Online Colouring Problems in Overlap Graphs and their Complements

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(the art of BBQ down under)

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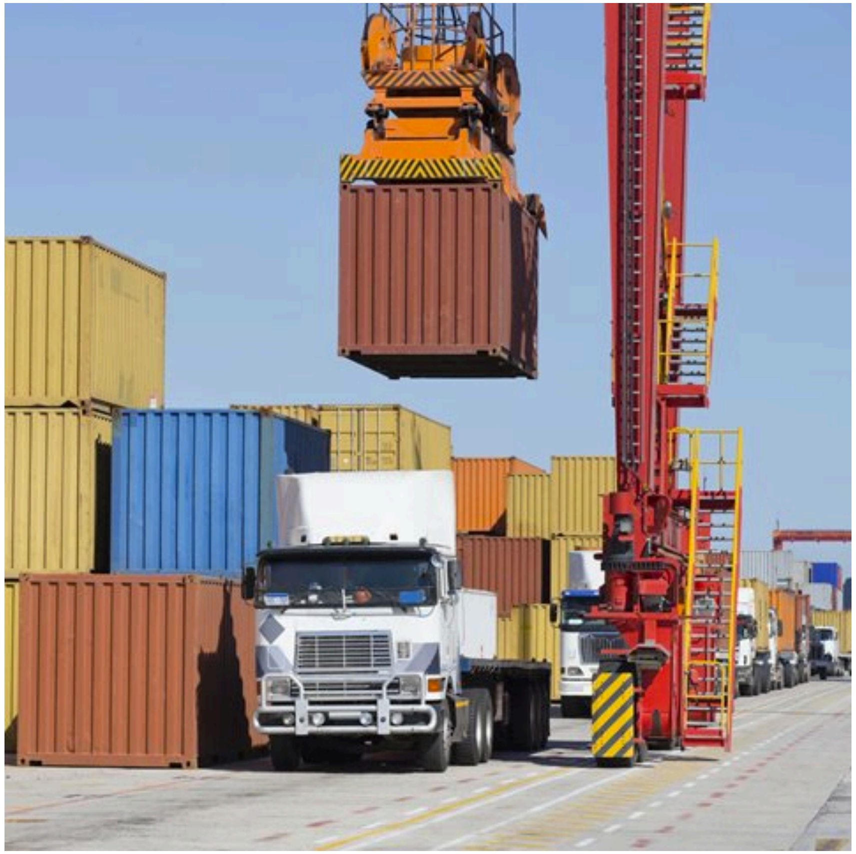
Motivation 1: stacking problem

Stacking problem



Stacking problem

Stacking problem

18/09

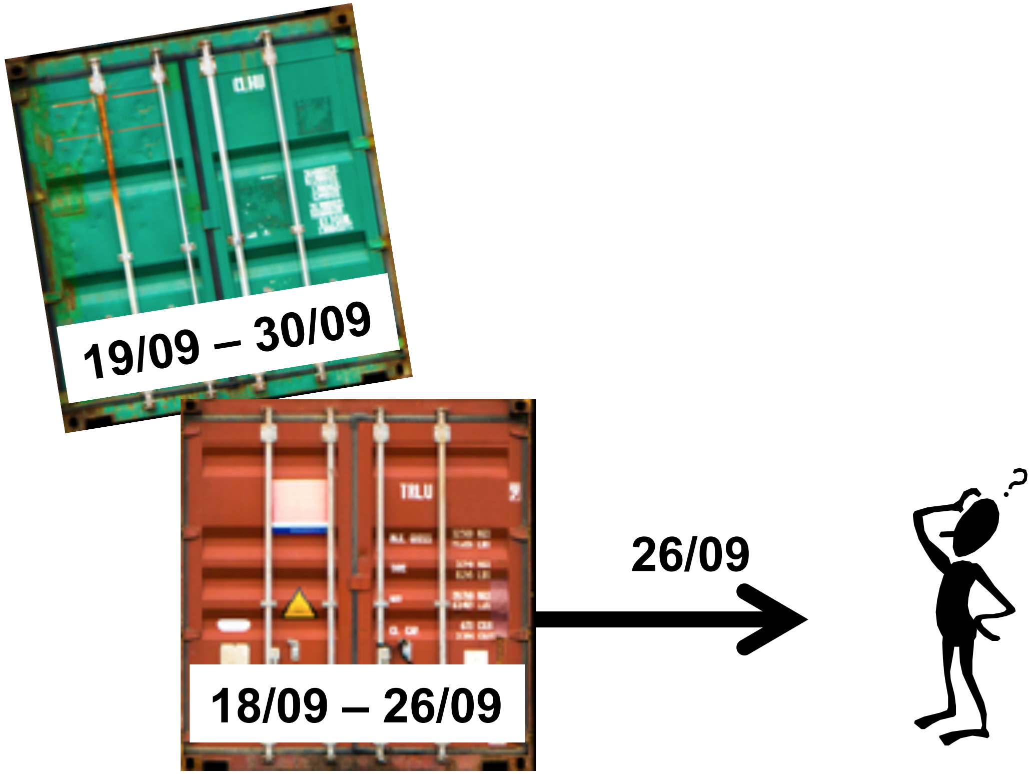
18/09 – 26/09

Stacking problem

19/09

Stacking problem

26/09



Stacking problem

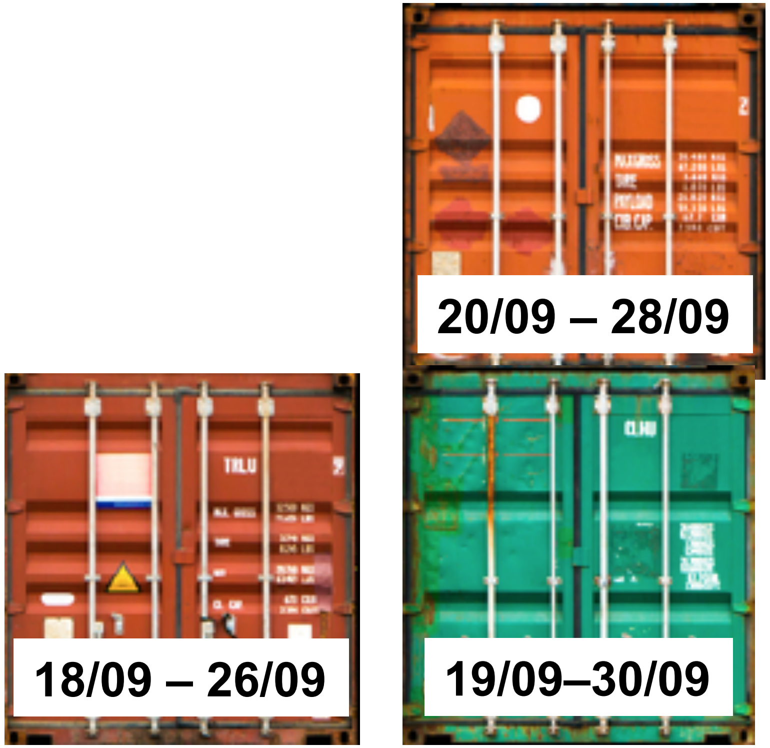
19/09

18/09 – 26/09

19/09–30/09

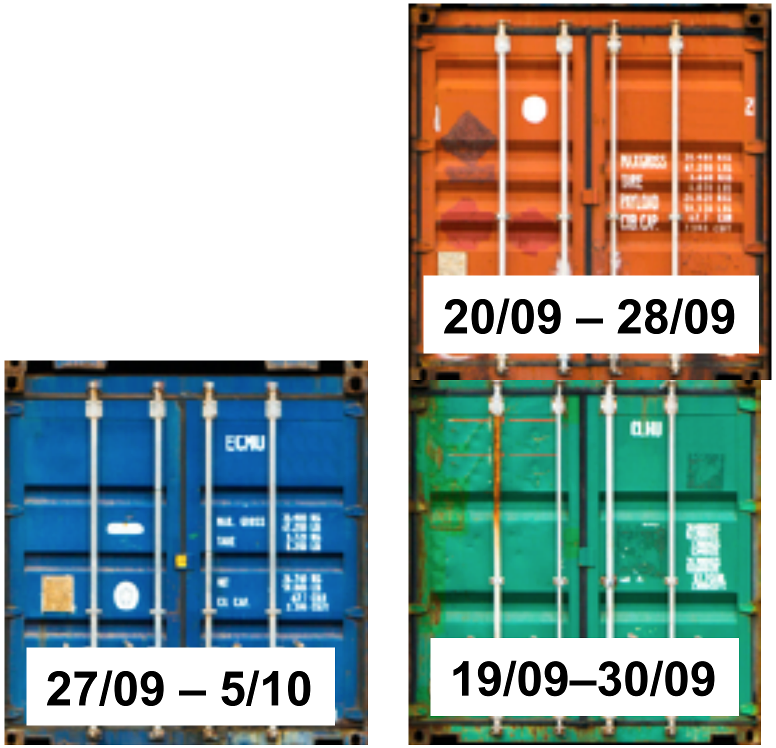
Stacking problem

20/09

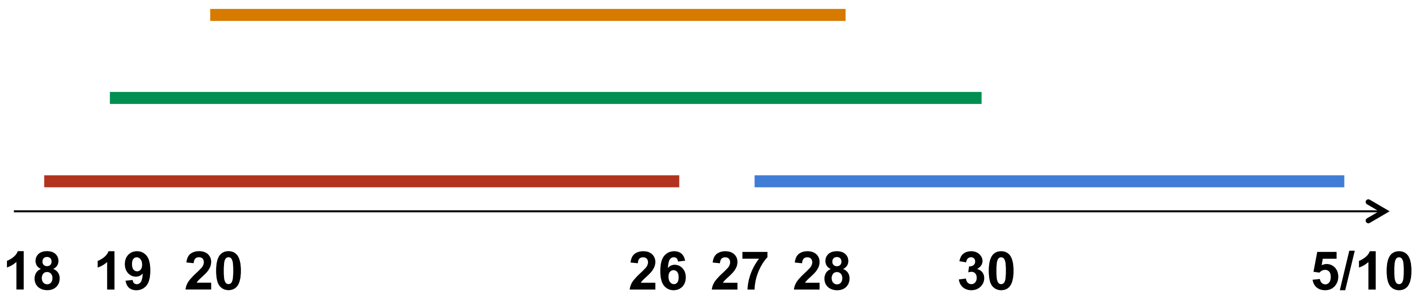


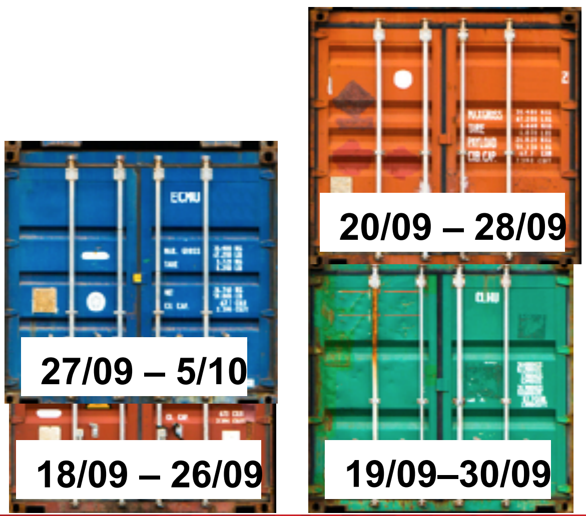
Stacking problem

27/09



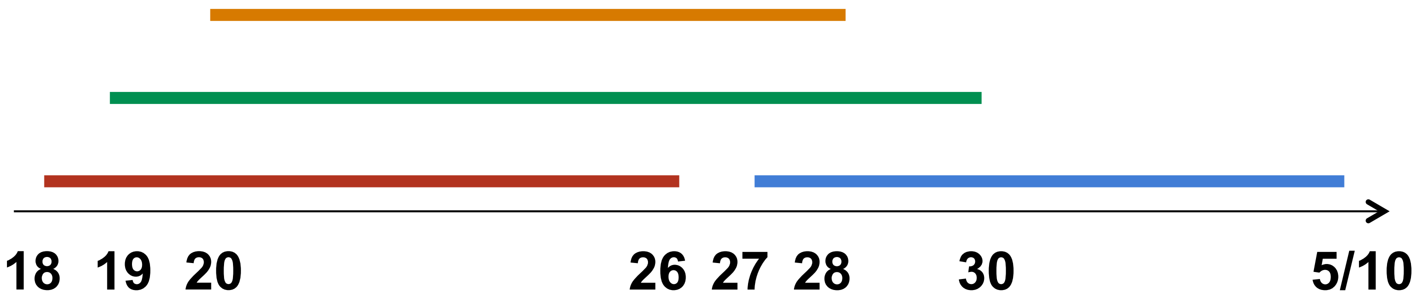
Related graph problem

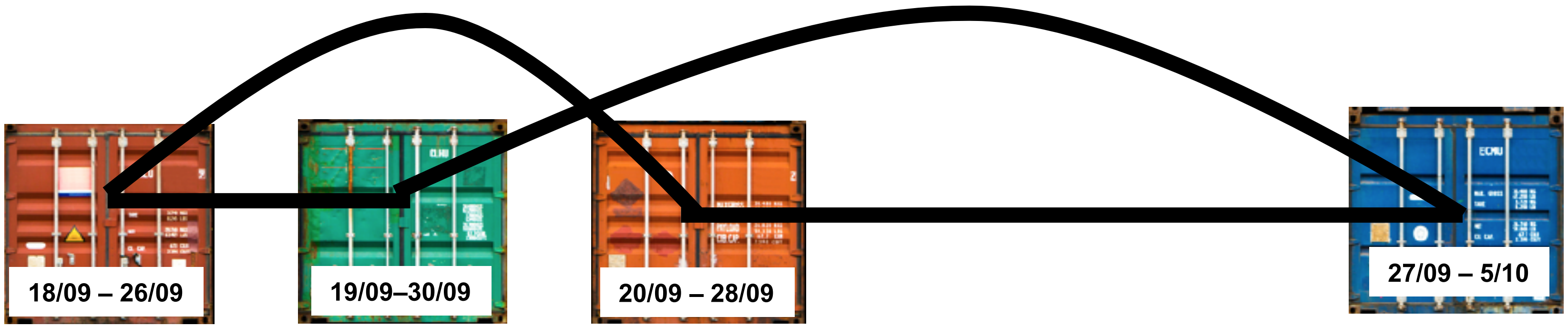




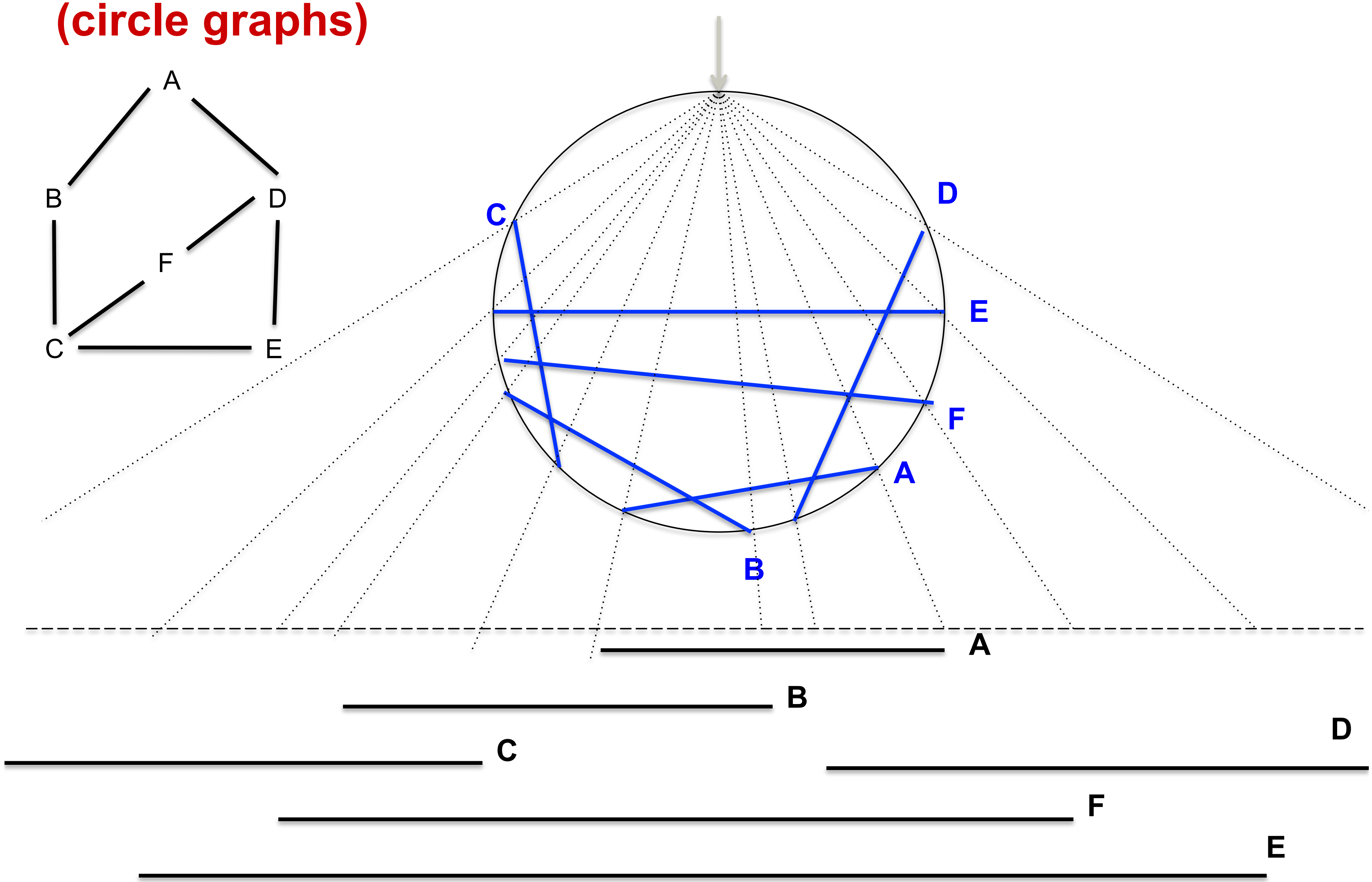
Related graph problem: incompatibility graph

Overlap graph



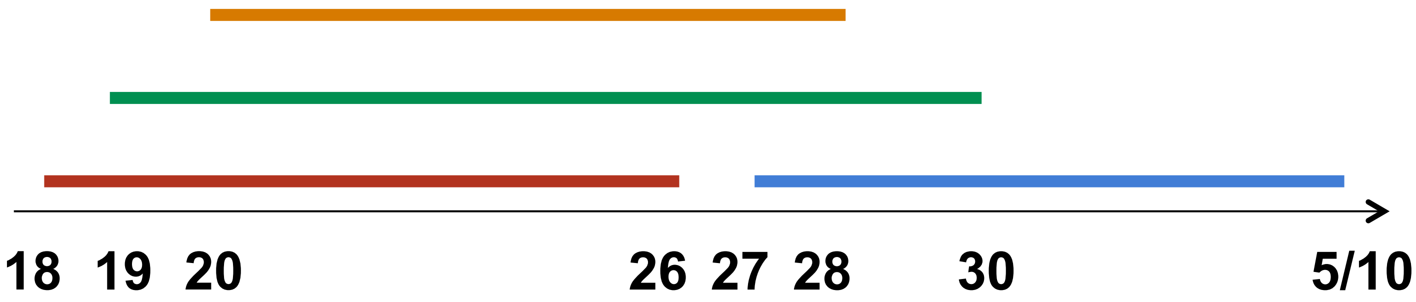


Overlap graphs as intersection graphs of chords (circle graphs)



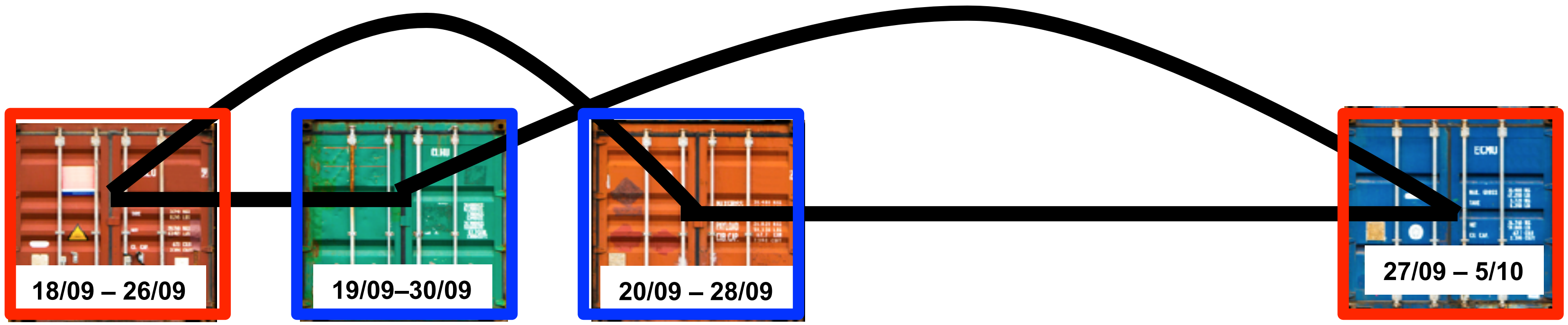
Related graph problem: min colouring

Overlap graph



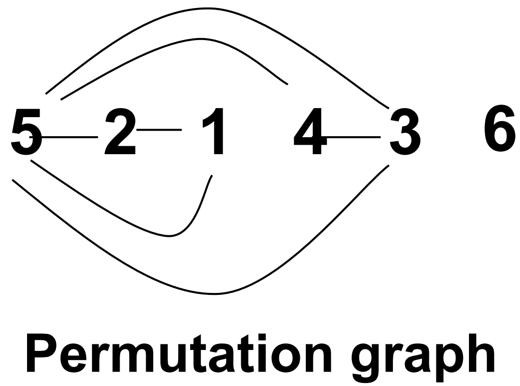
Stack 1

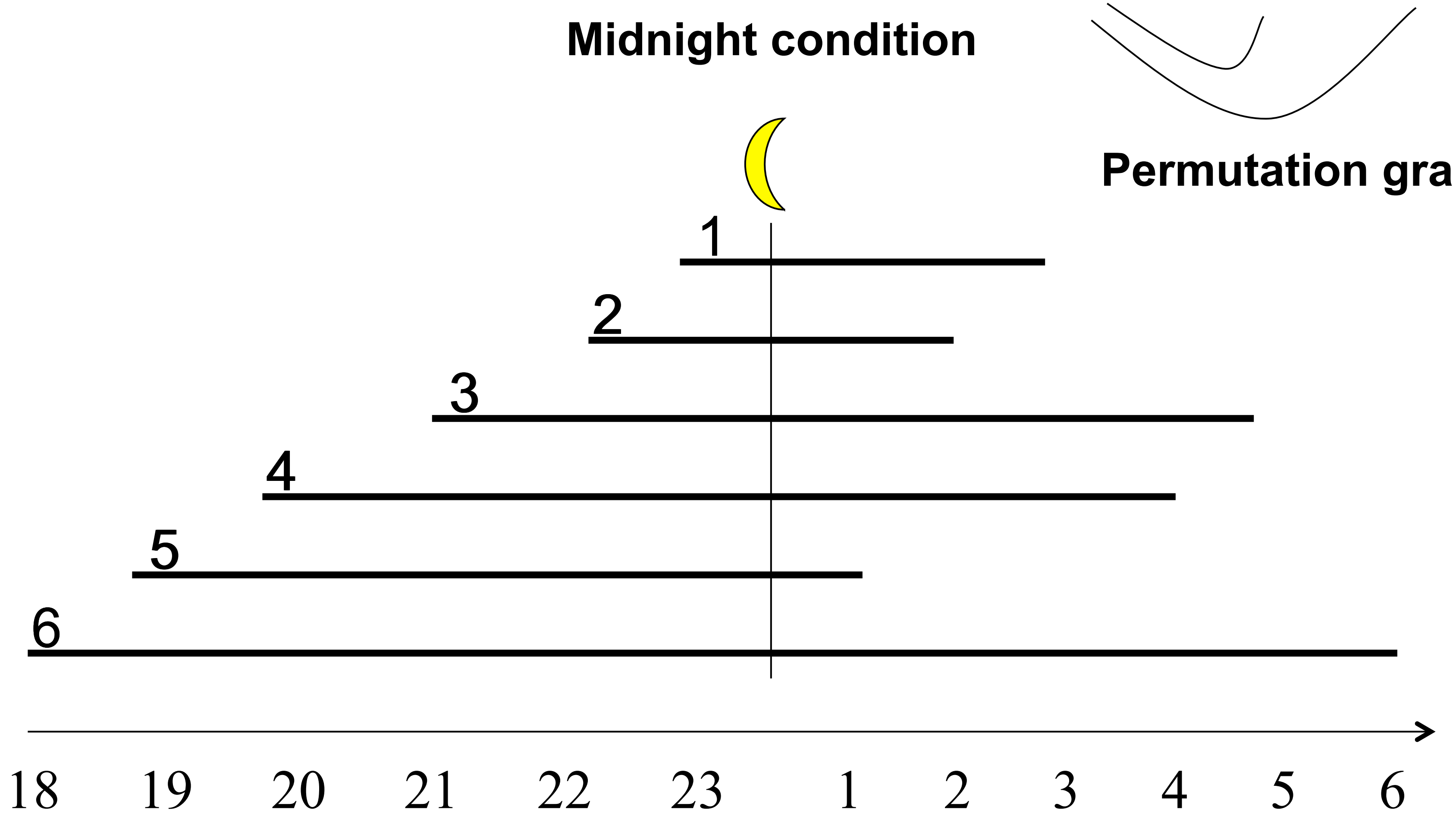
Stack 2



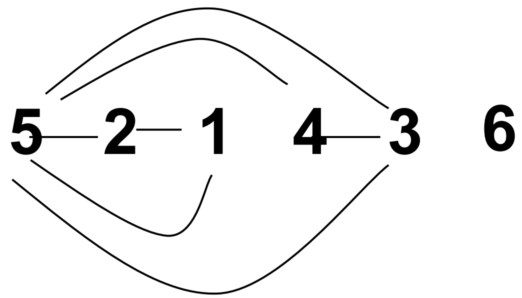
Motivation 2: track assignment

A particular case

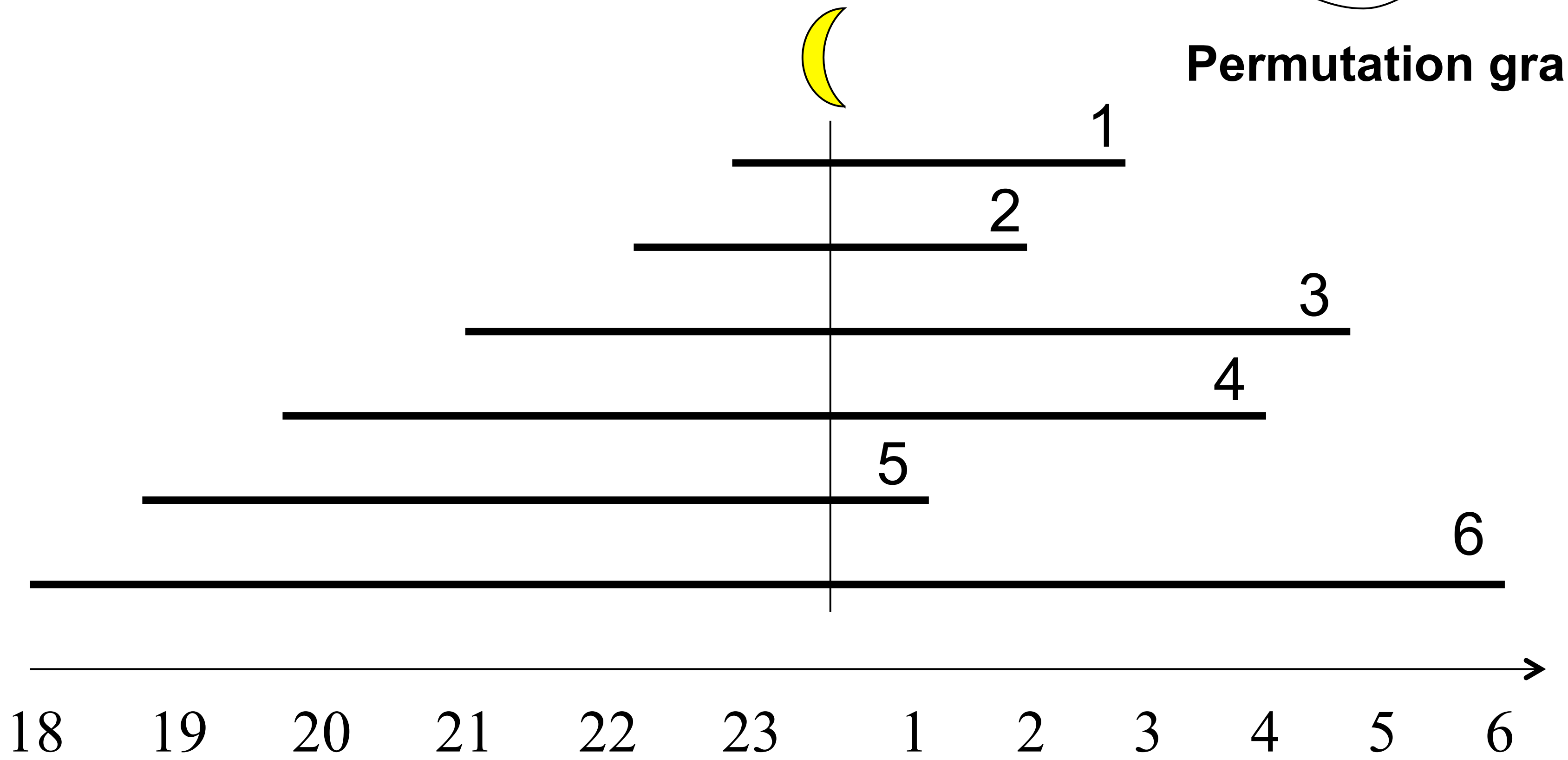




A particular case



Midnight condition



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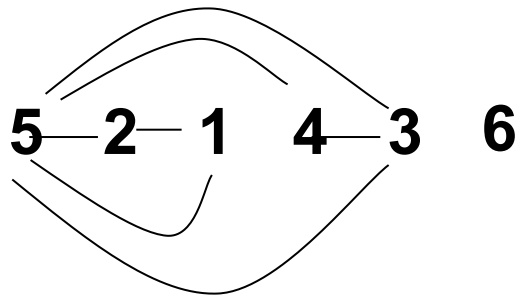
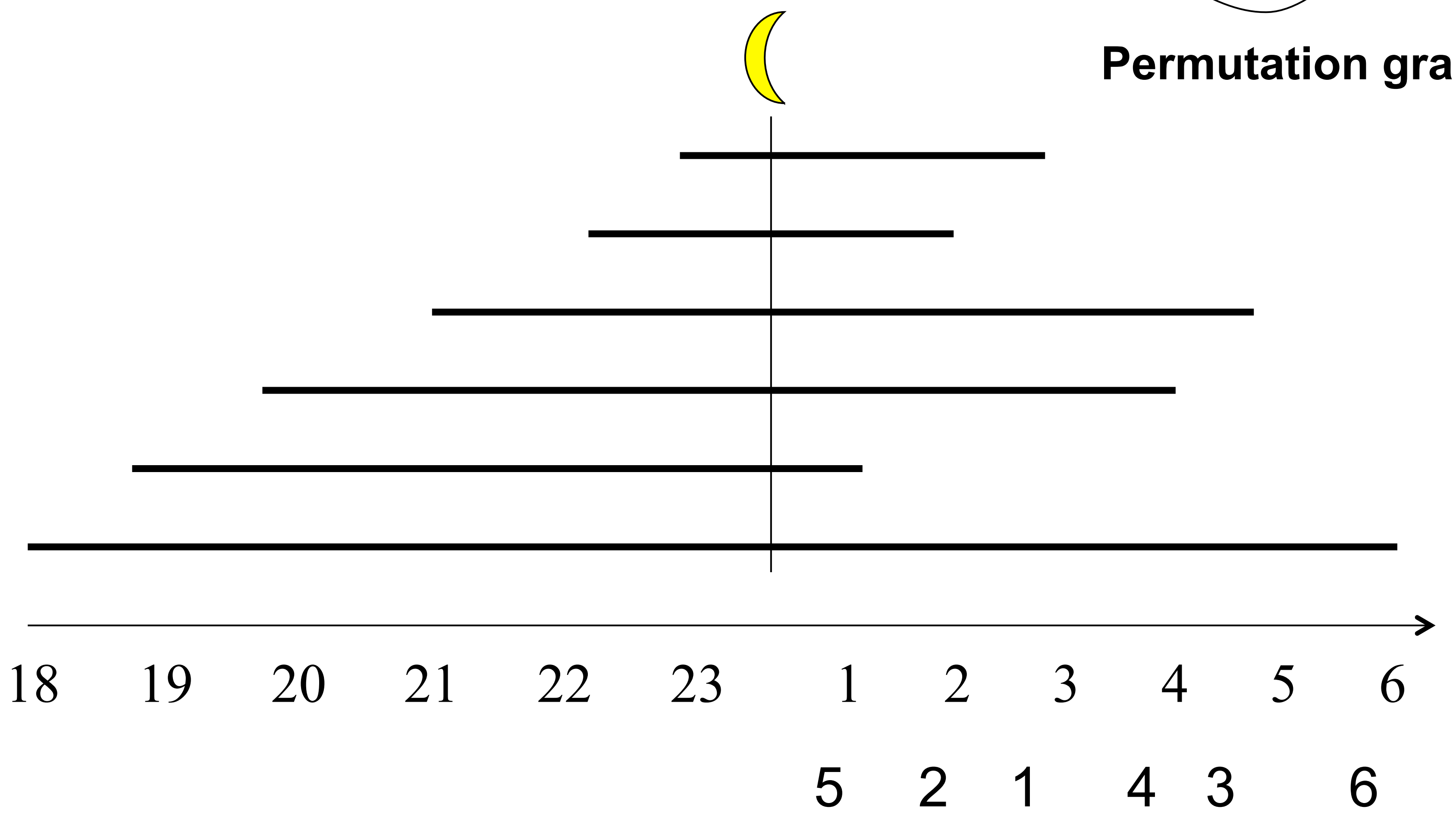




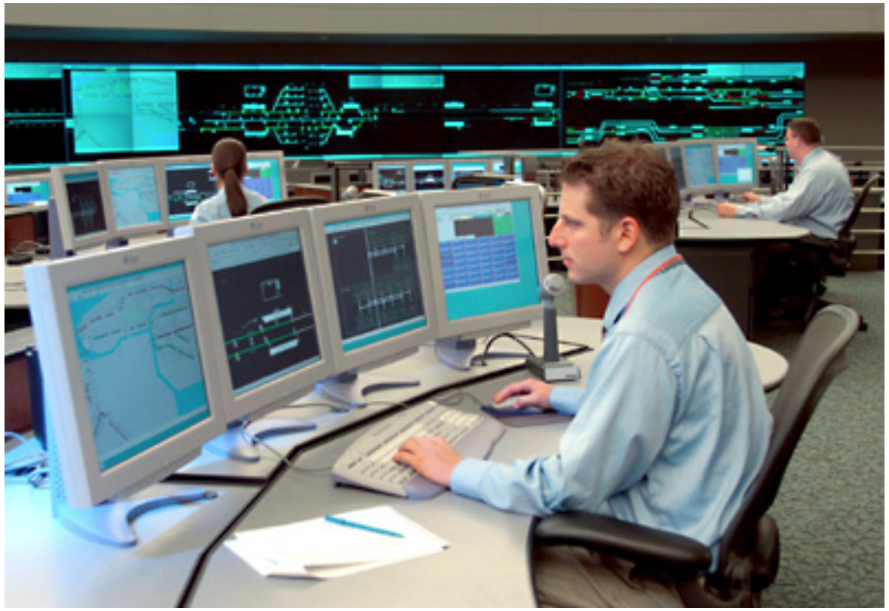
Stacking problem / track assignment

- Assign each item on a stack
- Allays put / remove items on the top a stack (Last in first out)
- The related incompatibility graph is an overlap / permutation graph
- Minimising the number of stacks

⇕

Minimum colouring overlap / permutation graphs

On-line stacking / track assignment



- Nothing is known about the future (items are known when they arrive)
- Departure time known at arrival
- Assign stack at arrival, never on top of an item leaving earlier

↕

On-line colouring overlap / permutation graphs

- Eventually additional constraints: e.g. fixed capacity for each stack
- Graph defines by intervals revealed from left to right

H-colouring in overlap graphs

R interval system
G_R: overlap graph associated with R
$\widetilde{G}_R$: interval graph associated with R
Load of R: clique number of $\widetilde{G}_R$
$\widetilde{\alpha}(R)$: independence number of $\widetilde{G}_R$
L: largest interval length, ℓ: smallest interval length

H-colouring in overlap graphs

Minimum H-colouring problem in an interval system

- H: hereditary property
- Instance: interval system R
- Solution: proper colouring of G_R, where each colour class satisfies H
- Objective: minimise the number of colours

H-colouring in overlap graphs

Minimum H-colouring problem in an interval system

- H: hereditary property
- Instance: interval system R
- Solution: proper colouring of G_R, where each colour class satisfies H
- Objective: minimise the number of colours

Problem	Property H satisfied by each colour class $R' \subset R$
colouring	$G_{R'}$ is an independent set (no overlap) .
b-bounded colouring	$G_{R'}$ is an independent set and $\lvert R' \rvert \leq b$.
clique covering	$G_{R'}$ is a clique (every two intervals overlap).
b-bounded clique covering	$G_{R'}$ is a clique and $\lvert R' \rvert \leq b$.
b-bounded load colouring	$G_{R'}$ is an independent set and R' is of load at most b.

Table 1: Examples of H-colouring problems in an overlap graph defined by R.

Problem	**Approximation**	**Competitive ratio (left to right)**
colouring	1	
clique covering		
b-bounded colouring	$2 - \frac{1}{\min(b, \chi_b)}$ (D,di Stefano,Leroy-B 2012)	
b-bounded clique covering		

Table 2: Known results in permutation graphs

Problem	**Approximation**	**Competitive ratio (left to right)**
colouring	$\log n$ (Cerný 2007)	$O(\frac{L}{\ell})$ (D,dS,L-B 2012) $O(\frac{\log^2 L}{\log\log L})$ $(\ell = 1)$
clique covering	$2(1 + \log \widetilde{\alpha}(R))$ Shahrokhi 2015	?
b-bounded colouring	?	?
b-bounded clique covering	?	?

Table 3: Best known results in overlap graphs

Problem	Approximation	Competitive ratio (left to right)
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b-bounded colouring	?	?
b-bounded clique covering	?	?

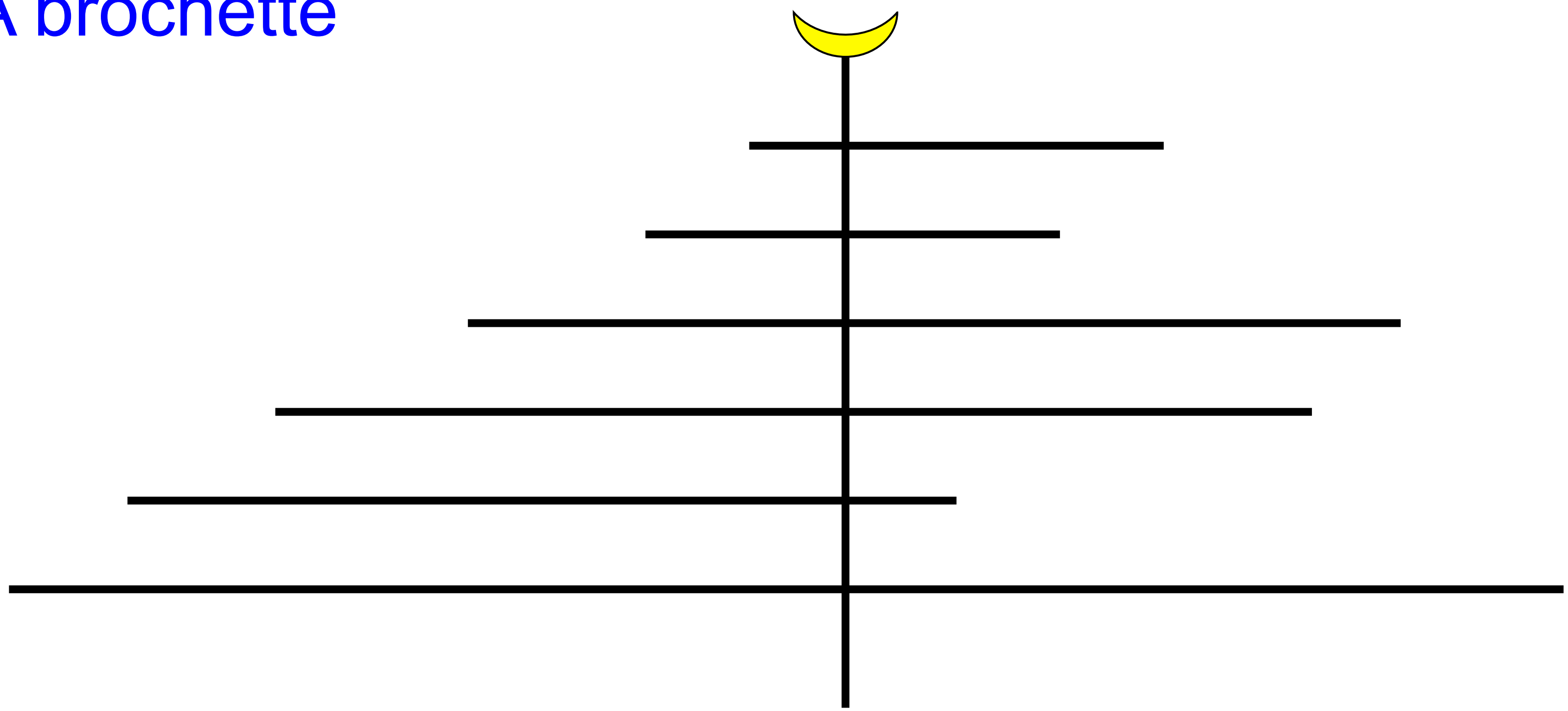
Table 3: Best known results in overlap graphs

Problem	Approximation	Competitive ratio (left to right)
colouring	$\log \widetilde{\alpha}(R) + c$	$2\lfloor \log_2(\frac{L}{\ell}) \rfloor + 7$
clique covering		
b-bounded colouring	$2(\log \widetilde{\alpha}(R) + c)$	$(2\lfloor \log_2(\frac{L}{\ell}) \rfloor + 7)(2 - \frac{1}{\sigma})$ $\sigma = \min\{b, \beta(R)\}$
b-bounded clique covering		
b-bounded load colouring		

Table 4: Our results

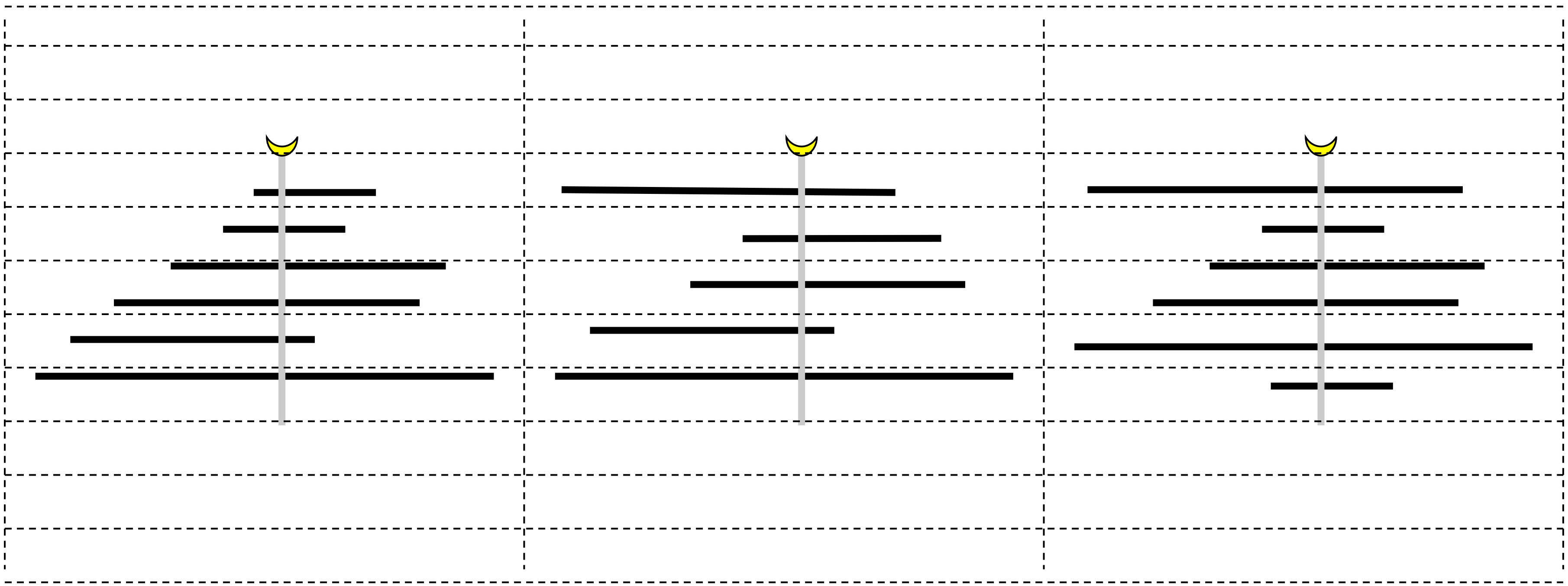
Main ingredient: the BBQ strategy

A brochette



Main ingredient: the BBQ strategy

A BBQ arrangement



Main ingredient: the BBQ strategy

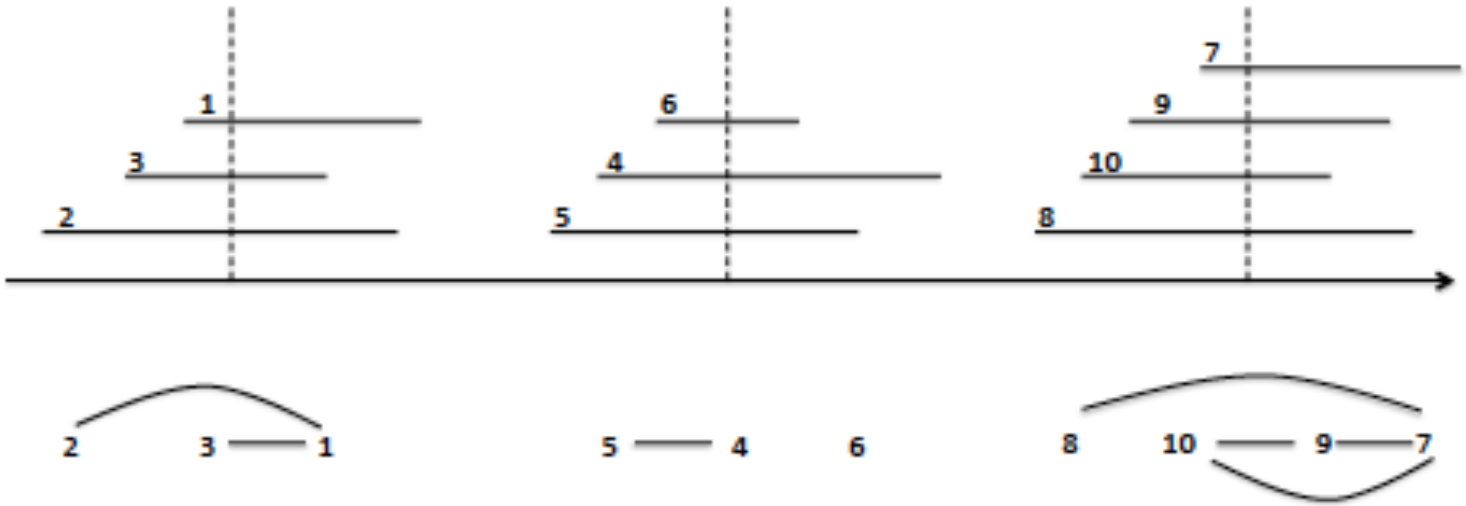
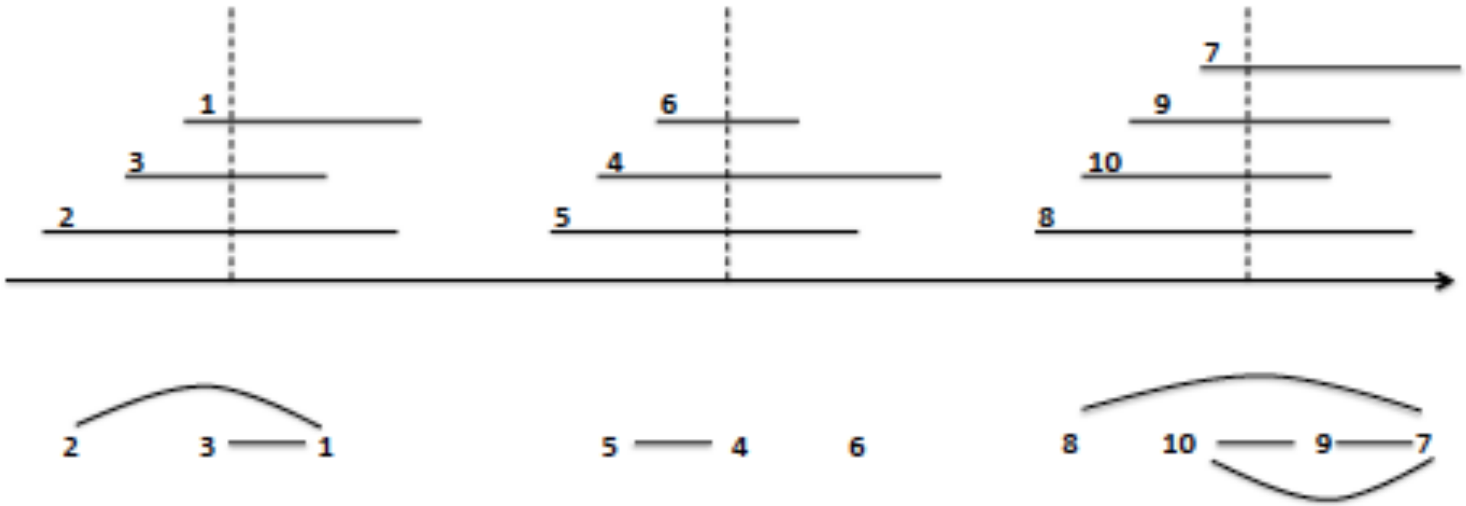


Fig. 1: BBQ arrangement B of intervals (above) and the related permutation graph (below) associated with the permutation $\pi_B = (2, 3, 1, 5, 4, 6, 8, 10, 9, 7)$ or equivalently, for instance, the list $Q_B = (0.5, 1.5, 0, 2.5, 2.1, 3.5, 4.5, 5.5, 5.3, 4.1)$.

Main ingredient: the BBQ strategy



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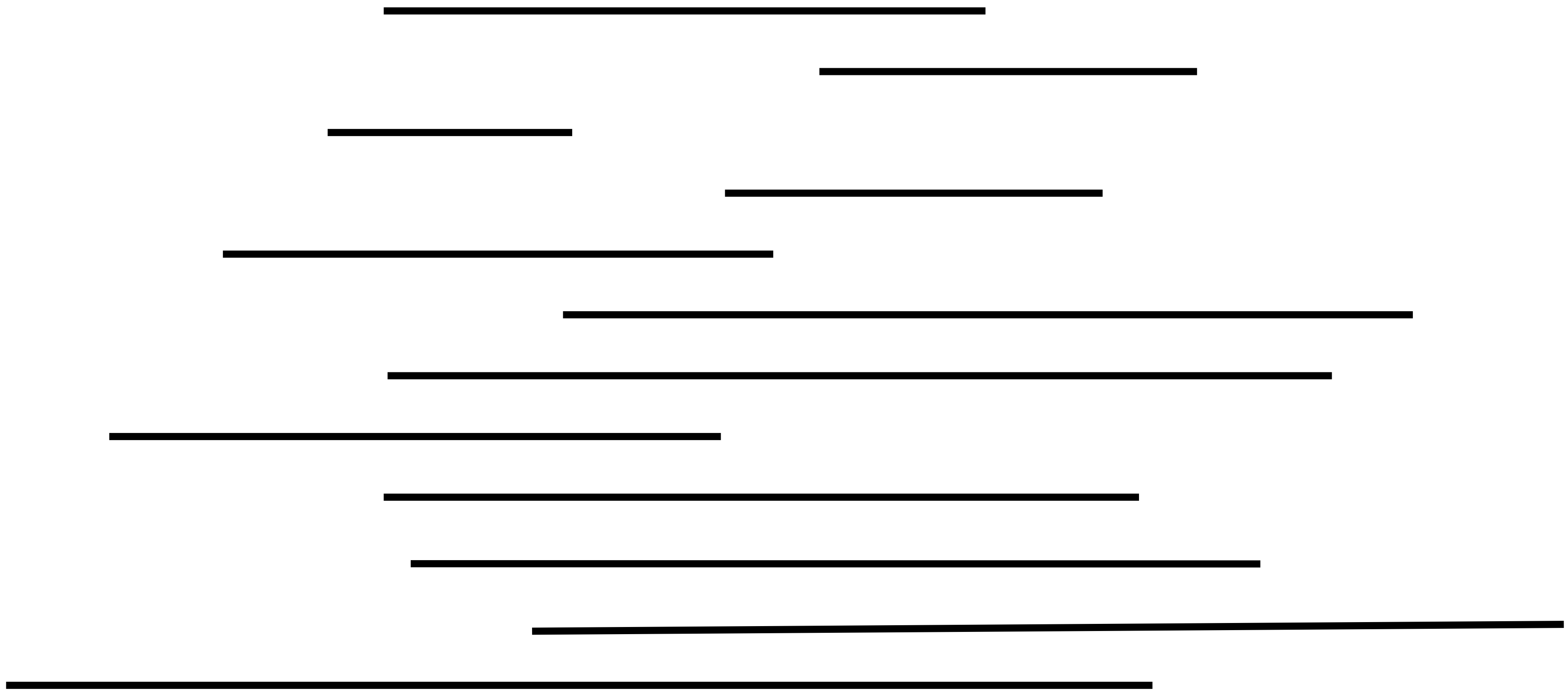
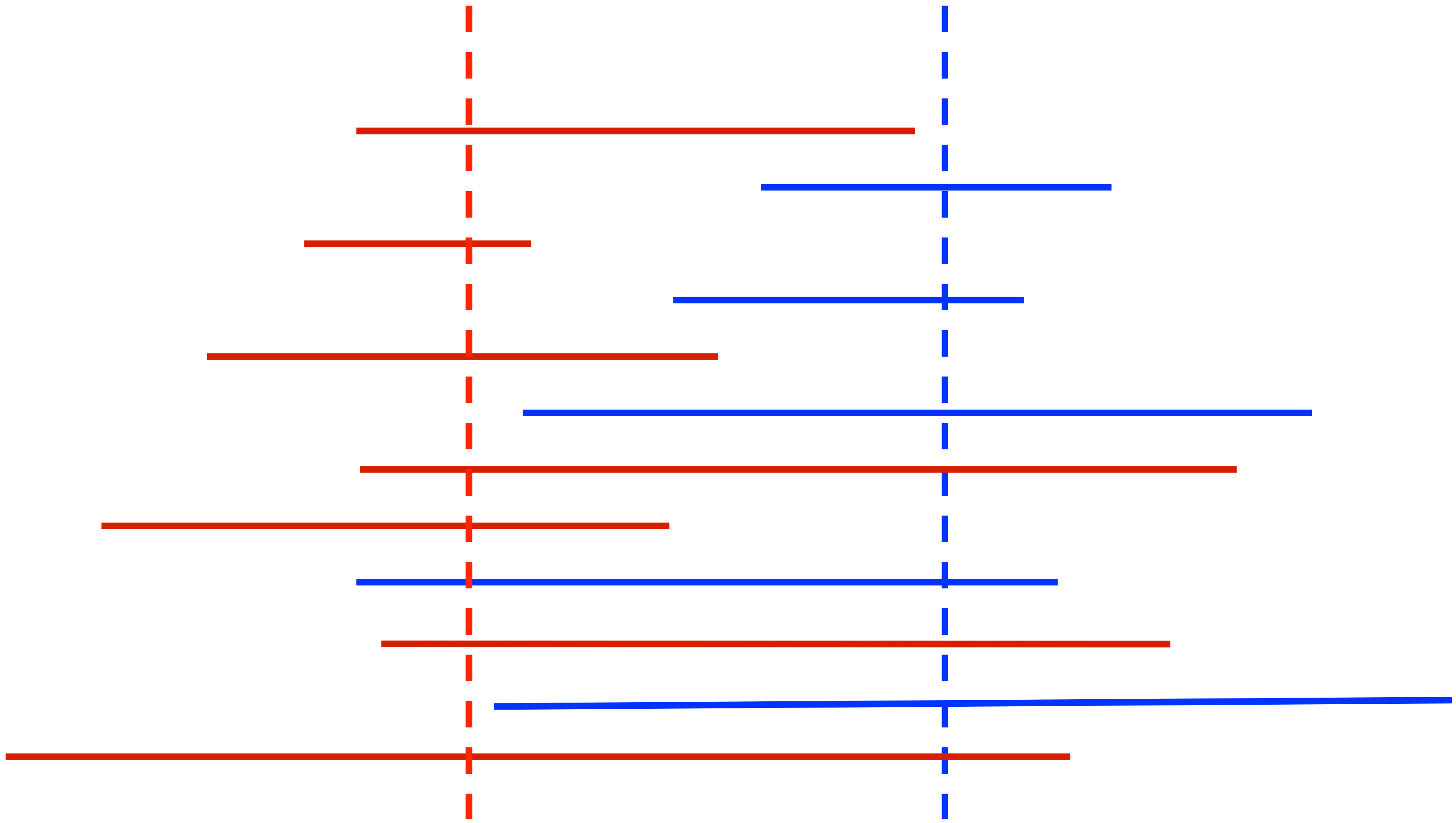
Proposition. *BBQ arrangements always induce permutation graphs.*

The BBQ strategy: main idea



The BBQ strategy: main idea

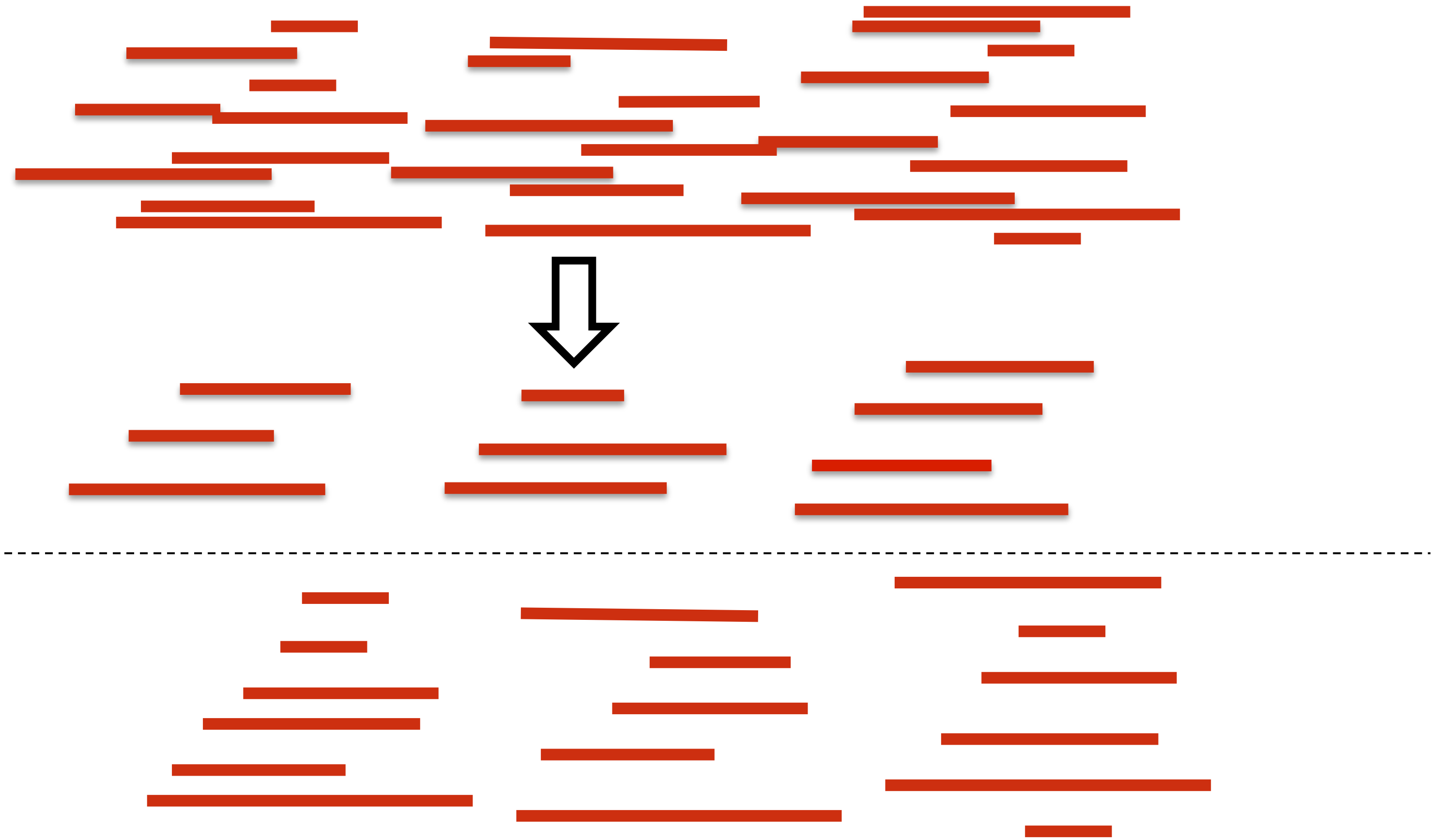


BBQ strategy: slice the steak into BBQ arrangements

Proposition. *(Partition into permutation graphs)*
Algorithm 1 described below is a polynomial online algorithm that partitions an overlap graph defined by an interval system presented from left to right into at most $\left(2\left\lfloor \log_2(\frac{L}{\ell})\right\rfloor + 7\right)$ *permutation graphs defined by a BBQ arrangement.*
Moreover, if L *is known in advance, a simplified version of the algorithm guarantees a number of at most* $\left(\left\lfloor \log_2(\frac{L}{\ell})\right\rfloor + 4\right)$ *permutation graphs in the decomposition.*

The BBQ strategy:

- Partition any instance into a minimum number of BBQ arrangements
- Solve independently each arrangement using a specific colour set



- Do it on-line

Decomposition strategy: (assume first L and ℓ are known)

define $k_L = \lceil \log_2 \left(\frac{L}{\lambda}\right) \rceil + 1$ and $k_\ell = \lfloor \log_2 \left(\frac{\lambda}{\ell}\right) \rfloor + 1$

For $-k_L \leq i \leq k_\ell$ we define the set S_i as follows: $S_i = \{k2^{-i}\lambda, k \in \mathbb{N}^*\}$
$S_i = \emptyset$ for $i < -k_L$ or $i > k_\ell$

$$S_{-k_L} \subset \ldots \subset S_0 \subset \ldots \subset S_{k_\ell}$$

Let us then define $\mathcal{P}_i$ as the set of intervals that intersect S_i but do not intersect S_{i-1}

$$\forall x_1, x_2 \in S_i, I_1, I_2 \in \mathcal{P}_i : [x_1 \in I_1 \wedge x_2 \in I_2 \wedge x_1 \neq x_2] \Rightarrow I_1 \cap I_2 = \emptyset$$

There are $k_L + k_\ell + 1 \leq \lfloor \log_2 \left(\frac{L}{\ell}\right) \rfloor + 4$ permutation graphs

Decomposition strategy

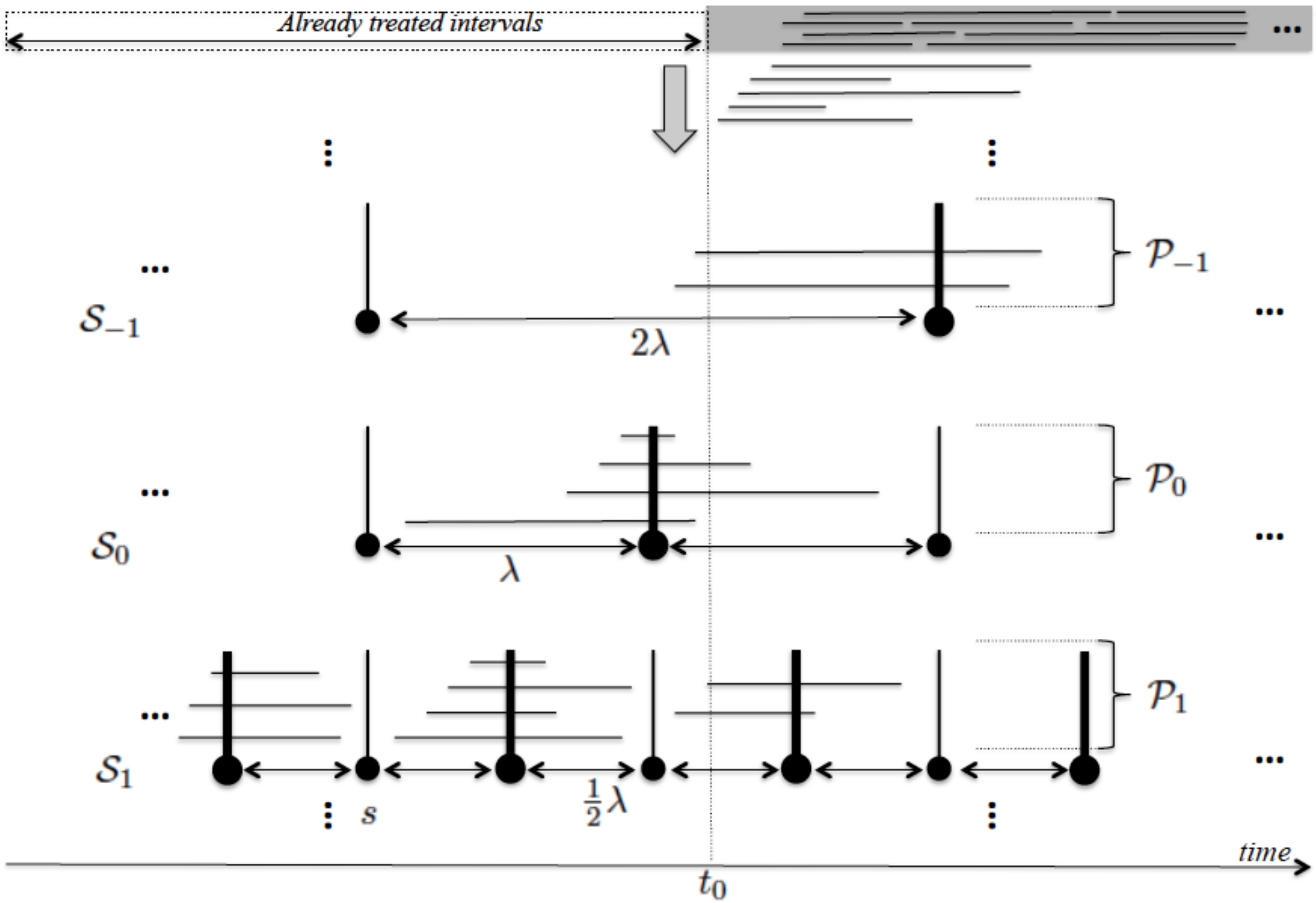


Fig. 2: The steak is sliced into intervals to be dropped on several layers $\mathcal{P}_i$ of brochettes. Active skews are thick, in particular $(s,1),(s,0)$ and $(s,-1)$ are inactive if $s \in \mathcal{S}_{-2}$. If the distance between the brochettes in the top layer is at least $2L$ then each $\mathcal{P}_i$ is a BBQ arrangement.

On-line algorithm: (unknown L and ℓ)

Algorithm 1 Partition into Permutation Graphs

Require: An overlap graph $G = (\mathcal{I}, E)$ presented online from left to right (the maximum length L is not known in advance).

Ensure: A partition of $\mathcal{I}$, $(\mathcal{P}_1, \ldots, \mathcal{P}_p)$ such that $G[\mathcal{P}_i]$ is a permutation graph.

$\mathcal{T}_i \leftarrow \emptyset, i \in \mathbb{Z}$

$\mathcal{R}_i \leftarrow \emptyset, i \in \mathbb{Z}$

$\ell \leftarrow \lambda, L \leftarrow \lambda$

When the first interval I is presented, set λ as its length and add I to $\mathcal{T}_0$

for each new interval $I = [a_I, b_I]$ **do**

  $\ell \leftarrow \min\{b_I - a_I, \ell\}, L \leftarrow \max\{b_I - a_I, L\}$

  $k_L \leftarrow \left\lceil \log_2\left(\frac{L}{\lambda}\right)\right\rceil + 1, k_\ell \leftarrow \left\lfloor \log_2\left(\frac{\lambda}{\ell}\right)\right\rfloor + 1$

  $j \leftarrow \min\left\{i \in \{-k_L, \ldots, k_\ell\} : \left\lceil \frac{2^i a_I}{\lambda}\right\rceil \leq \left\lfloor \frac{2^i b_I}{\lambda}\right\rfloor\right\}$

  if $j = -k_L$ **then**

    Add I to $\mathcal{R}_j$

  else

    Add I to $\mathcal{T}_j$

  end if

end for

The final partition is $(\mathcal{T}_{-k_L+1}, \ldots, \mathcal{T}_{k_\ell}) \cup (\mathcal{R}_{-k_L}, \ldots, \mathcal{R}_0)$

On-line reduction

Theorem. *(Online reduction)*
For any online algorithm for a H-Colouring problem guaranteeing a competitive ratio of ρ on permutation graphs defined by a BBQ arrangement presented from left to right, there is an online algorithm for the same problem on overlap graphs defined by an interval system presented from left to right guaranteeing the competitive ratio $\left(2\left\lfloor \log_2(\frac{L}{\ell})\right\rfloor + 7\right)\rho$. If L is known in advance the ratio is $\left(\left\lfloor \log_2(\frac{L}{\ell})\right\rfloor + 4\right)\rho$.
Moreover, if the former online algorithm is polynomial, then the latter is polynomial as well.

Problem	Approximation	Competitive ratio (left to right)
colouring	$\log n$ (Cerný 2007)	$O(\frac{L}{\ell})$ (D,dS,L-B 2012) $O(\frac{\log^2 L}{\log\log L})$ $(\ell = 1)$
clique covering	$2(1 + \log \widetilde{\alpha}(R))$ Shahrokhi 2015	?
b-bounded colouring	?	?
b-bounded clique covering	?	?

Table 3: Best known results in overlap graphs

Problem	Approximation	Competitive ratio (left to right)
colouring	$\log \widetilde{\alpha}(R) + c$	$2\left\lfloor \log_2(\frac{L}{\ell}) \right\rfloor + 7$
clique covering		
b-bounded colouring	$2(\log \widetilde{\alpha}(R) + c)$	$(2\left\lfloor \log_2(\frac{L}{\ell}) \right\rfloor + 7)(2 - \frac{1}{\sigma})$
b-bounded clique covering	$\sigma = \min\{b, \beta(R)\}$	
b-bounded load colouring		

Table 4: Our results

Application to approximation

Proposition. *Given an interval system R of size n and $\varepsilon > 0$, we can modify R in $O(n \log n)$-time, preserving the relative position of intervals (containment, overlapping and disjoint relation), so that in the new system R', the maximum length $L(R')$ and the minimum length $\ell(R')$ satisfy $\frac{L(R')}{\ell(R')} \leq (2+\varepsilon)\widetilde{\alpha}(R)$.*

Hardness result

Theorem. *(D, di Stefano, Leroy-Beaulieu 2012).*
There exists a constant κ *such that there is no* $\left(\kappa \log\log \frac{L}{\ell}\right)$*-competitive online algorithm for colouring overlap graphs defined by an interval system presented from left to right.*

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Theorem *There is no* $\left(\frac{\log\left(\frac{L}{\ell}\right)}{2\log\log\left(\frac{L}{\ell}\right)}\right)$*-competitive algorithm exists, even on bipartite instances.*

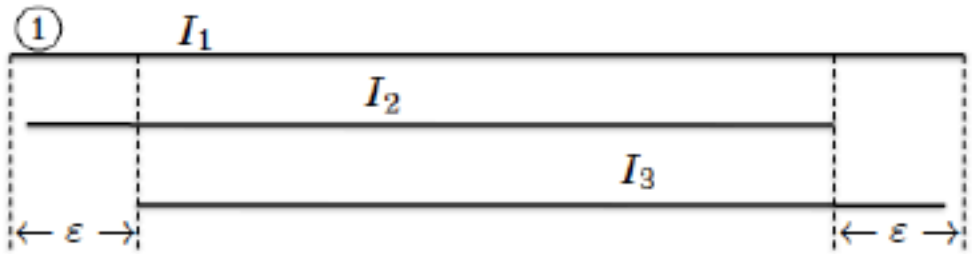
Hardness result

Lemma 13. *For any $K \geq 3$, $\varepsilon > 0$ and any online algorithm for colouring overlap graphs defined by an interval system presented from left to right, there is an interval system R_K such that it is possible to force K different colours on an independent set $B_K \subset R_K$ of at least K interval such that:*

1. *G_{R_K} is bipartite and moreover there is a 2-colouring of G_{R_K} for which B_K is monochromatic;*
2. *B_K is a brochette and there is $t_{R_K} \in \bigcap_{I \in B_K} I$ such that intervals in $R_K \setminus B_K$ have a right endpoint less than t_{R_K};*
3. *for all $K \geq 3$, $W(R_K) = \gamma_K \times K!$ with $(\gamma_K)_{K \geq 3}$ an increasing sequence varying from $\frac{1}{3} + \frac{\varepsilon}{6}$ to $\frac{\varepsilon}{6} + e - \frac{7}{3}$, $\ell(R_K) > 1$ and $L(R_K) \leq W_k - 2(K-3)$.*

Hardness result

Case $\bar{K} = 3$



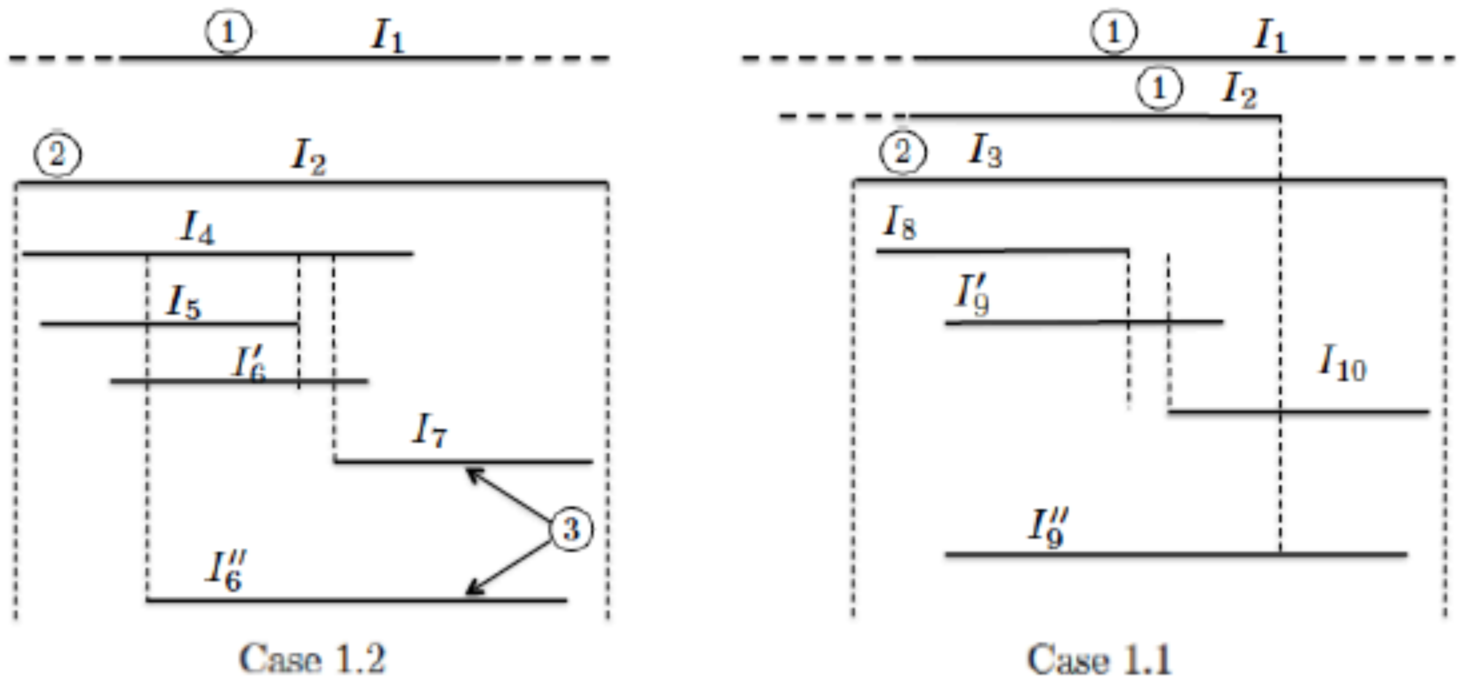
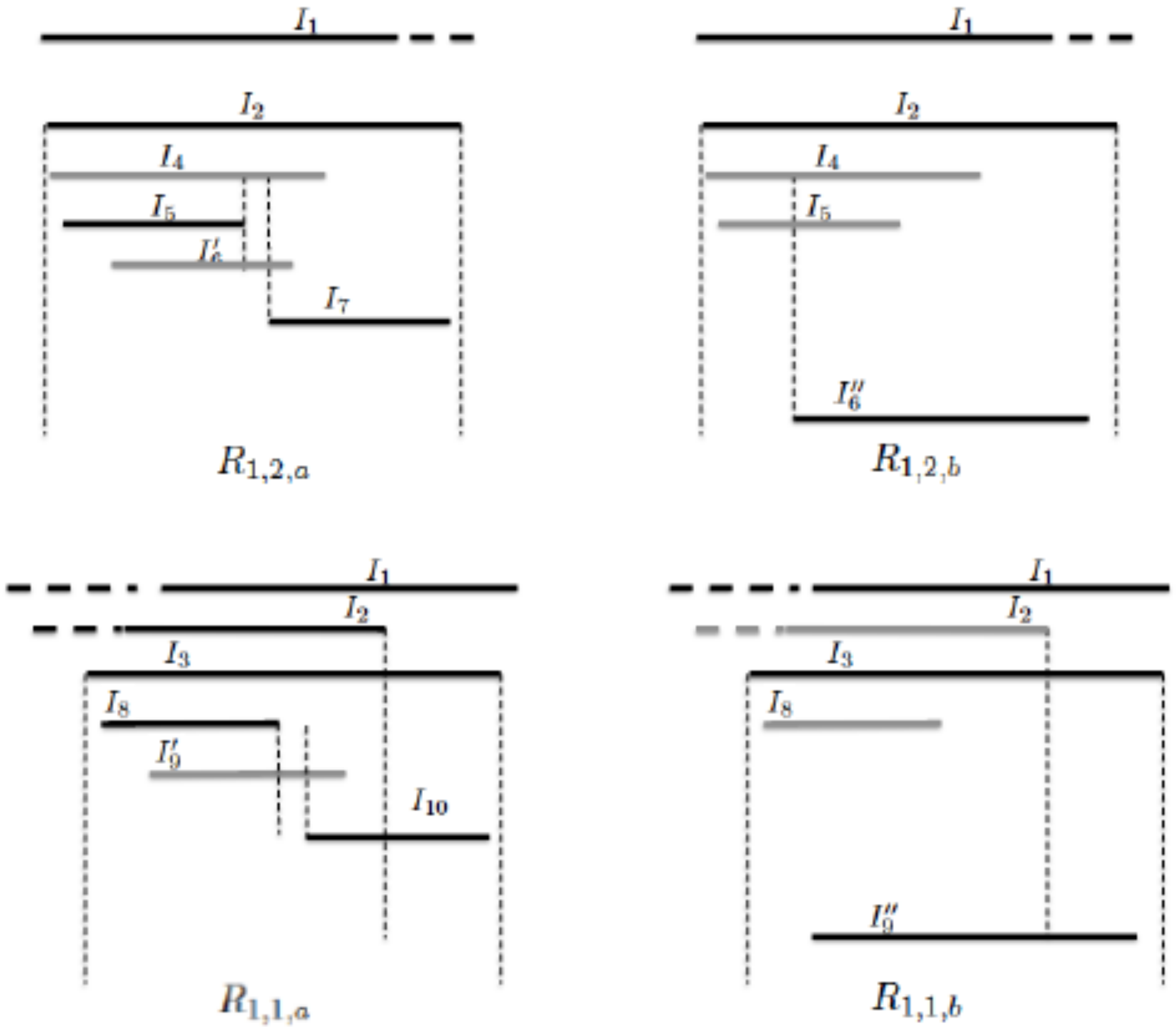
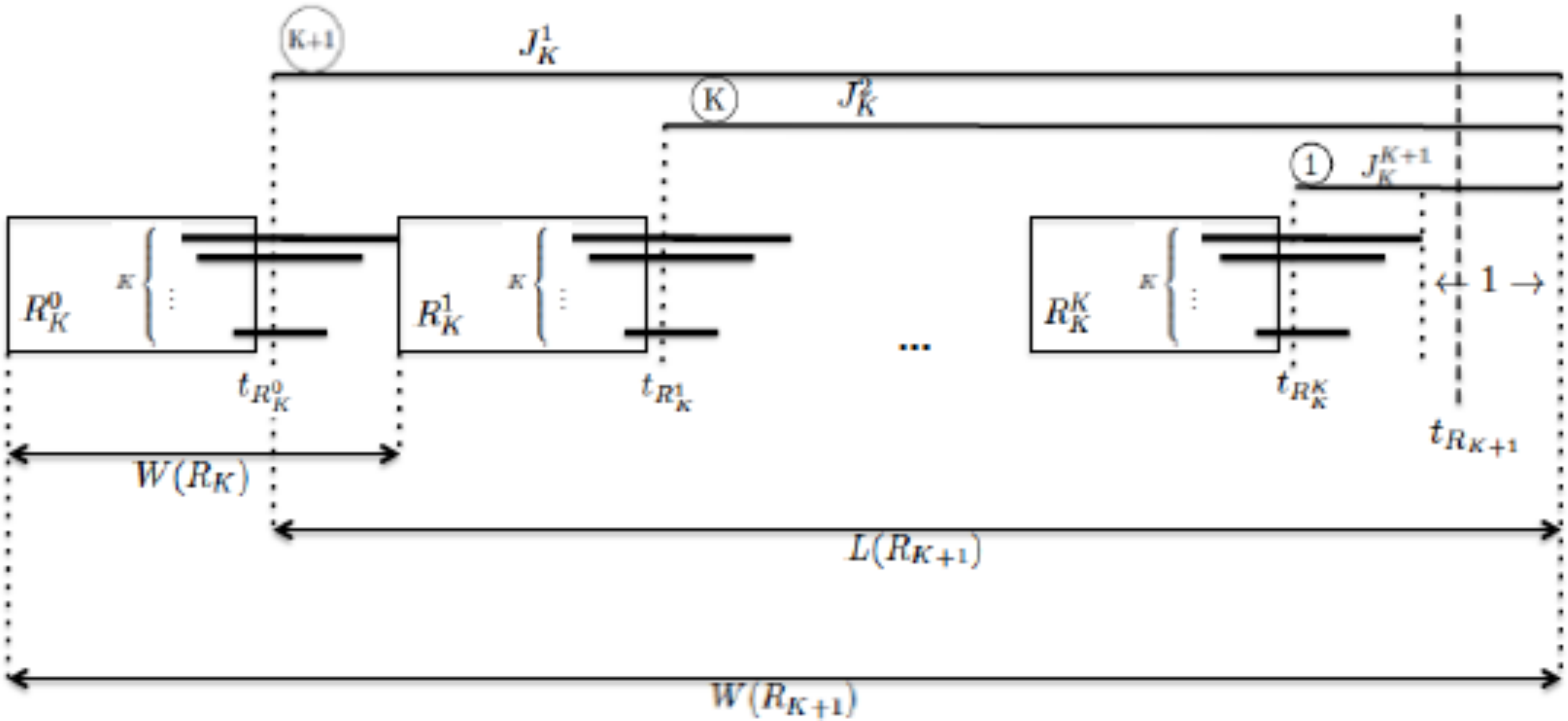


Figure 4: The interval system R_3 to force three colours in a bipartite overlap graph. In the Case 1.2 the algorithm colours I_1 with 1 and I_2 with 2 - in the Case 1.1 the algorithm colours both I_1 and I_2 with the same colour 1.

Hardness result Case $\bar{K} = 3$ Bipartition



Hardness result Induction step



Hardness result

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$$\begin{cases} W(R_{K+1}) & = & (K+1)W(R_K) + 1, K \geq 3 \\ W(R_3) & = & (2 + \varepsilon) \end{cases}$$

Hardness result

Lemma 13. *For any* $K \geq 3$, $\varepsilon > 0$[9] *and any online algorithm for colouring overlap graphs defined by an interval system presented from left to right, there is an interval system* R_K *such that it is possible to force* K *different colours on an independent set* $B_K \subset R_K$ *of at least* K *interval such that:*

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2. B_K *is a brochette and there is* $t_{R_K} \in \bigcap_{I \in B_K} I$ *such that intervals in* $R_K \setminus B_K$ *have a right endpoint less than* t_{R_K}*;*
3. *for all* $K \geq 3$, $W(R_K) = \gamma_K \times K!$ *with* $(\gamma_K)_{K \geq 3}$ *an increasing sequence varying from* $\frac{1}{3} + \frac{\varepsilon}{6}$ *to* $\frac{\varepsilon}{6} + e - \frac{7}{3}$, $\ell(R_K) > 1$ *and* $L(R_K) \leq W_k - 2(K-3)$.

$$\begin{cases} W(R_{K+1}) & = & (K+1)W(R_K) + 1, K \geq 3 \\ W(R_3) & = & (2+\varepsilon) \end{cases}$$

$$\forall K \geq 3, W(R_K) = K!\gamma_K \text{ with } \gamma_K = \left(\frac{2+\varepsilon}{6} + \sum_{i=4}^{K} \frac{1}{i!} \right)$$

Hardness result

Theorem *There is no* $\left(\frac{\log\left(\frac{L}{\ell}\right)}{2\log\log\left(\frac{L}{\ell}\right)}\right)$*-competitive algorithm exists, even on bipartite instances.*

$N > e^e$ $\qquad \frac{\varepsilon}{6} + e - \frac{7}{3} = 1$ $\qquad K \times K! > N$

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$$\frac{L(R_K)}{\ell(R_K)} \geq \frac{1}{2}(2+\varepsilon) \times K \times K! > N \qquad \frac{L(K_R)}{\ell(K_R)} \leq K!$$

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$$\frac{\log\left(\frac{L(K_R)}{\ell(K_R)}\right)}{\log\log\left(\frac{L(K_R)}{\ell(K_R)}\right)} \leq \frac{\log(K!)}{\log\log(K!)} \leq \frac{K\log K}{\log K + \log\log K} < K$$

Problem	Approximation	Competitive ratio (left to right)
colouring	$\log n$ (Cerný 2007)	$O(\frac{L}{\ell})$ (D,dS,L-B 2012) $O(\frac{\log^2 L}{\log\log L})$ $(\ell = 1)$
clique covering	$2(1 + \log \widetilde{\alpha}(R))$ Shahrokhi 2015	?
b-bounded colouring	?	?
b-bounded clique covering	?	?

Table 3: Best known results in overlap graphs

Problem	Approximation	Competitive ratio (left to right)
colouring	$\log \widetilde{\alpha}(R) + c$	$2\left\lfloor \log_2(\frac{L}{\ell}) \right\rfloor + 7$
clique covering		
b-bounded colouring	$2\left(\log \widetilde{\alpha}(R) + c\right)$	$\left(2\left\lfloor \log_2(\frac{L}{\ell}) \right\rfloor + 7\right)\left(2 - \frac{1}{\sigma}\right)$
b-bounded clique covering	$\sigma = \min\{b, \beta(R)\}$	
b-bounded load colouring		

Table 4: Our results

Polynomial Approximation

Proposition 16. *Given an interval system R of size n and $\varepsilon > 0$, we can modify R in $O(n \log n)$-time, preserving the relative position of intervals (containment, overlapping and disjoint relation), so that in the new system R', the maximum length $L(R')$ and the minimum length $\ell(R')$ satisfy $\frac{L(R')}{\ell(R')} \leq (2+\varepsilon)\widetilde{\alpha}(R)$.*

Conclusion

- Improves the competitive ratio for online colouring overlap graphs from linear to a logarithmic factor
- Competitive results for new colouring problems
- Narrows the gap between competitive results and hardness result
- New problem: partionning an overlap graph into permutation graphs
- Competitive-preserving online reduction
- Works on the graph and its complement
- Future work: other generalised colouring problems (split & cocouloring)

On going project: excluding some interval configurations

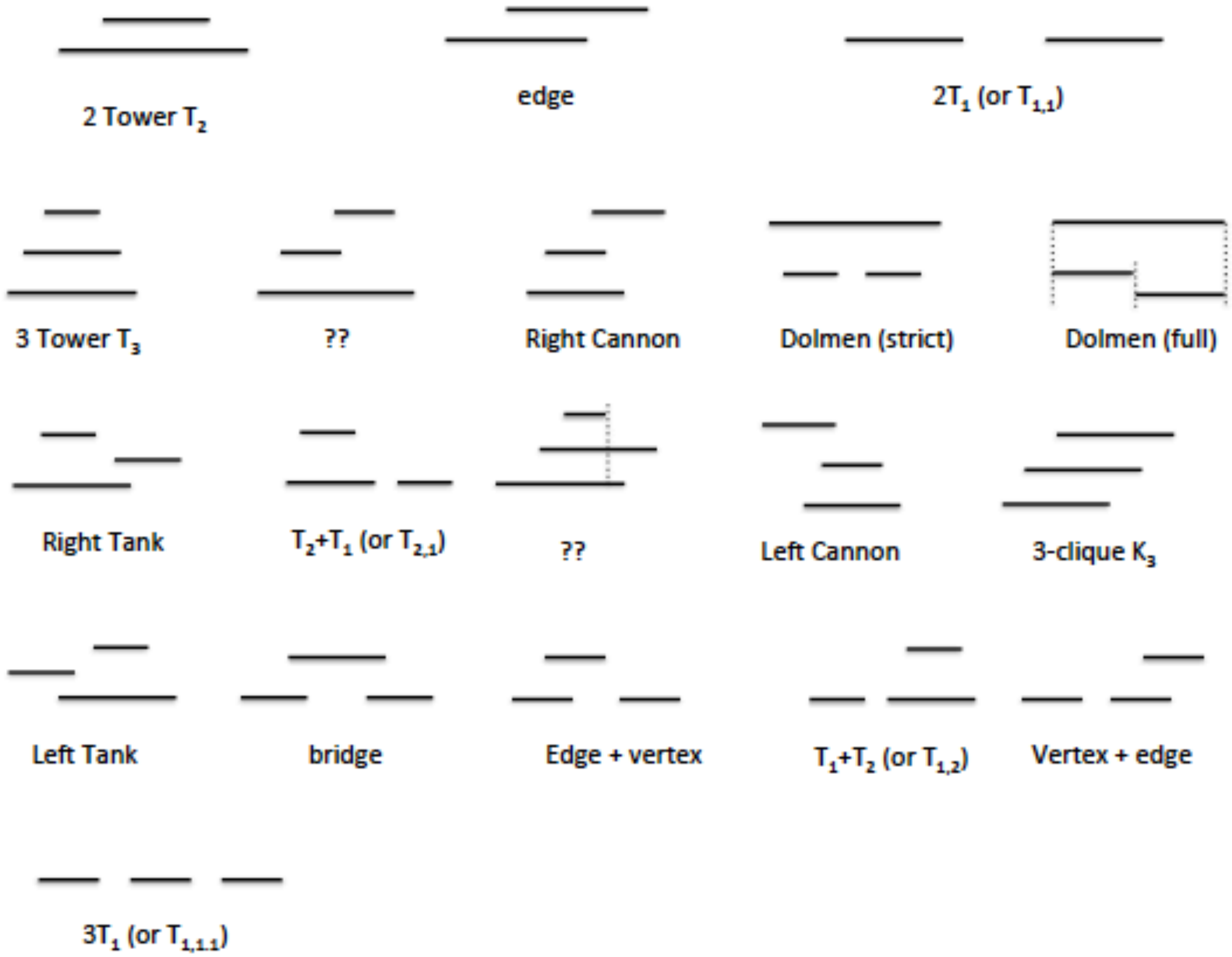

Figure 1: The different systems of size two and three.

On going project: excluding some interval configurations

R	G_R	$\frac{L}{\ell}$	R-free class	left to right competitivity of the R-free class
T_2 (2-Tower)	2-independent set	any	Unitary interval graphs	1-competitive
Edge	K_2	any	Independent set	1-competitive
$T_{1,1}$	2-independent set	any	brochette (permutation graph)	1-competitive
T_3	3-independent set	any	not perfect (includes C_5)	not $< \frac{3}{2}$-competitive (Proposition 1) 2-competitive (Proposition 6) 3/2-competitive in bipartite case (Remark 2)
3-stair	edge+vertex	any	perfect (Proposition 7)	1-competitive (Proposition 7)
Bridge	P_2	any	permutation	1-competitive
Right cannon	P_2	any	perfect (Proposition 8)	1-competitive (Proposition 8)
Strict dolmen	3-independent set	> 2	not perfect (includes C_5)	not $< \frac{3}{2}$-competitive (Proposition 1) ω-competitive (Proposition 9) 3/2-competitive in bipartite)

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