uv \in E$ then $c(u) \neq c(v)$.

List coloring: each vertex v has a list $L(v)$ of possible colors.

List k-coloring: list coloring with k colors: $\forall v \in V, L(v) \subset K$

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Choosability

Given a graph $G = (V, E)$ and a function $f : V \longrightarrow \mathbb{N}$,

G is called *f-choosable* if it has a list coloring for every list system L satisfying $\forall v \in V, |L(v)| = f(v)$.

If $\forall v \in V, f(v) = \ell$, G is just called ℓ-choosable.

If $G = (B \cup W, E)$ is bipartite and f is defined by:

$$f(v) = \begin{cases} p & \text{for} \quad v \in B \\ q & \text{for} \quad v \in W \end{cases}$$

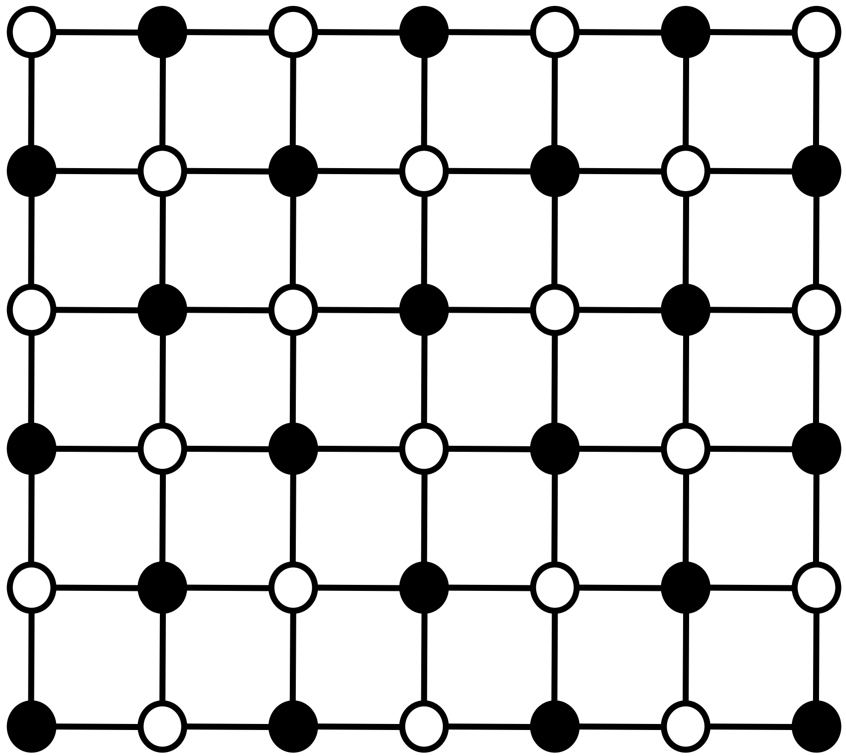
G is called (p, q)-choosable.

Choosability: first remarks

It is a hereditary property:
if G is k-choosable, then any subgraph is k-choosable.

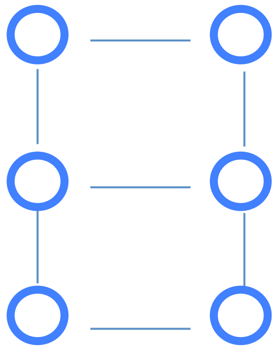
k-choosable $\Longrightarrow$ k-colorable.

First examples in grid graphs



A grid $G(6,7)$.

First examples in grid graphs

 is not 2-choosable.

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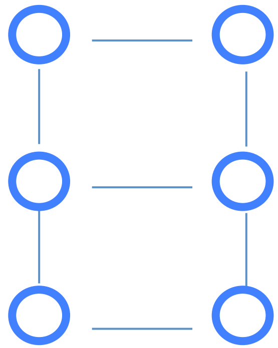

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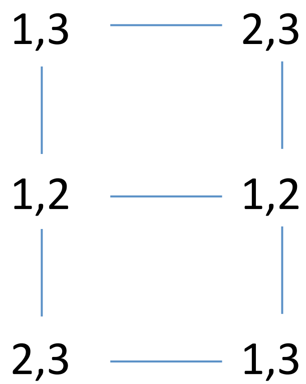

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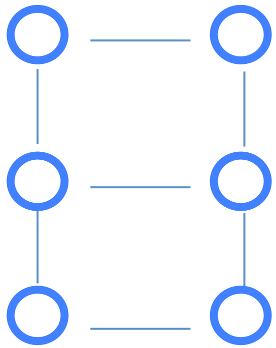

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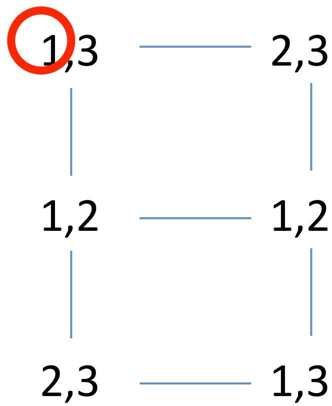

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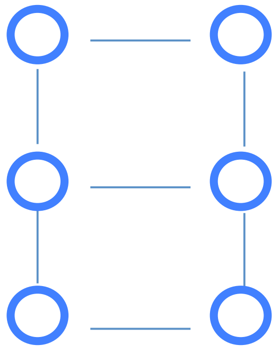

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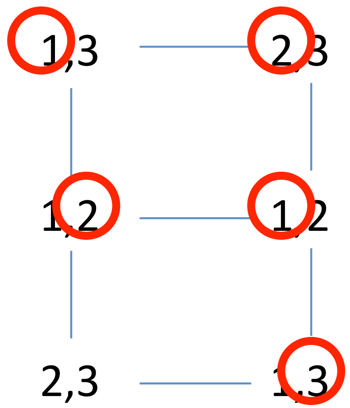

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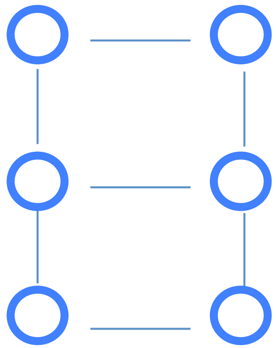
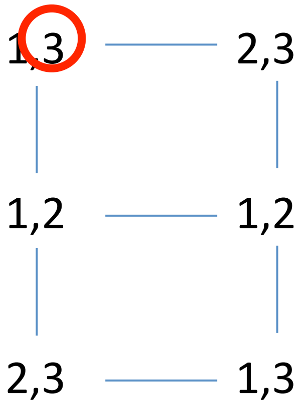
 is not 2-choosable.



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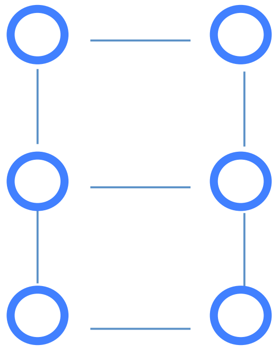

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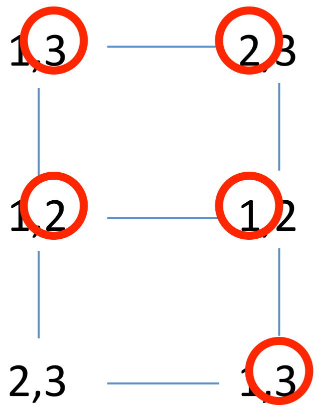

Grids are not (2,3)-choosable

(MD, de Werra, 2013)

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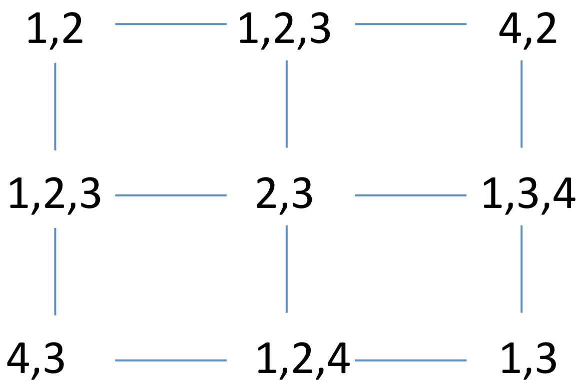

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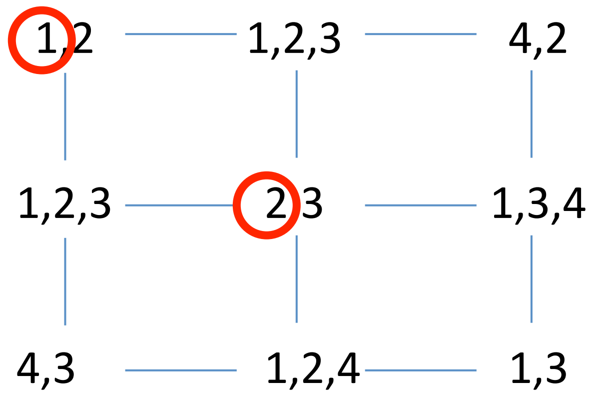

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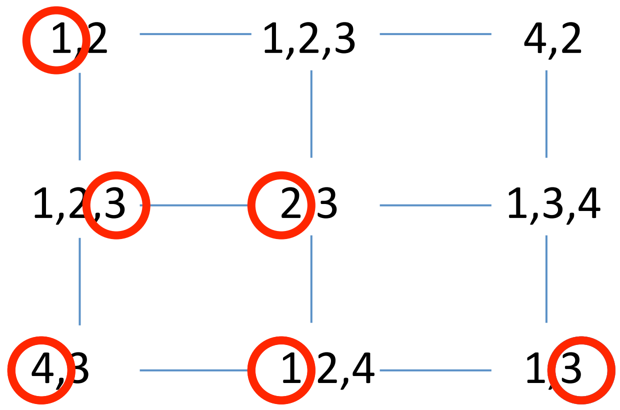

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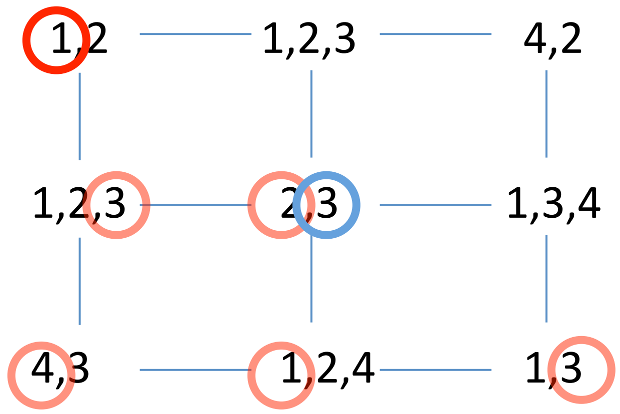

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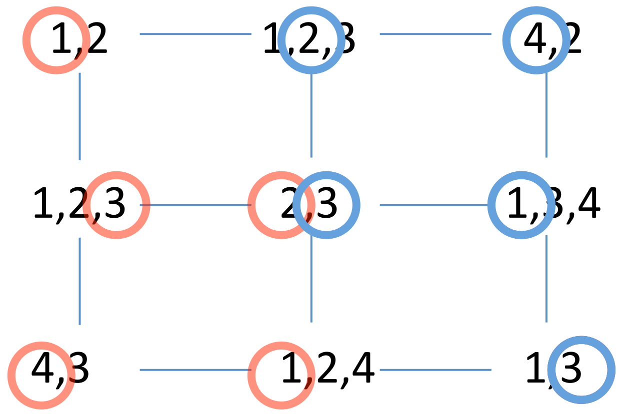

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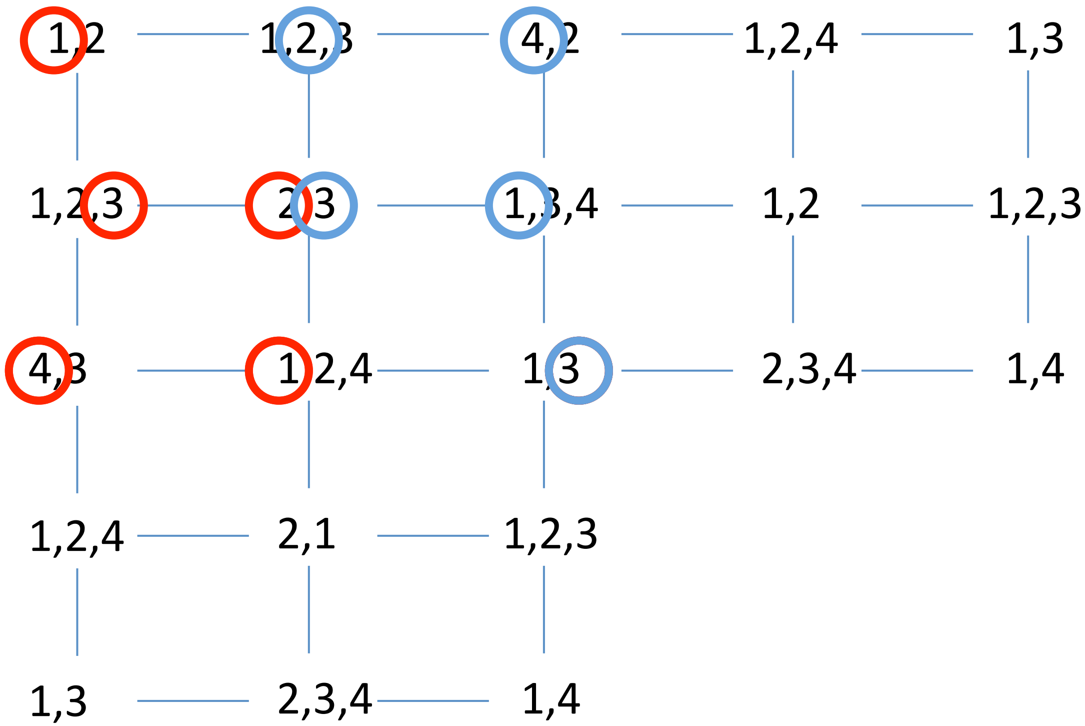

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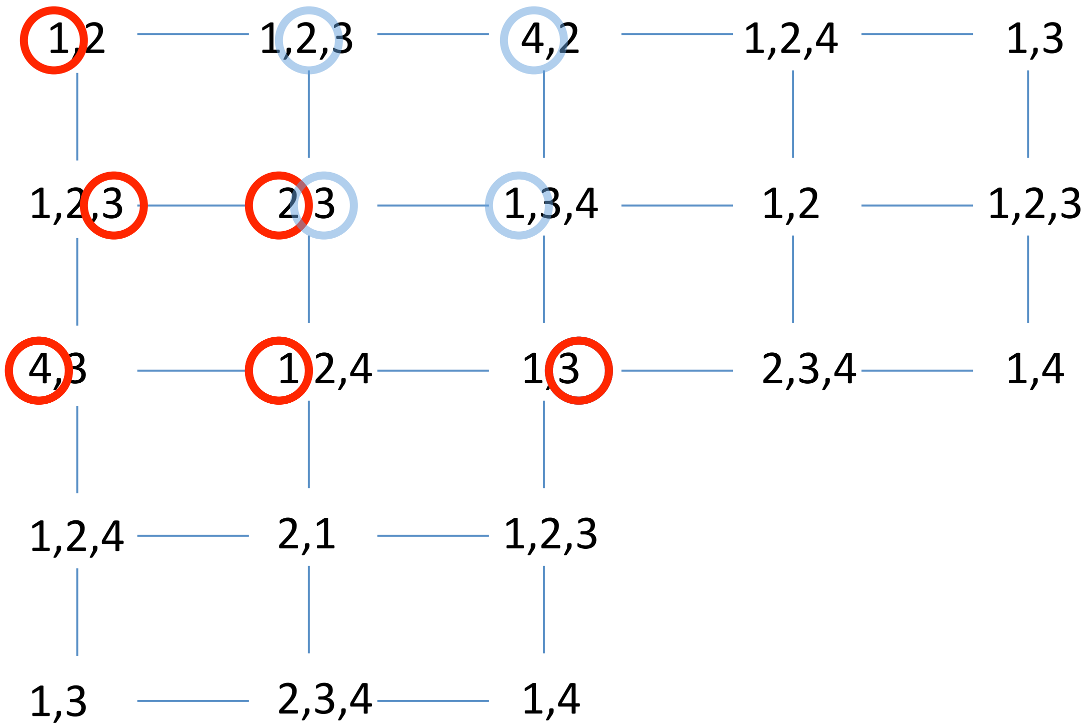

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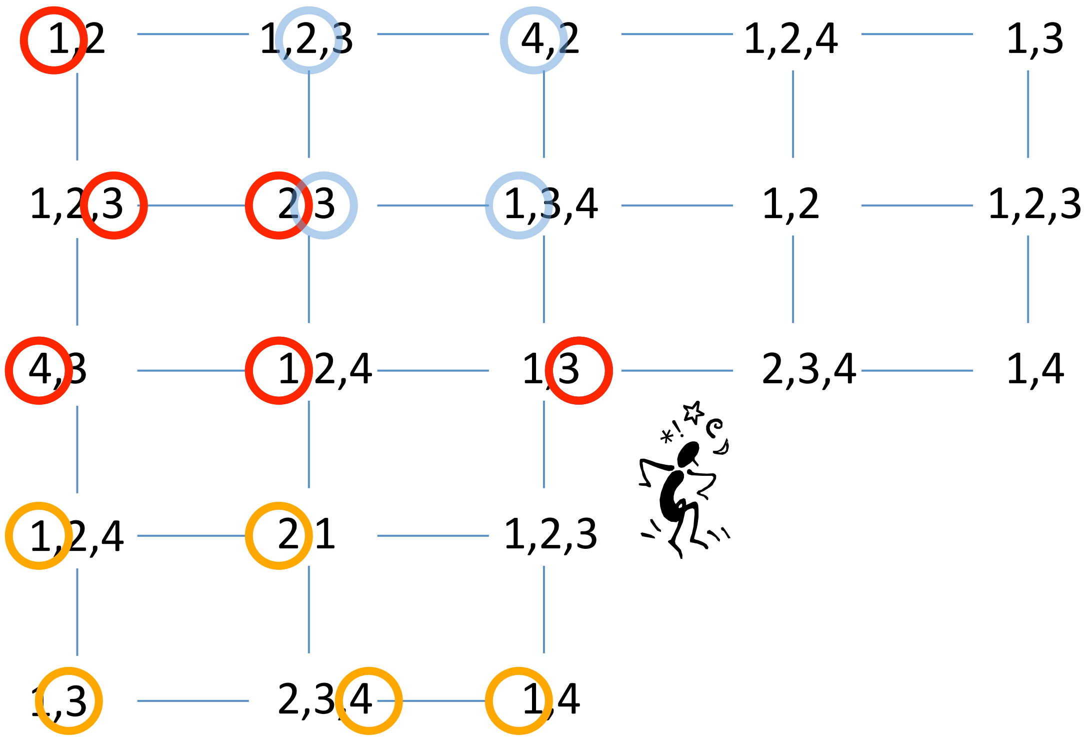

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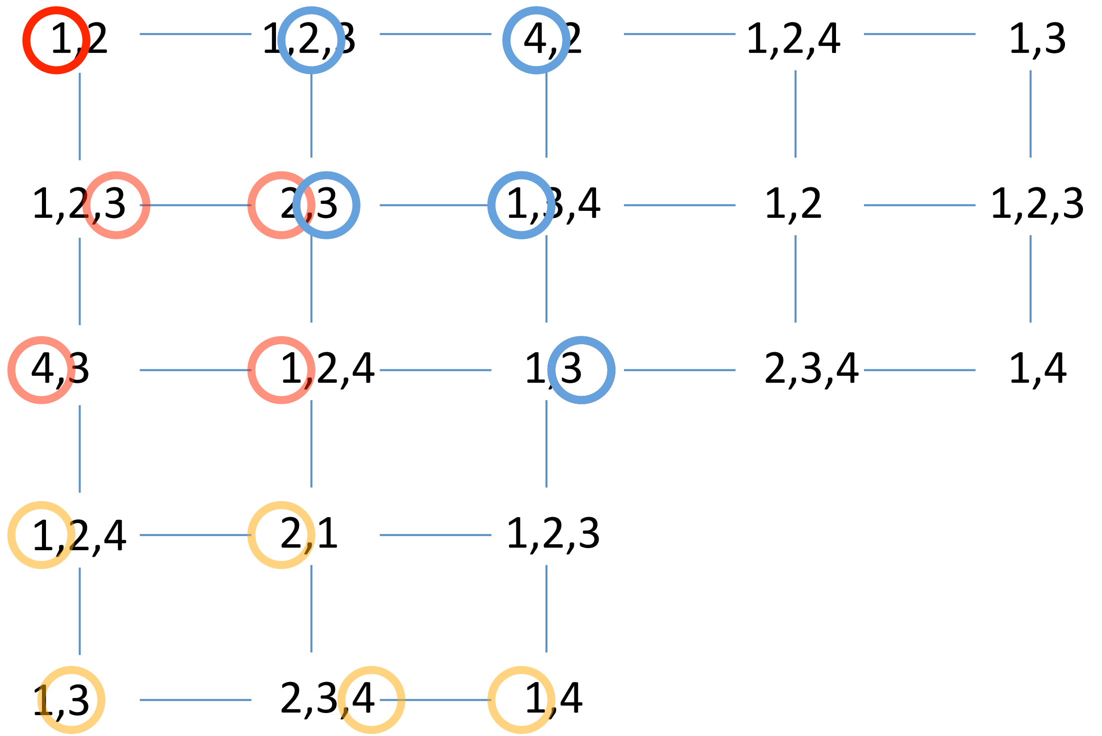

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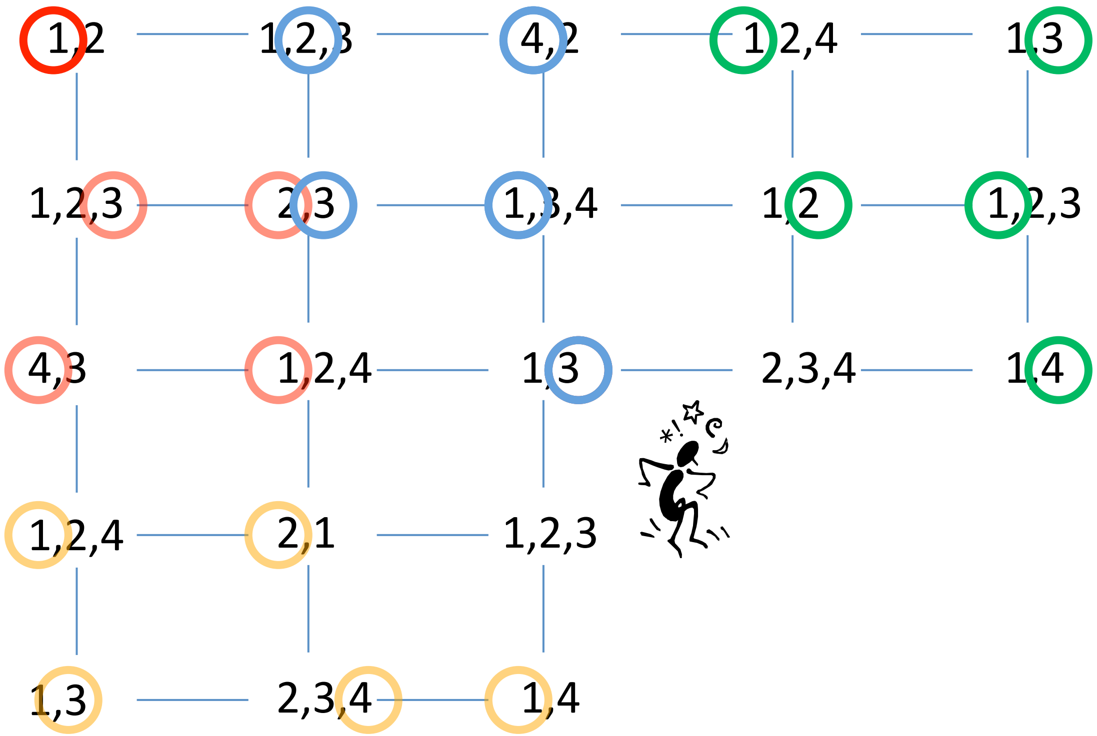

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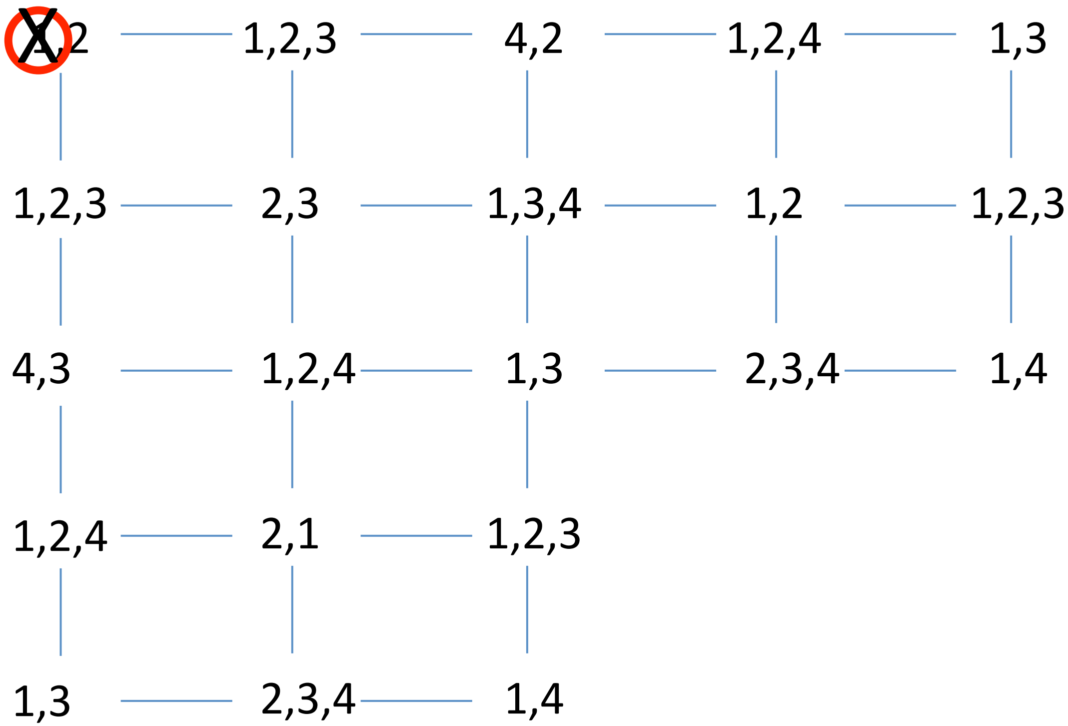

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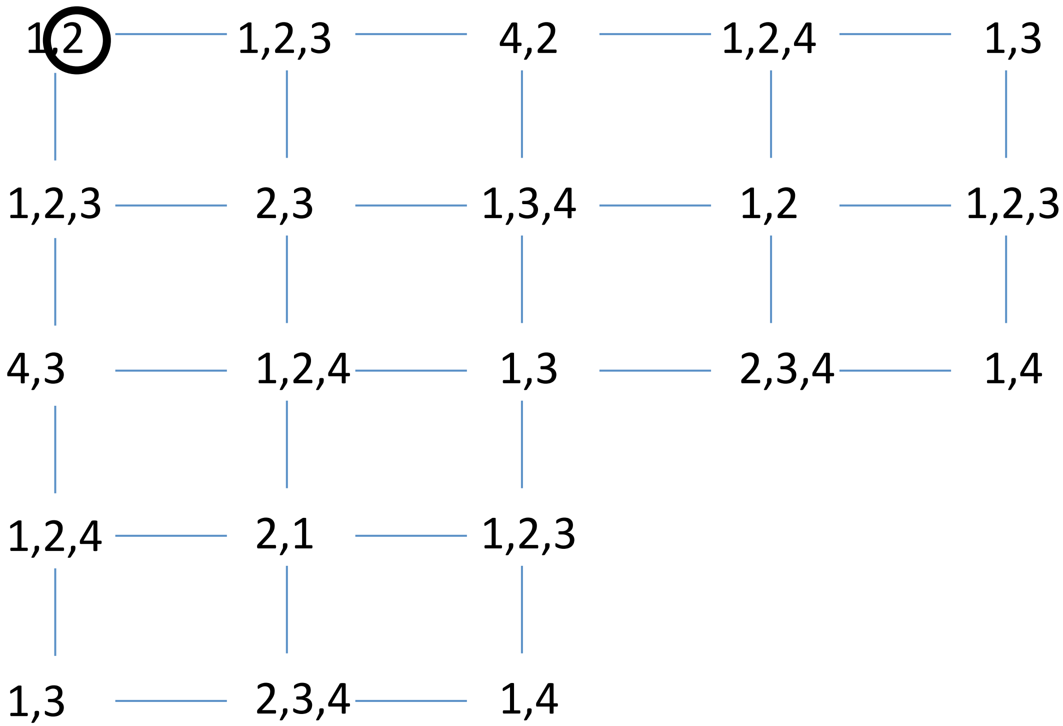

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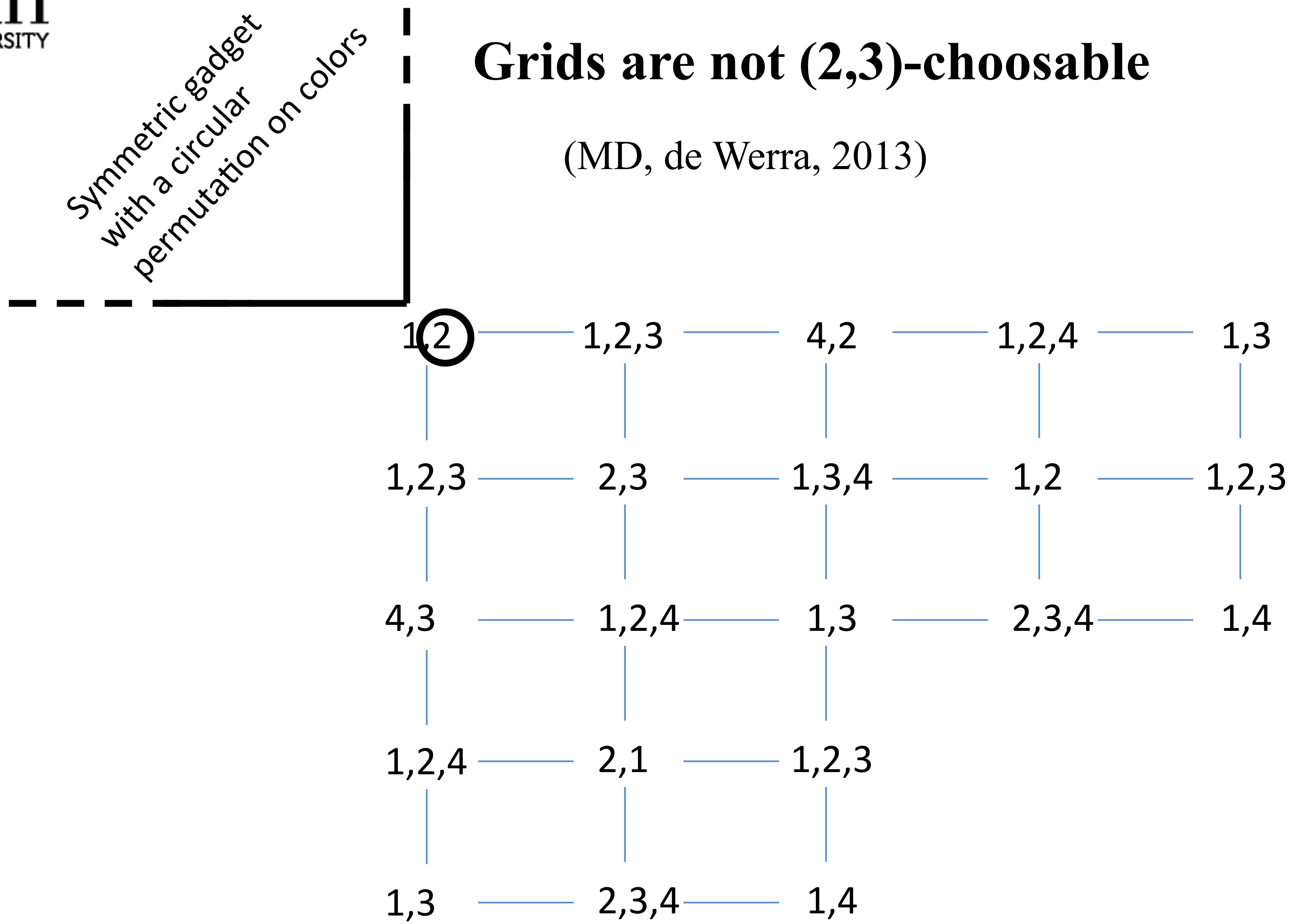

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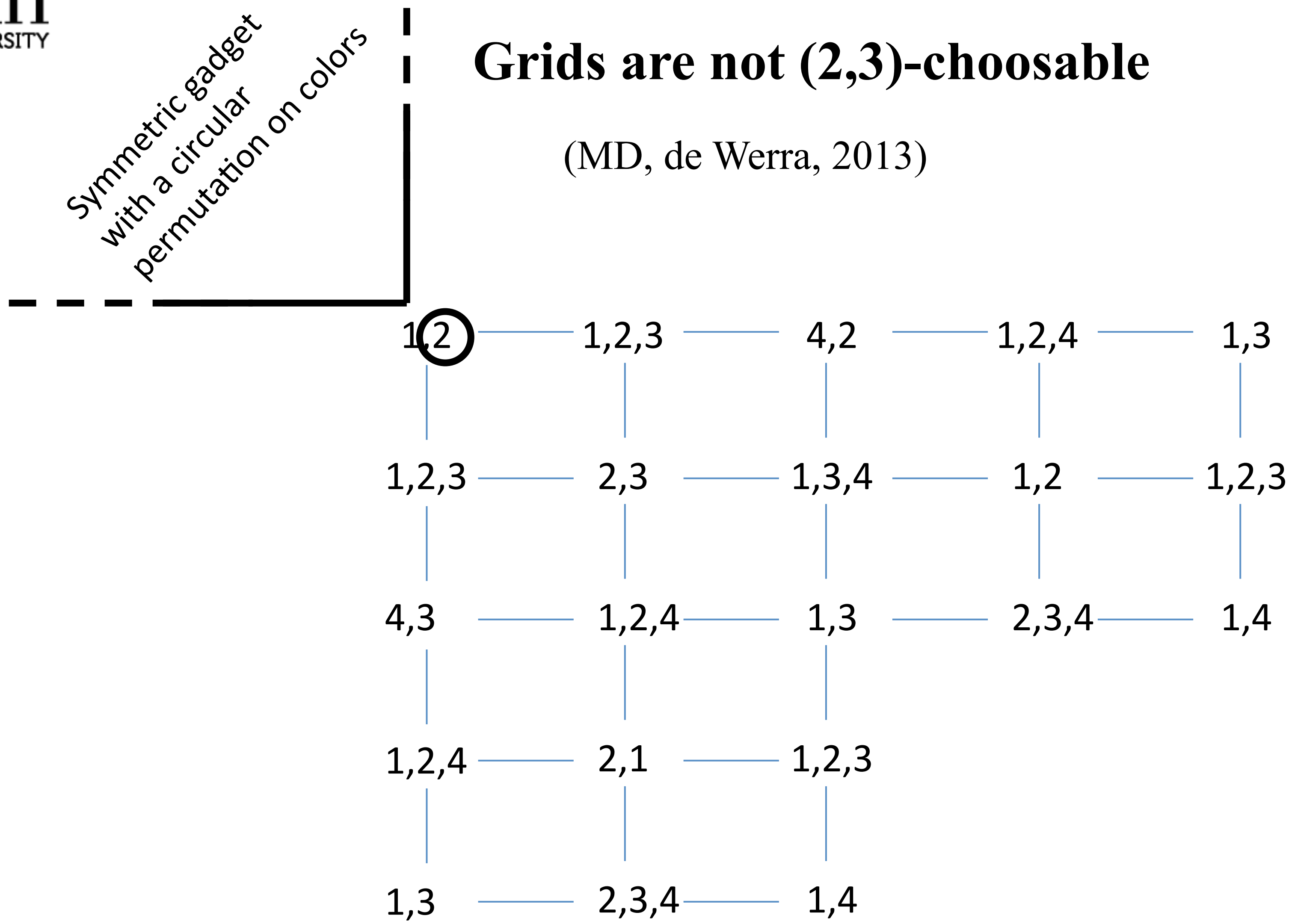


? Find an example with less than 41 vertices

Grids are (2,4)-choosable

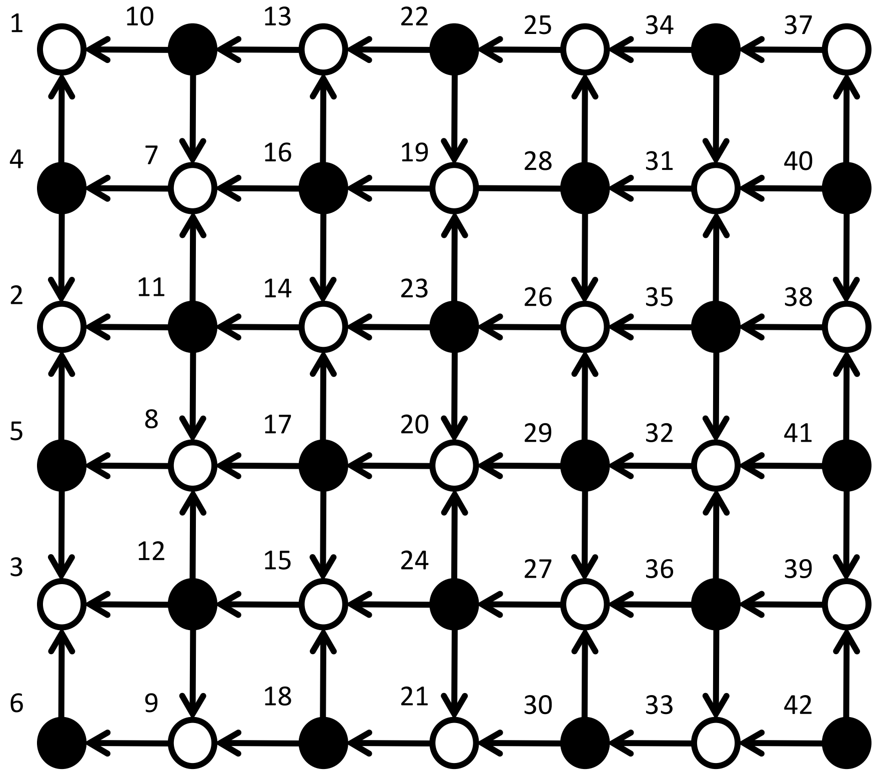
(MD, de Werra, 2013)

Acyclic orientation s.t.

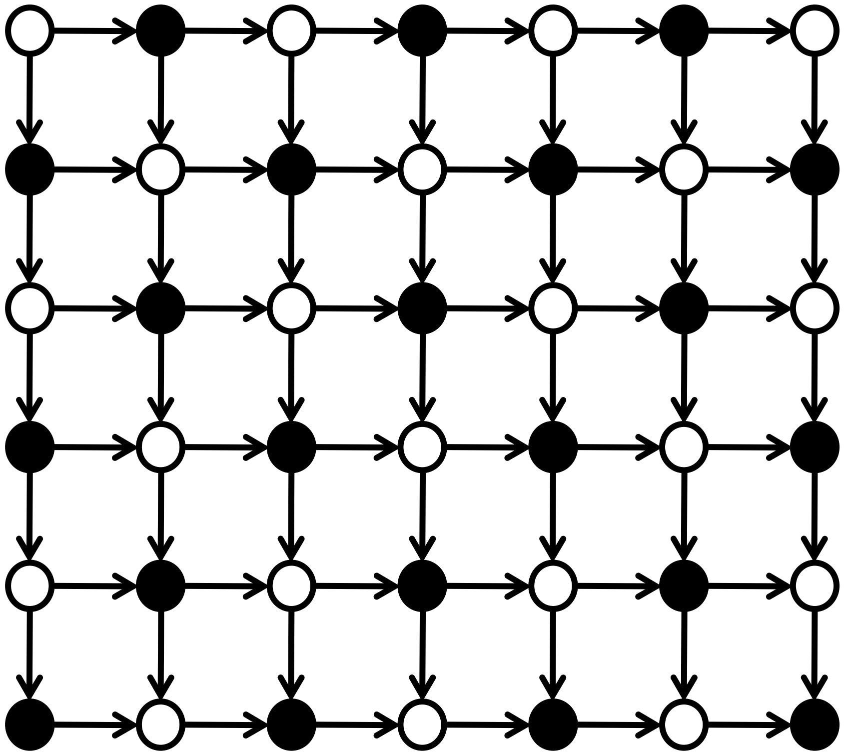
$$d^+(x) \le 1, x \in V_1$$

$$d^+(x) \le 3, x \in V_2$$

Number vertices by eliminating
Vertices s.t. $d^+(x) = 0$



Grids are 3-choosable



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Choosability of bipartite graphs

Theorem (Alon, Tarsi, Combinatorica 12, 1992)

A bipartite graph of maximum degree Δ is $\left(\left\lceil \frac{\Delta}{2} \right\rceil + 1, \left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right)$-choosable

N. Alon, M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (2) (1992) 125–134.

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Characterization of 2-choosability

P. Erdös, A.L. Rubin, H. Taylor, Choosability in graphs, in Proc. of West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, Congressus Numerantium, 26, (1979), 125–157.

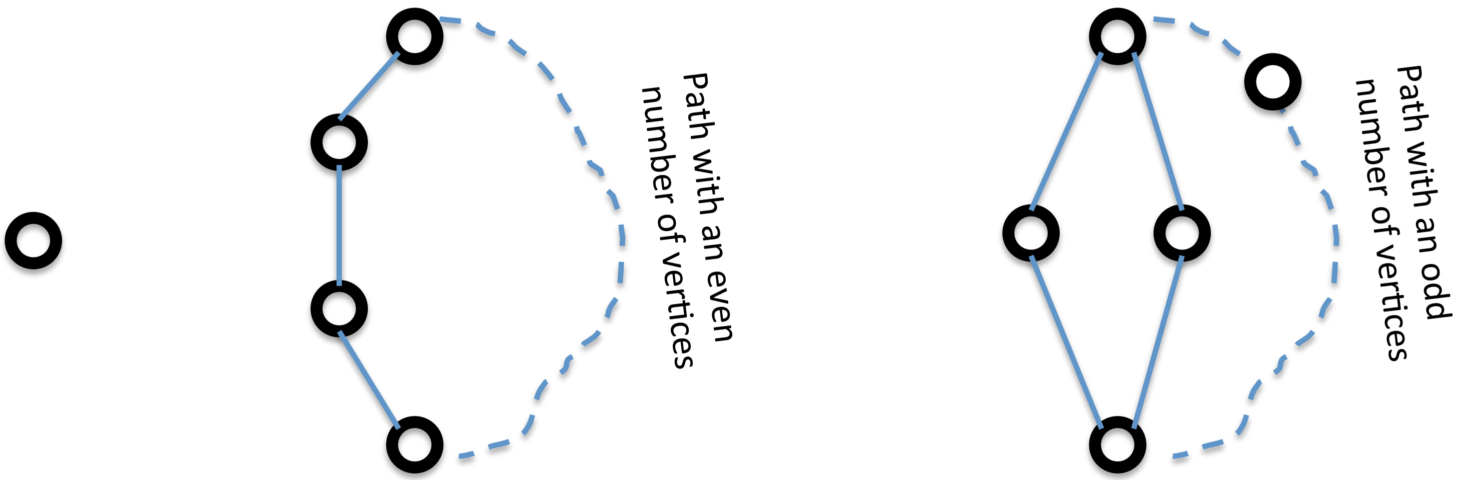
The *core* of G is obtained by repeatedly removing a vertex of degree 1 together with its incident edge until the graph contains only isolated vertices and vertices of degree at least 2

G is 2-choosable if and only if its core is 2-choosable.

Characterization of 2-choosability

P. Erdös, A.L. Rubin, H. Taylor, Choosability in graphs, in Proc. of West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, Congressus Numerantium, 26, (1979), 125–157.

G is 2-choosable if and only if its core is one of the following graphs:



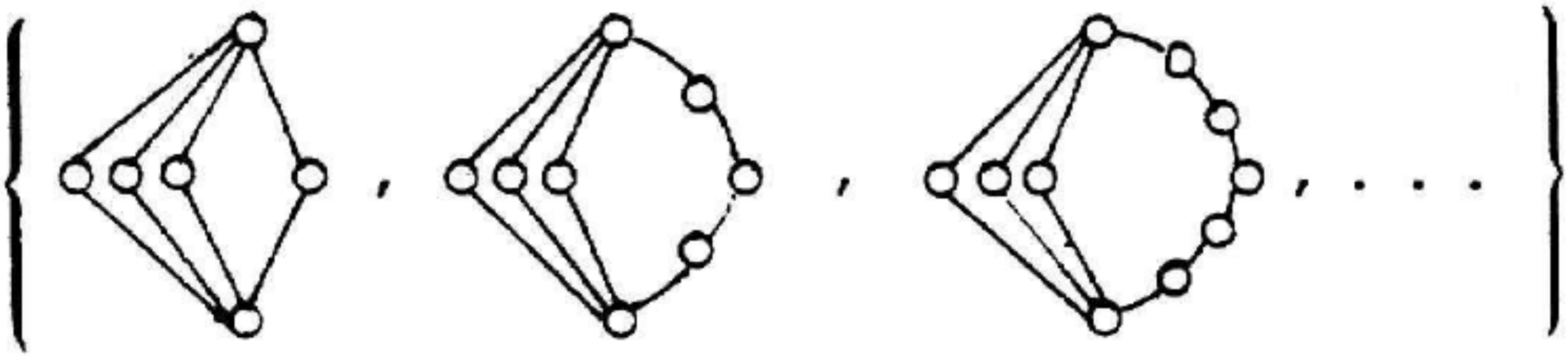
K_1 $\quad$ $C_{2m+2}, m \geq 1$ $\quad$ $\theta_{2,2,2m}, m \geq 1$

$$T = \{K_1, C_{2m+2}, \theta_{2,2,2m}, m \geq 1\}$$

Characterization of 2-choosability

Sketch of proof:

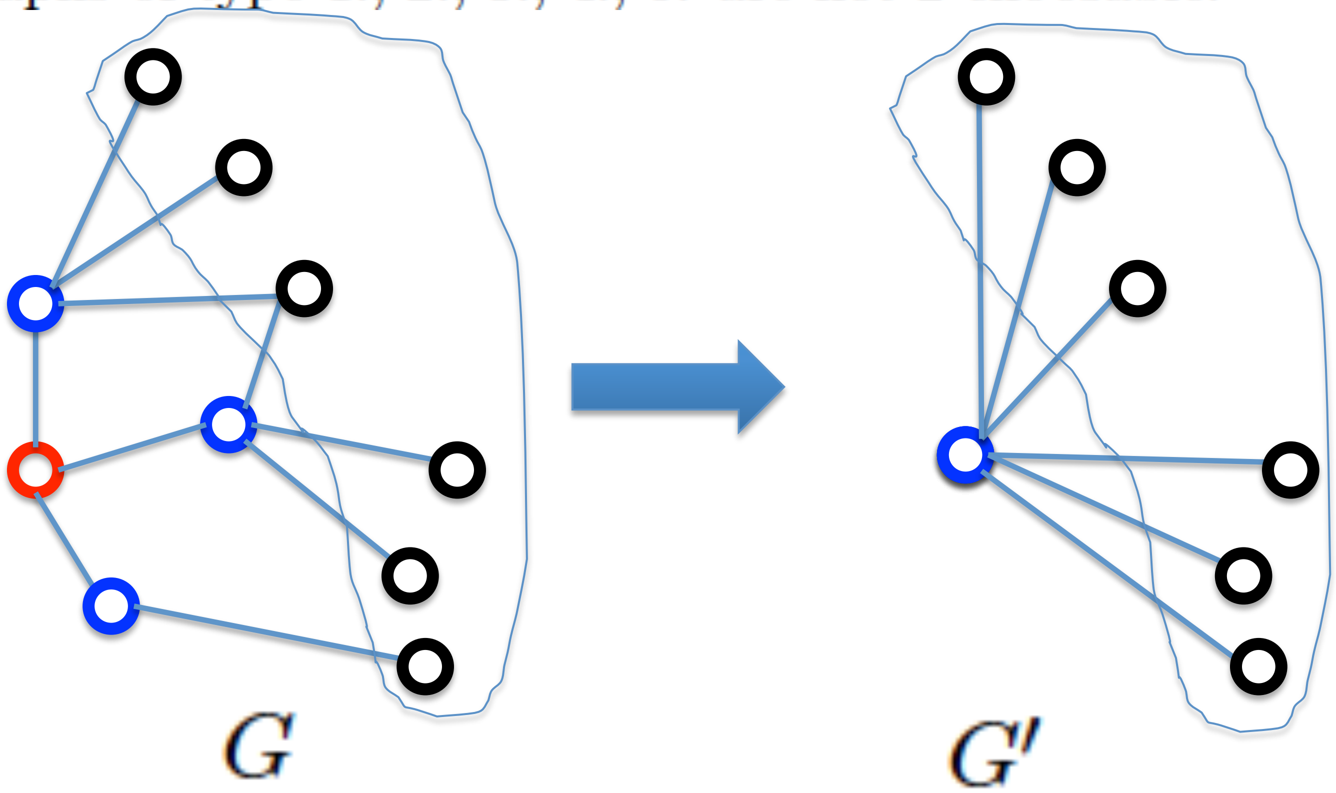
1 G connected and without vertices of degree less than 2, if G is not in T then it contains either:

1. An odd cycle.
2. Two node disjoint even cycles connected by a path.
3. Two even cycles having exactly one node in common.
4. $\Theta_{a,b,c}$ where $a \neq 2$ and $b \neq 2$.
5. 

Characterization of 2-choosability

Sketch of proof (cont.):

2 Graphs of type 1., 2., 3., 4., 5. are not 2-choosable.

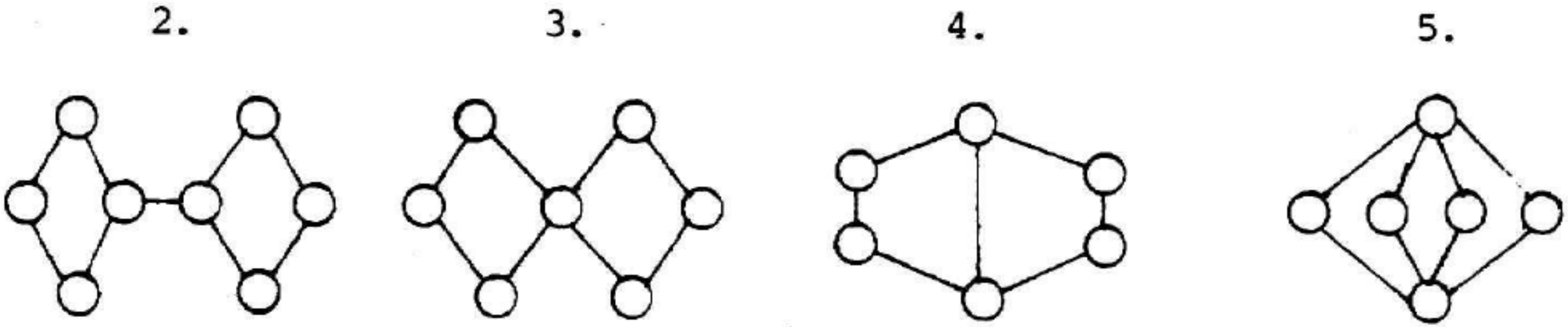


G' not 2-choosable $\Longrightarrow$ G not 2-choosable

Characterization of 2-choosability

Sketch of proof (cont.):

3 It remains to show that the following graphs are not 2-choosable



Characterization of 2-choosability

Sketch of proof (cont.):

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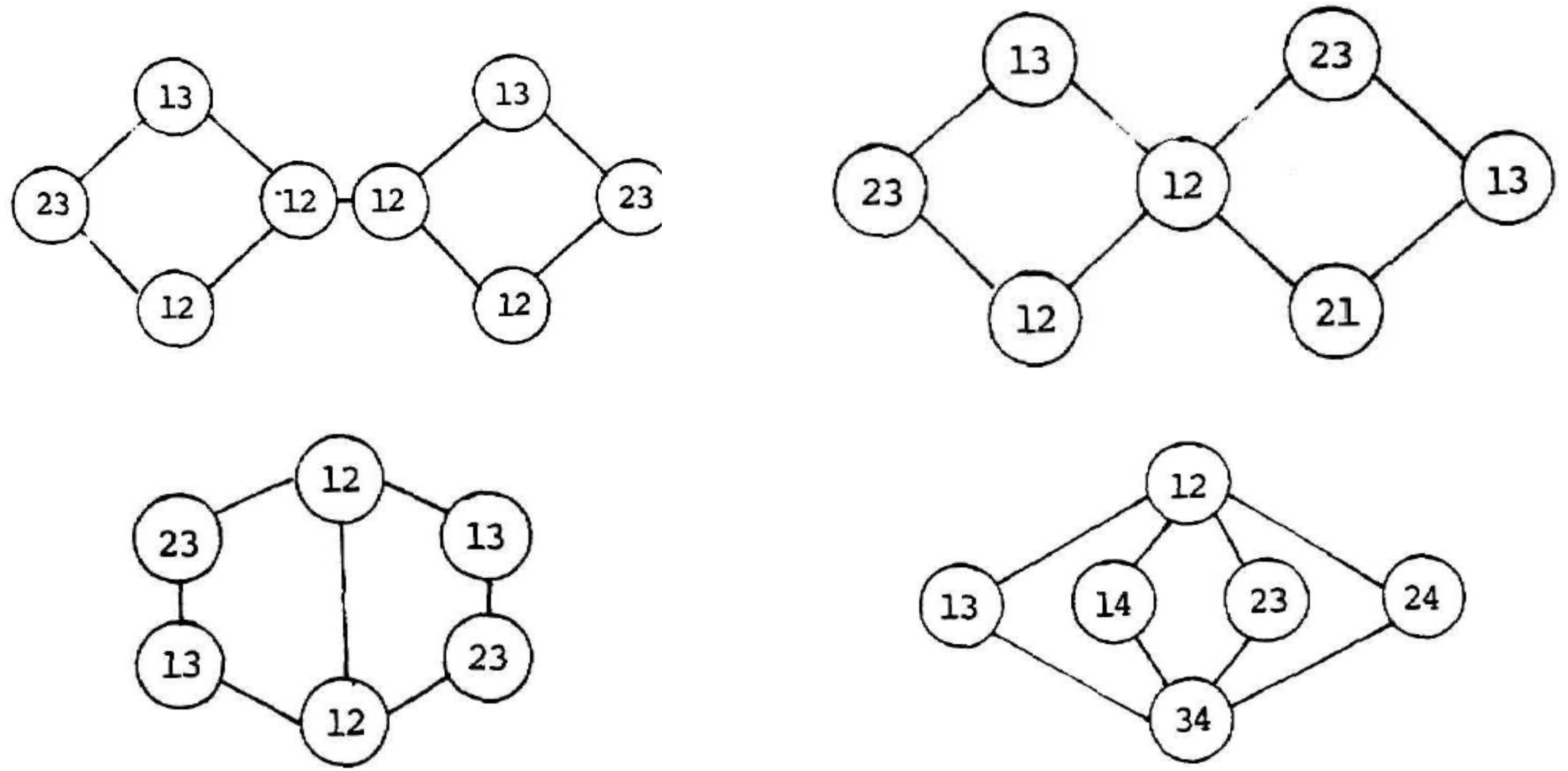

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Choosability of planar graphs

Planar: 4-colorable (four colors theorem, Appel and Haken 1977)

Triangle-free planar: 3-colorable (Grotzsch's theorem, 1959)

Planar: 5-choosable

C. Thomassen, Every planar graph is 5-choosable, J. Combin. Theory Ser. B 62 (1994) 180–181.

Planar+bipartite:3-choosable

N. Alon, M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (2) (1992) 125–134.

Planar+triangle free:4-choosable

Jan Kratochvíl and Zsolt Tuza. Algorithmic complexity of list colorings. *Discrete Appl. Math.*, 50(3):297–302, 1994.

Planar + no 3- and 4-cycles:3-choosable

C. Thomassen, 3-list-coloring planar graph of girth 5, J. Combin. Theory Ser. B 64 (1995) 101–107.

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Complexity of choosability
Which relevant complexity class?

At least as hard as colorability

Not always in $NP = \exists^P P$, but in $\Pi_2^p = \forall^P \exists^P P$.

SAT problem Π_2^p-complete:

$$\forall U_1 \cdots \forall U_k \exists V_1 \cdots \exists V_r \Phi$$

Φ is a formula in conjunctive normal form

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Complexity of choosability
Main known results

$\{2,3\}$-CH is Π_2^p-complete in bipartite graphs of maximum degree 4.

P. Erdös, A.L. Rubin, H. Taylor, Choosability in graphs, in Proc. of West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, Congressus Numerantium, 26, (1979), 125–157.

$\{2,3\}$-CH is Π_2^p-complete in planar bipartite graphs of maximum degree 5.

S. Gutner, The complexity of planar graph choosability, Discrete Mathematics, 159: 119-130, 1996.

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Choosability with fixed number of colors

(MD., D. de Werra, 2015)

Given a graph $G = (V, E)$ and a function $f : V \longrightarrow \mathbb{N}$,

G is called $[f, k]$-*choosable* if it has a list k-coloring

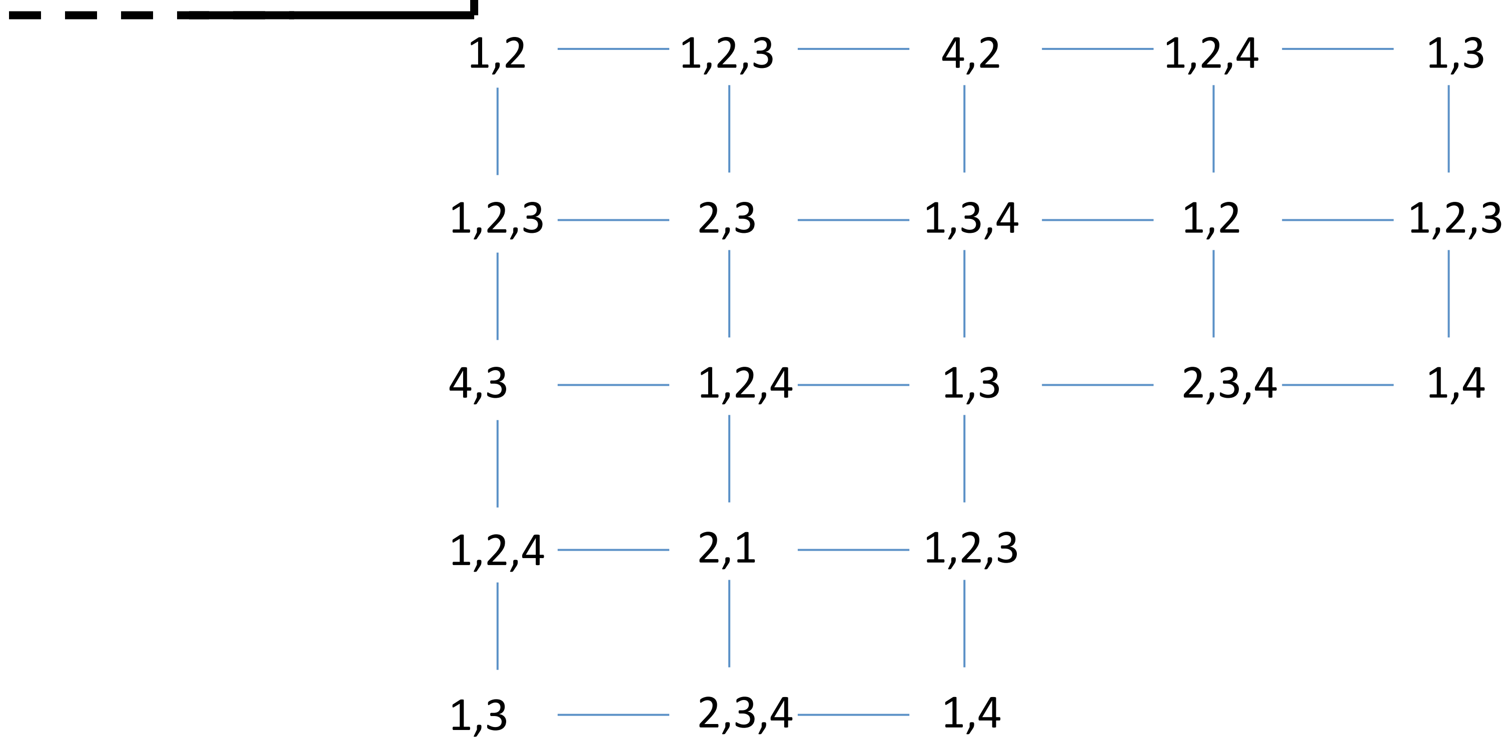
for every list system L satisfying $\forall v \in V, |L(v)| = f(v)$.

If $\forall v \in V, f(v) = \ell$, then G is simply called $[\ell, k]$-*choosable*.

$[k, k]$-choosable $\Longleftrightarrow$ k-colorable

Grids are not [(2,3),4]-choosable

(MD, de Werra, 2013)



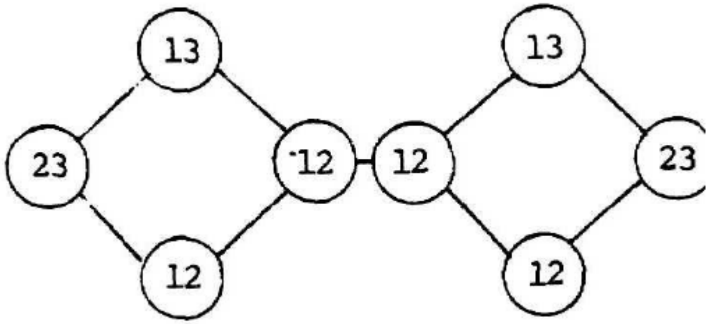
Bipartite graphs are [(2,3),3]- choosable

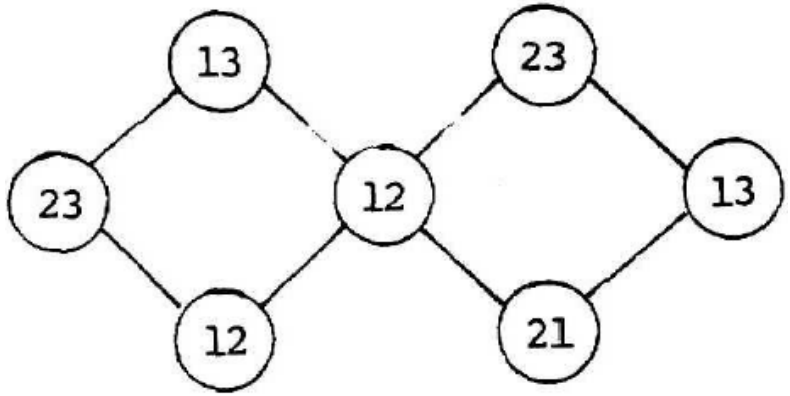
2-choosability revisited

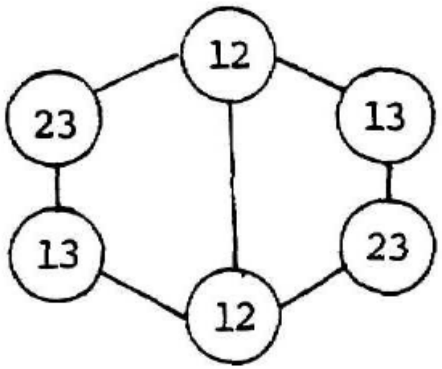
G is 2-choosable if and only if its core is in

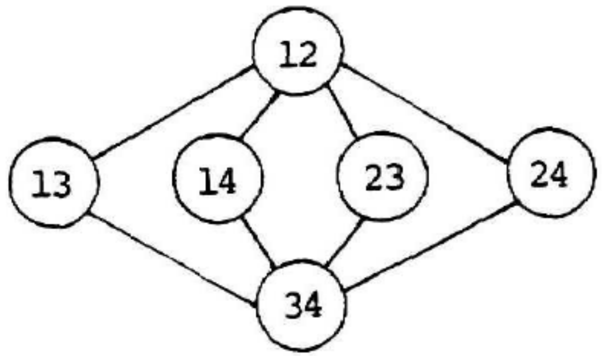
$$T = \{K_1, C_{2m+2}, \theta_{2,2,2m}, m \geq 1\}$$

Characterization of 2-choosability (remind …)









2-choosability revisited

G is 2-choosable if and only if its core is in

$$T = \{K_1, C_{2m+2}, \theta_{2,2,2m}, m \geq 1\}$$

$\iff$ $[2,4]$-choosable

2-choosability revisited

G is 2-choosable if and only if its core is in

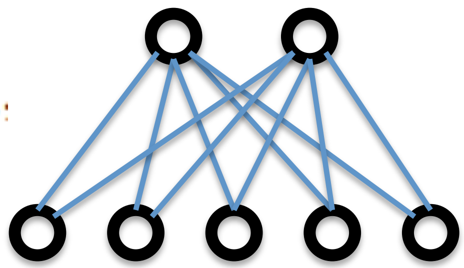
$$T = \{K_1, C_{2m+2}, \theta_{2,2,2m}, m \geq 1\}$$

$\iff$ [2, 4]-choosable

A graph is [2, 3]-choosable if and only if its core belongs to

$$\{K_1, C_{2m+2}, \theta_{2,2,2m}, K_{2,m}, m \in I\!N\}$$

$K_{2,m}$:



k-choosability vs $(k+1)$-choosability

$$\begin{aligned}[2,k]\text{-choosable, } k \geq 4 \quad &\iff \quad [2,4]\text{-choosable} \\ &\iff \quad 2\text{-choosable}\end{aligned}$$

What about ℓ-choosability, $\ell > 2$?

k-choosability vs $(k+1)$-choosability

$$[2,k]\text{-choosable},\ k \geq 4 \iff [2,4]\text{-choosable} \iff 2\text{-choosable}$$

What about ℓ-choosability, $\ell > 2$?

For any $\ell \geq 3$ *and any* $k \geq 2\ell - 2$, *there is a bipartite graph*

that is $[\ell, k]$*-choosable but not* $[\ell, k+1]$*-choosable*

Complexity with few colors

$\{2,3\}$-CH is Π_2^p-complete in planar bipartite graphs of maximum degree 5.

S. Gutner, The complexity of planar graph choosability, Discrete Mathematics, 159: 119-130, 1996.

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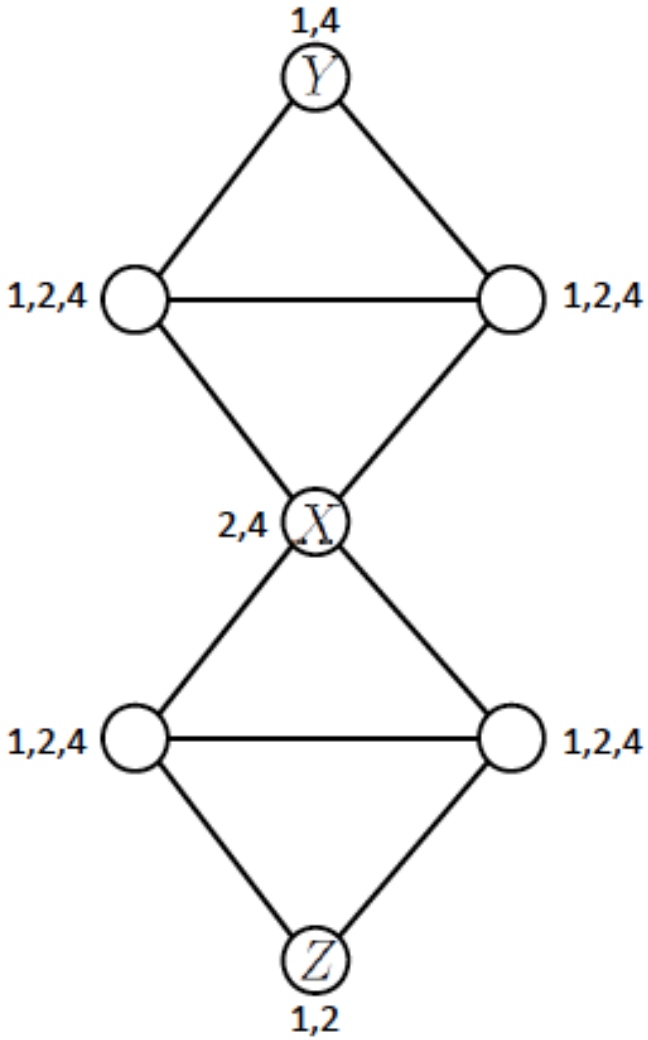
For any $k \geq 3$, $[\{2,3\},k]$-CH is Π_2^p-complete in subgrids.

For any $k \geq 5$, $[\{2,3,5\},k]$-CH is Π_2^p-complete in grids.

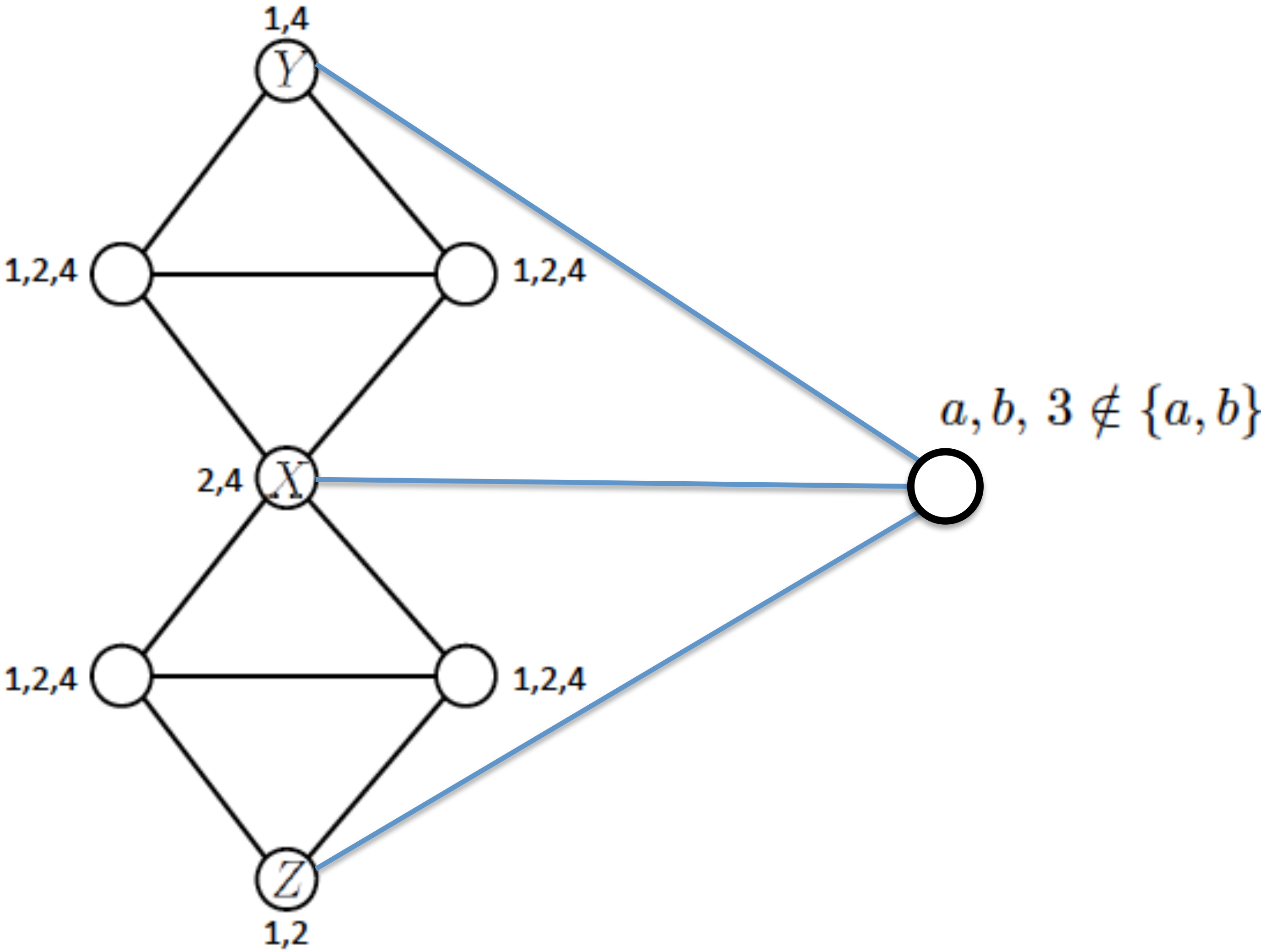
Complexity with few colors: [3,*k*]-choosability

No list coloring



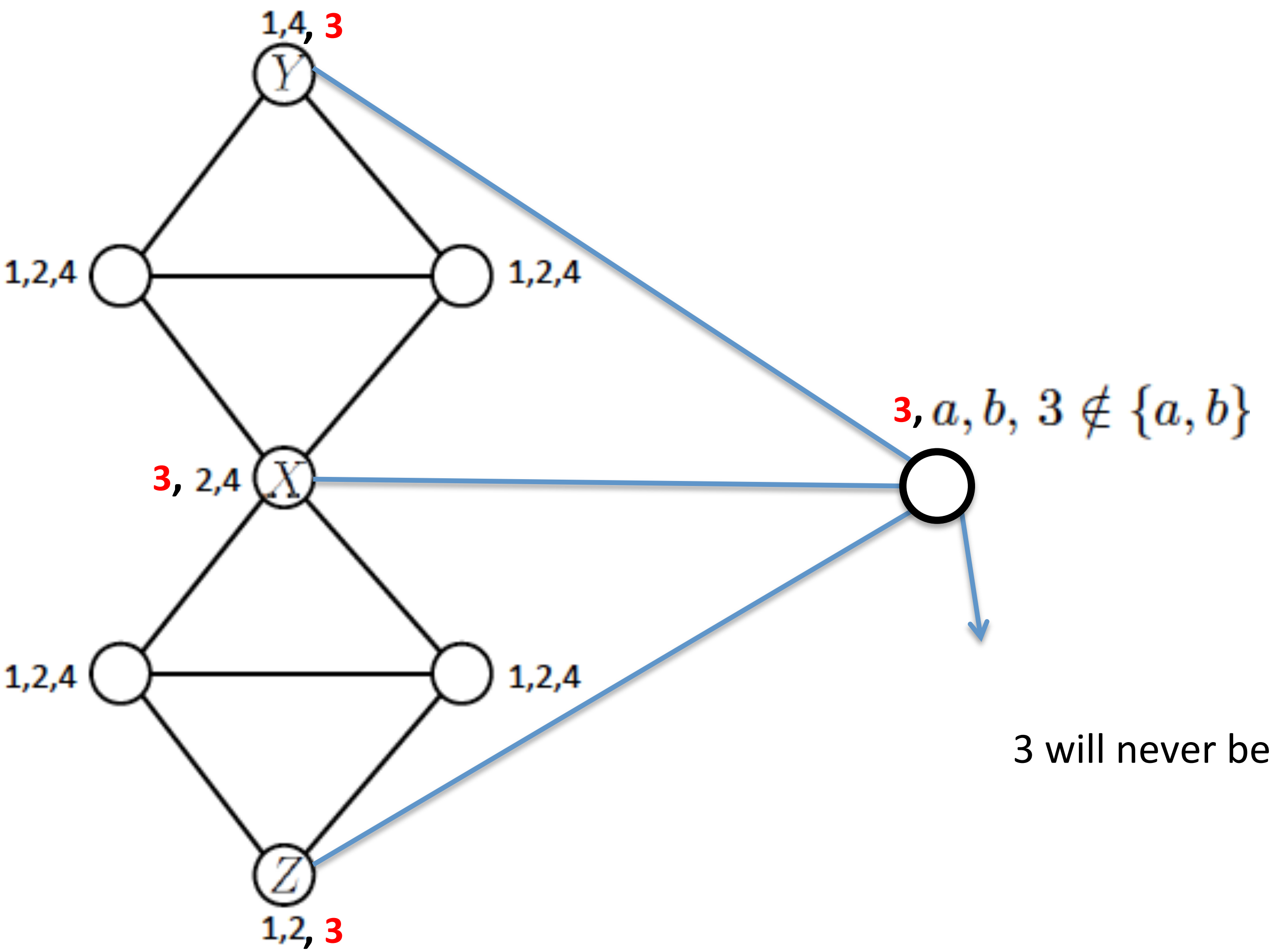
Complexity with few colors: [3,*k*]-choosability

No list coloring



Complexity with few colors: [3,*k*]-choosability

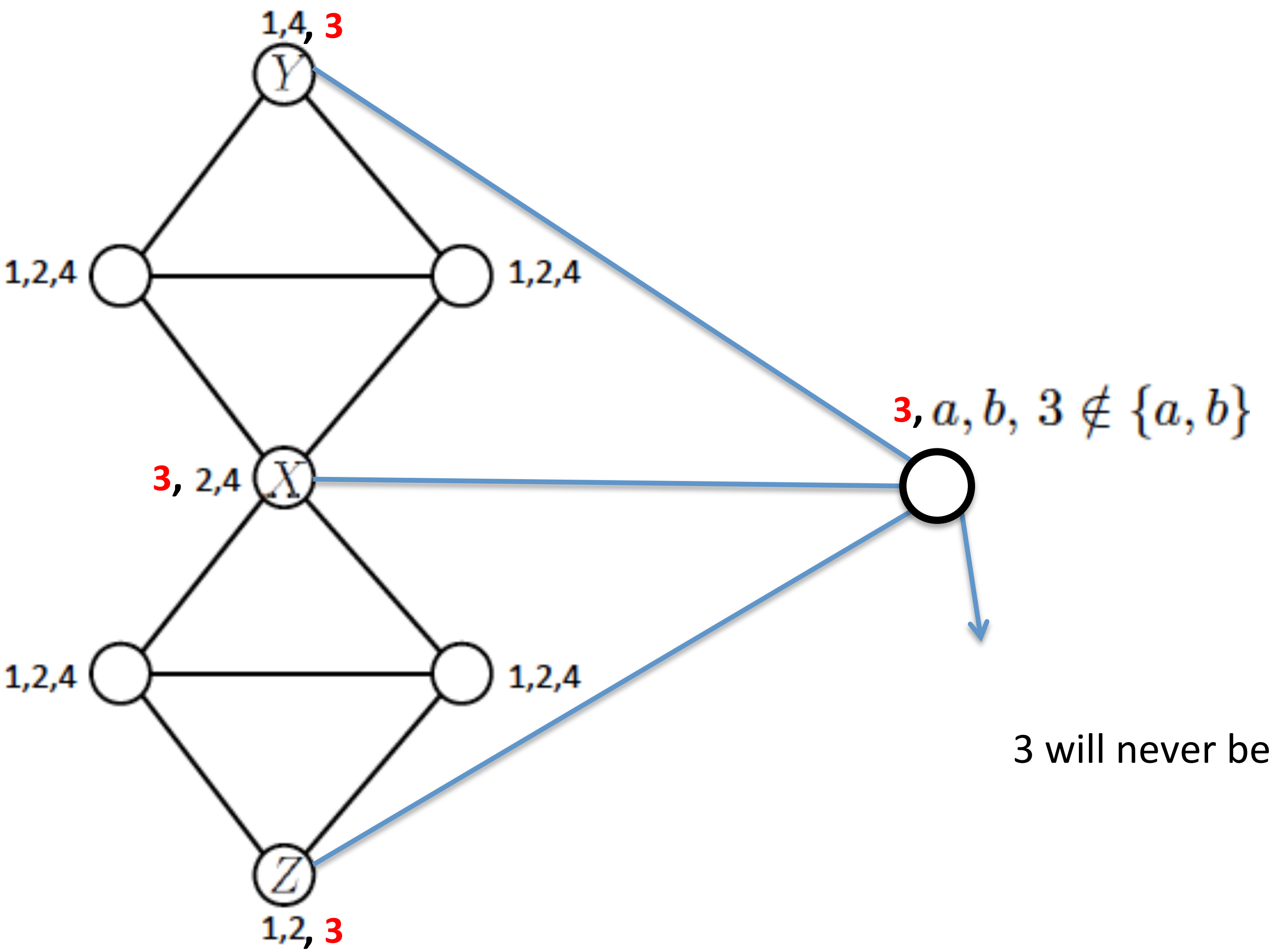
No list coloring



3 will never be used

Complexity with few colors: [3,*k*]-choosability

No list coloring

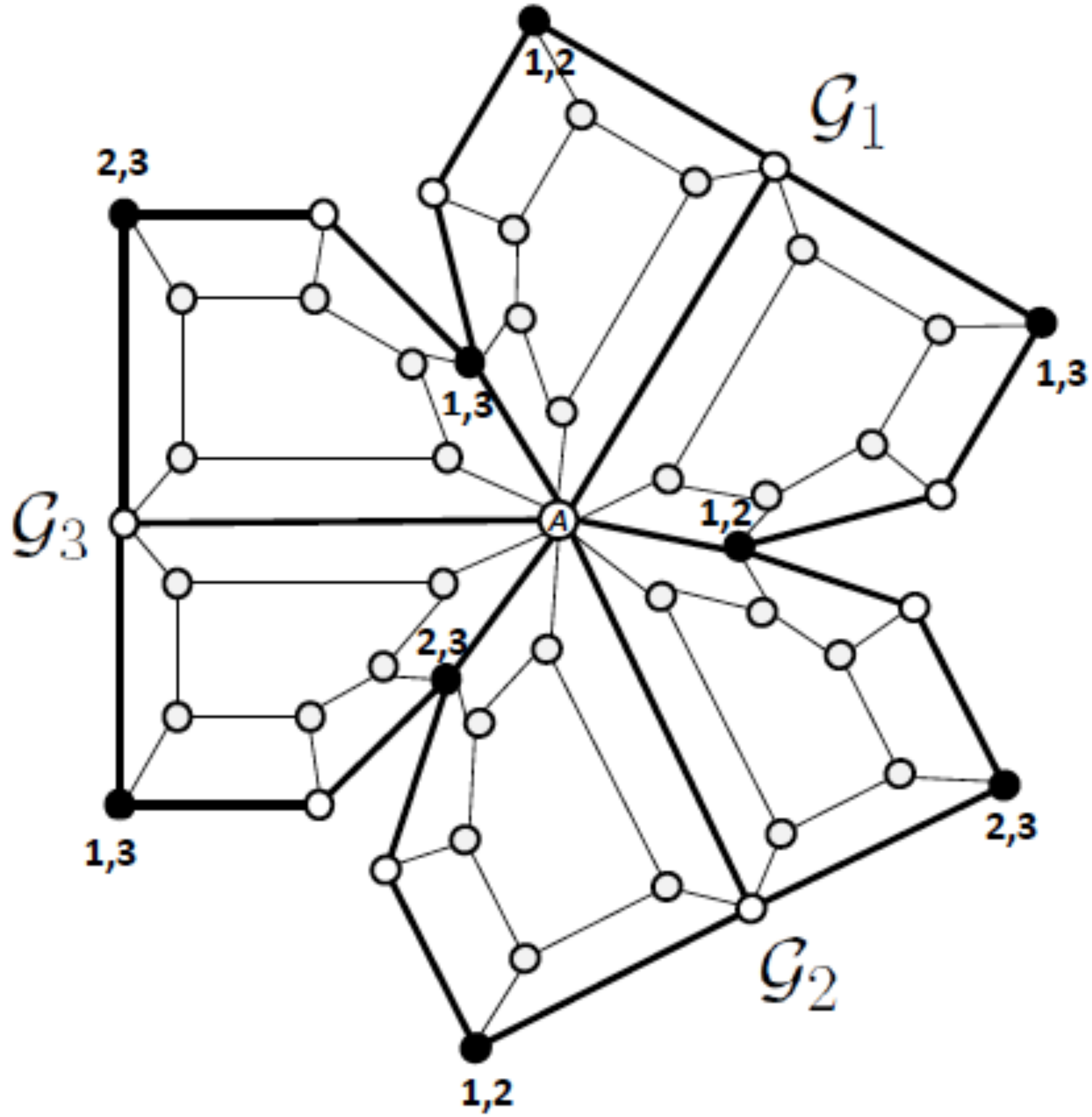


3 will never be used

> $[3,4]$-*CH is* Π_2^p-*complete in 3-colorable planar graphs of degree 7,* C_{2i+1}-*free for* $i \geq 4$ *and such that every odd cycle* C_5 *or* C_7 *has 2 short chords.*

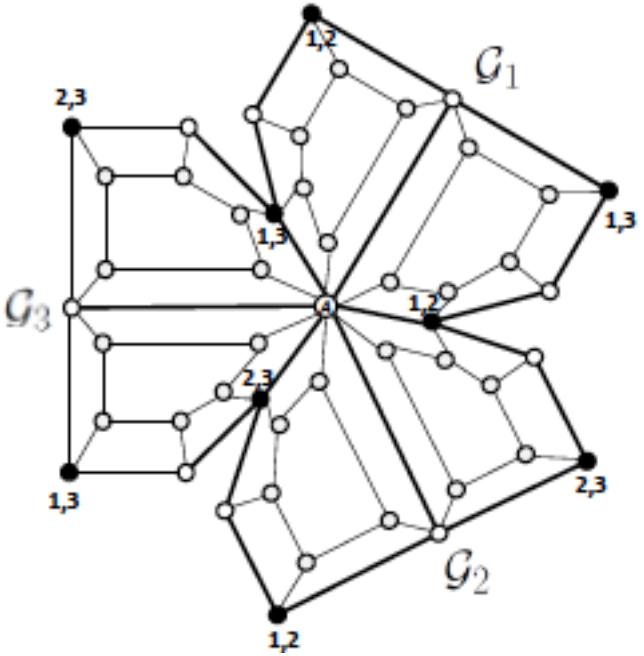
Complexity with few colors: triangle-free planar graphs

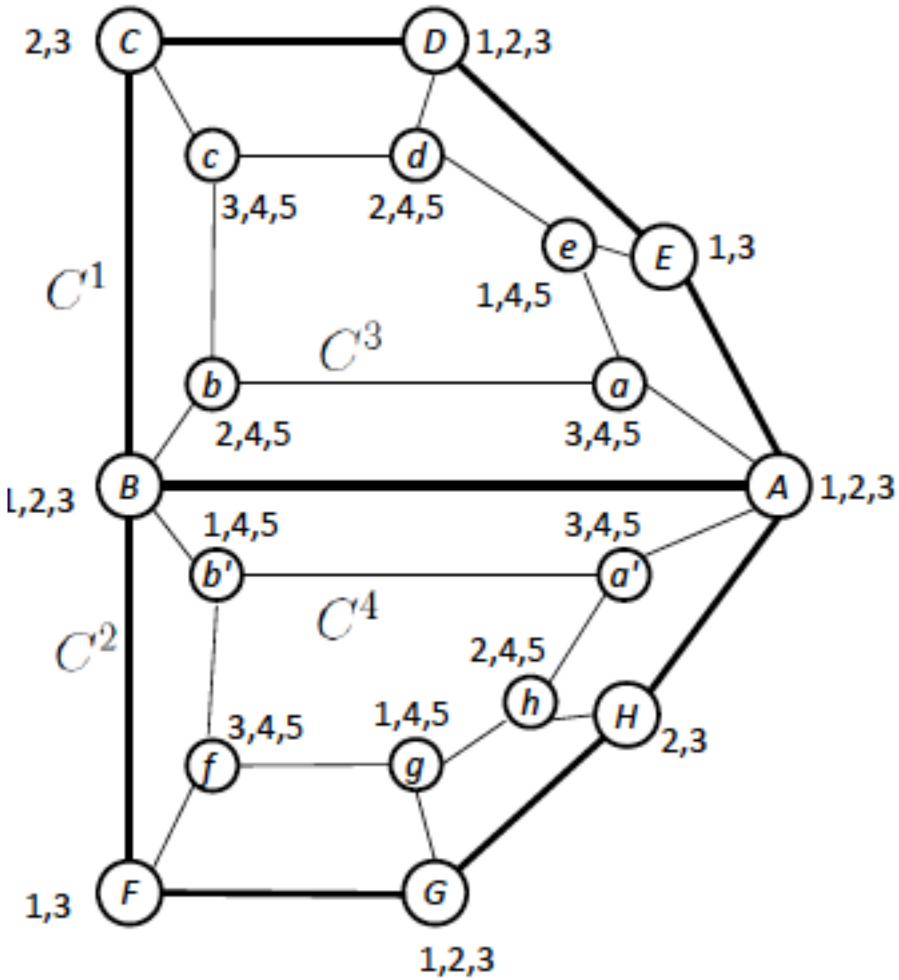
No list coloring



Complexity with few colors: triangle-free planar graphs

Color 3 forbidden for *A*



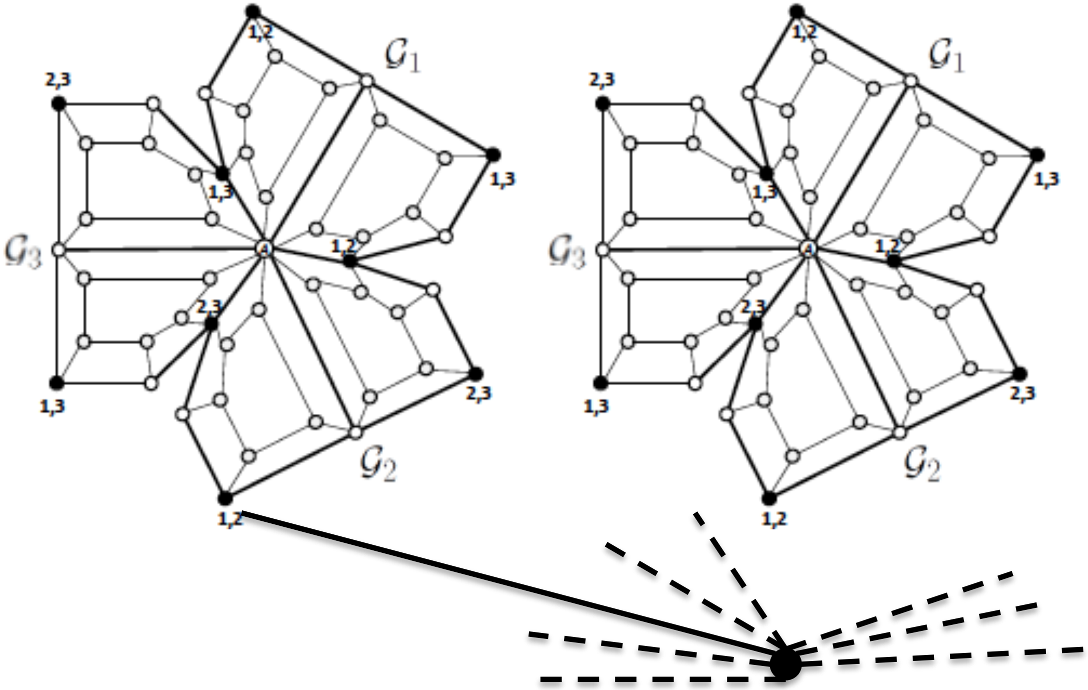


$[3,5]$-CH is Π_2^p-complete in triangle-free planar graphs of maximum degree 13.

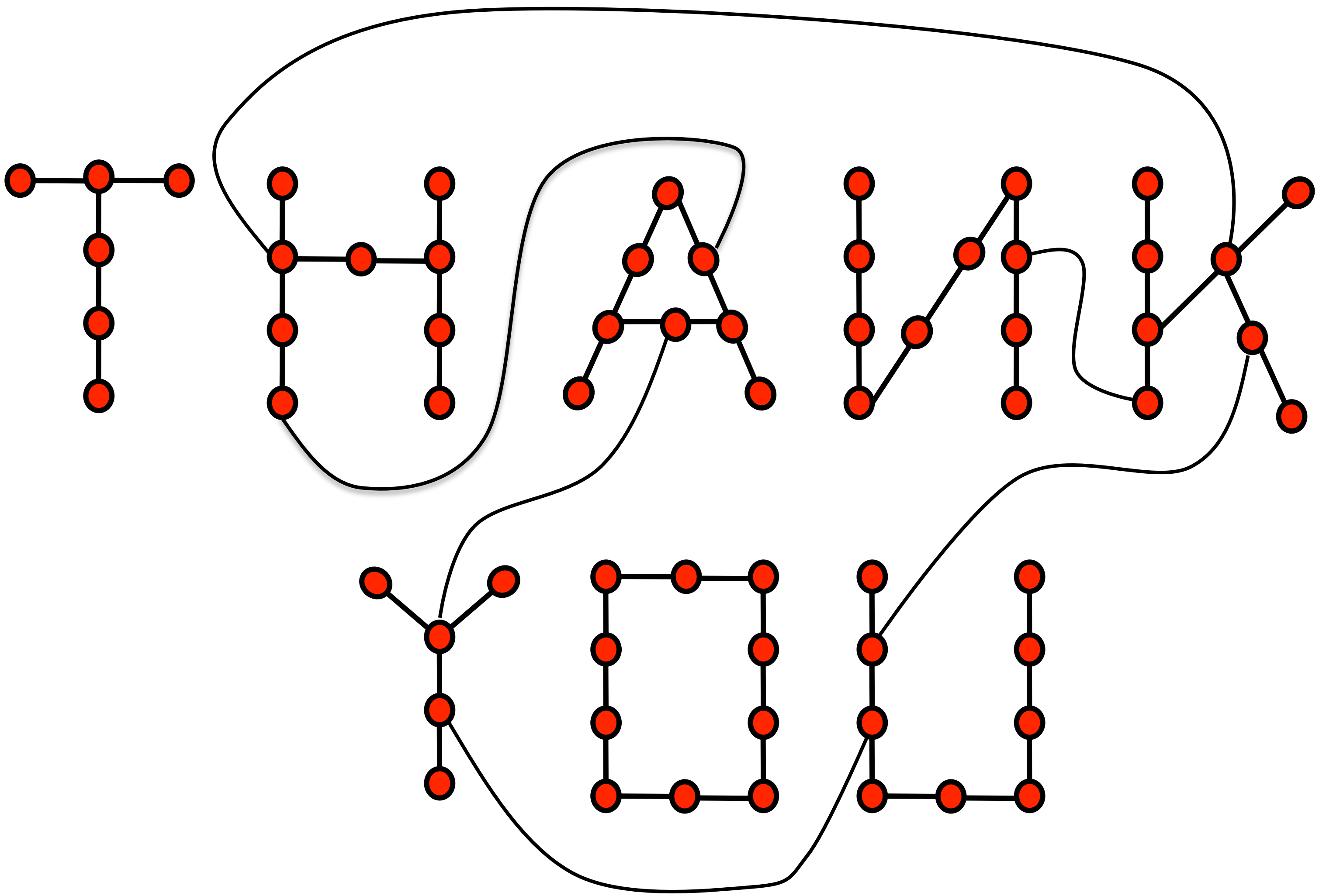
Last remark

This allows to construct a not [3,5]-choosable triangle-free planar graph with 148 vertices.

Gutner gave a triangle-free graph with 164 vertices not 3-choosable.



An exercise: is this graph 2-choosable?



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