The Metric Dimension Problem.

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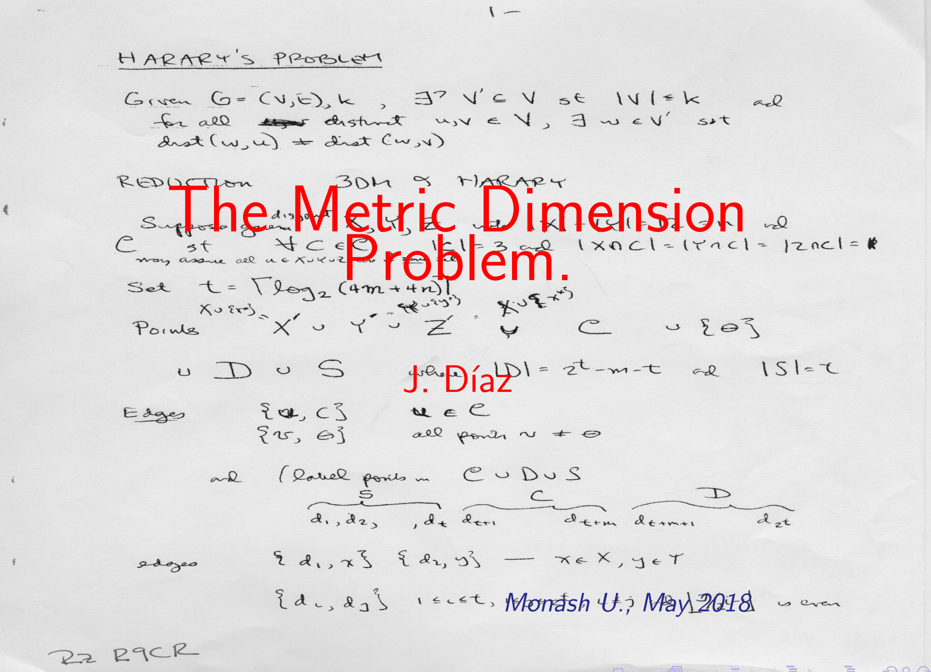
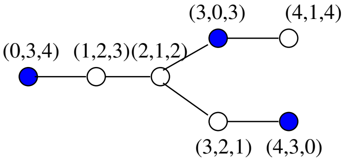
Monash U., May 2018



The Metric Dimension problem

Given $G(V, E)$ its metric dimension, $\beta(G)$ is the cardinality of the smallest $L \subset V$ s.t. $\forall x, y \in V$, $\exists z \in L$ with $d_G(x, z) \neq d_G(y, z)$. The set L is called a resolving set.

Harary, Melter, (1976), Slater, (1974)



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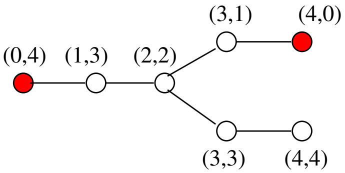
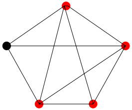


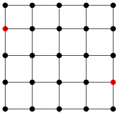
Characterizations of MD for some particular graphs

- $\beta(G) = 1$ iff G is a path.

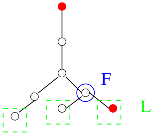
- $\beta(G) = n - 1$ iff G is a n-clique.



- If $\beta(G) = 2 \Rightarrow G$ does not contain $K_{3,3}$ or K_5 Khuller,Raghavachari,Rosenfeld (1996)



- If T a tree, L the set of leaves and F the set of fathers of L with degree $\geq 3 \Rightarrow$ $\beta(T) = |L| - |F|$. (Slater 1975)



MD and graph properties

- Metric dimension of certain Cartesian product of graphs:
  For different examples of G and H produce UB and LB to the MD of $G\square H$. They gave an example of a G with bounded MD, where $G\square G$ has unbounded MD.
  Caceres,Hernandez,Mora,Pelayo,Puertas,Sera,D.Wood (2007)

- If G has diameter D, $n \leq D^{\beta(G)-1} + \beta(G)$.
  Khuller,Raghavachari,Rosenfeld (1996)

- Let $\mathcal{G}_{\beta,\mathcal{D}}$ be the class of graphs with MD$=\beta$ and diameter$=D$, the authors determine the max. number of vertices for $G \in \mathcal{G}_{\beta,\mathcal{D}}$.
  Hernando,Mora,Pelayo,Seara,Wood (2010)

Complexity of Metric Dimension

- NPC for general graphs, Garey,Johnson (1979)
- P for trees, Khuller,Raghavachari,Rosenfeld (1996)
- NPC for bounded degree planar graphs
  Díaz, Pottonen, Serna, Van Leeuwen, (2012)
- NPC for Gabriel graphs   Hoffman, Wanke (2012)
  G is Gabriel $\forall u, v \in V(G)$ are adjacent if the closed disc of which line segment uv is diameter contains no $w \in V(G)$.
  $\Rightarrow$ Unit Disks Graphs are NPC
- NPC for weighted MD for a variety of graphs
  Epstein, Levin, Woeginger (2012)

NPC for bounded degree planar graphs:Sketch

Consider the 1-Negative Planar 3-SAT problem: Given a sat formula ϕ s.t.

- every variable occurs exactly once negatively and once or twice positively,
- every clause contains two or three distinct variables,
- every clause with three distinct variables contains at least one negative literal,
- the clause-variable graph G_ϕ is planar.

decide if it is SAT.
1-Negative Planar 3-SAT problem is NPC: reduction from Planar-SAT.

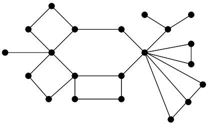
1-Negative Planar 3-SAT problem $\leq_p$ decisional MD bounded degree planar graphs.

Aproximability to MD

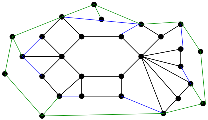
- There is a $2\log n$-approximation for general graphs, Khuller*
- If P$\neq$ NP, there is not a $o(2\log n)$-approximation, Beliova,Eberhard,Erlebach,Hall,Hoffmann,Mihálak,Ram (2006)
- $\forall \epsilon > 0$, There is no $(1-\epsilon)\log n$ for general graphs, unless NP$\subseteq$ DTIME $(n^{\log\log n})$, Hauptmann,Scmhied,Viehmann(12)
- If P$\neq$ NP, not $o(\log n)$-approximation for general graphs with maximum degree 3, Hartung,Nichterlein (2013)

MD is in P for outerplanar graphs

An undirected G is said to be an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing.



For $k > 1$, G is said to be an *k*-outerplanar graph if removing the vertices on the outer face results in a $(k-1)$-outerplanar embedding.



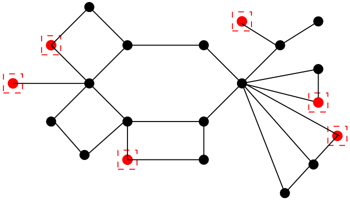
MD $\in$ P for outer-planar graphs

1. Characterize the resolving sets by giving 2 conditions: one over the vertices and another over the faces
2. Define a T where the vertices are the cut vertices and faces of G and the edges in T correspond to inner edges and bridges (separators) of G. Notice as size of an inner face could be arbitrarily large, the width of T could be arbitrary.
   Explore T in bottom-up fashion using two data structures:
   2.1 Boundary conditions
   2.2 Configurations

Algorithm for outerplanar

Even the number of vertices in G represented by $v \in V(T)$ could be unbounded, the total number of configurations is polynomial.

The algorithm works in $O(n^8)$ (plenty of room for possible improvement)



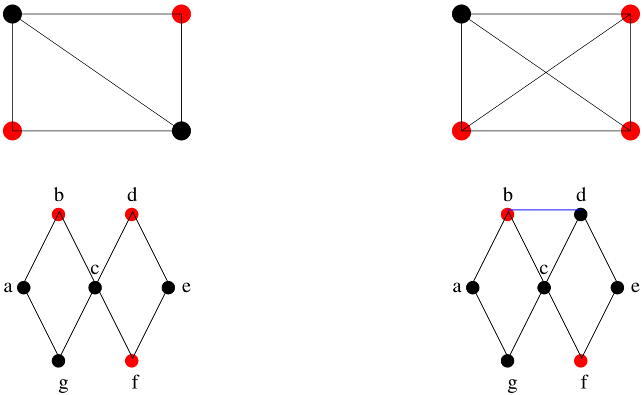
Open probelms on the complexity of MD

Prob. 1: Find if MD for K-outerplanar graphs is in P or in NPC.

Baker's Technique (1994):
The technique aims to produce FPTAS for problems that are known to be NPC on planar graphs. They decompose the planar realization into k-outerplanar, get an exact solution for each k-outerplanar slice and combine them. Solving for each k-outerplanar using DP on a tree decomposition, that for each vertex separator of size at most $2k$.
Prob. 2: We know that unless NP$\subseteq$ DTIME $(n^{\log \log n})$, MD has tor PTAS in planar graphs $\in$ PTAS for planar graphs. Is it in APX-hard?

Why MD is difficult? 1

- *Strongly non-local*. A vertex in L can resolve vertices very far away.
- *Non-closed under vertex addition, subtraction, or subdivision.*



Why MD is difficult?

- *MD does not have the bidimensionality behavior.*

A problem is bidimensional if it does not increase when performing certain operations as contraction of edges, and the solution value for the problem on a $n \times n$-grid is $\Omega(n^2)$ Demaine, Fomin, Hajiaghayi, Thilikos (2005)

Bidimensionality has been used as a tool to find PTAS for bidimensional problems that are NPC on planar graphs. Demaine, Hajiaghayi (2005).

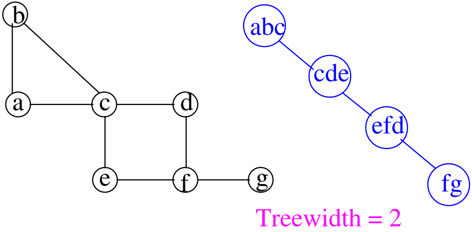
Examples: feedback vertex set, minimum maximal matching, face cover, edge dominating set ....

Background on parametrized complexity

The Tree-width of $G = (V, E)$ is a tree $(\{X_i\}, T\})$:

- $\cup X_i = V$
- $\forall e \in E, \exists i : e \in X_i$
- If $v \in X_i \cap X_j$ then $\forall X_k \in X_i \rightsquigarrow X_j$ we have $v \in X_k$

The tree width of a graph G is the size of its largest set $|X_i| - 1$.



Parametrized complexity

Classify the problems according to their difficulty with respect to the input size n an input parameter k of the problem.
Downey, Fellows (1999)

Fixed parameter tractable: FPT is the class of problems solvable in time $f(k)\text{poly}(n)$ (where $f(k) = 2^k$)

- Ex. (k-vertex cover) Given (G, k), does G have a VC $\leq k$?
  Time of k-VC $= (kn + 1.2^k)$. $\therefore$ k-VC $\in$ FPT.

- Another ex. SAT with m clauses and k variables it can be checked in time $O(m2^k)$.

$$\text{P} \subseteq \text{FPT} \subseteq \text{W[1]} \subseteq \text{W[2]} \subseteq \cdots \subseteq \text{XP}$$

Metric Dimension and parametrized complexity

- W[2]-complete for general graphs, Hartung, Nichterlein (2013)

Courcelle's Theorem *Any problem definable by Monadic Second Order Logic is* FPT *when parametrized by tree width and the length of the formula.*

So far, it seems to be difficult to formulate MD as an MSOL-formula $\Rightarrow$ Courcelle's Theorem can't apply.

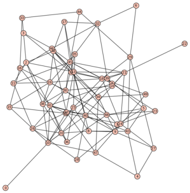
Prob. 3: Prove formally that MD can not be expressed as an MSOL formula.

Prob. 4: Show if MD $\in$ P (or not) for bounded tree-width graphs.

Prob. 5: Study the parametrized complexity of MD on planar graphs.

Binomial Graphs $G(n, p)$

$G \in G(n, p)$ if given n vertices $V(G)$, each possible edge e is included independently with probability $p = p(n)$.

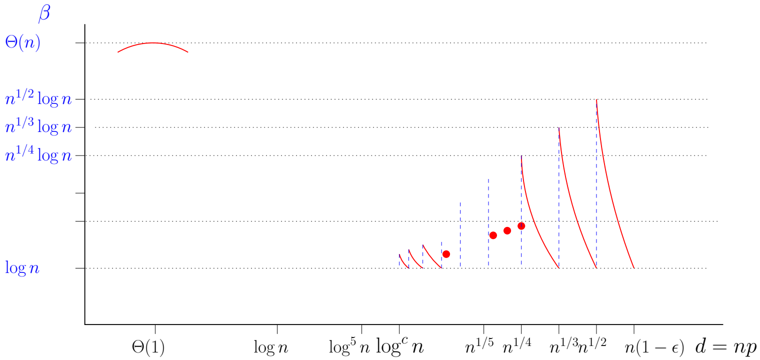


Whp $|E(G)| = p\binom{n}{2}$ and the expected degree of a vertex: $d = np$.
Giant component threshold: $p_t = (1+\epsilon)\frac{1}{n}$.
Connectivity threshold: $p_c = (1+\epsilon)\frac{\log n}{n}$.

Expected $\beta(G)$ in $G(n, p)$

Bollobas, Mitsche, Pralat (2013)

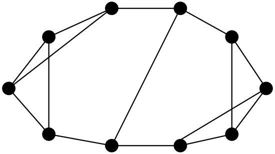
Given $G \in G(n,p)$, choose randomly the resolving set $L \subseteq V$ and bound $\mathbf{Pr}[\exists u, v \text{ not separated by } L]$.



Prob. 6: Find if there is a $\mathbf{E}[\beta(G)]$ for $\Theta(1/n) < p < \log^5 n/n$

Random t-regular Graphs $\mathcal{G}(n, t)$

$G \in \mathcal{G}(n, t)$ if it is uniformly sampled from the set of all graphs with n vertices and degree t. Assume $t = \Theta(1)$.



Let $G \in \mathcal{G}(n, t)$:

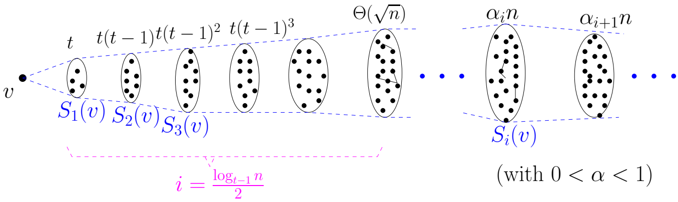
- For $t \geq 3$ aas G is *strongly connected* Cooper (93).
- For $t \geq 3$ aas G is *Hamiltonian* Robinson,Wormald (92,93), Cooper, Frieze (94).
- For $t \geq 3$ aas the diameter of $G = \log_{t-1} + o(\log n)$ Bollobas, Fernandez de la Vega (81)
- For $t \geq 3$, G is an *expander*, i.e. $\exists c > 1$ s.t. $\forall S \subset V(G)$ with $1 \leq |S| \leq \frac{n}{2}$, $\mathcal{N}(S) \geq c|S|$.

Expected $\beta(G)$ for $\mathcal{G}(n,t)$

Given $G \in \mathcal{G}(n,t)$, $|V| = n$ and $2 < t = \Theta(1)$, then whp

$$\mathbf{E}[\beta(G)] = \Theta(\log n)$$

Given $G \in \mathcal{G}(n,t), v \in V(G)$, let $S_i = \{u \in V(G) \mid d_G(v,u) = i\}$



Given $v \in V(G)$, for any pair $(u,w) \in V^2$:
v does not separate u and w if $u, w \in S_i$, and
v separates u and w if $u \in S_i$ & $w \in S_{i+1}$ (or vice versa).

Expected $\beta(G)$ for $\mathcal{G}(n,t)$

Therefore, $\mathbf{Pr}[v \text{ separates } u\&w] \geq 2\alpha_i\alpha_{i+1}$, and
$\mathbf{Pr}[v \text{ does not separate } u\&v] \geq \alpha_i^2 + \alpha_{i+1}^2$,
where α_i and α_{i+1} are constants between 0 and 1.

$$\underbrace{(1-\alpha_i\alpha_{i+1})}_{\alpha} \geq \mathbf{Pr}[v \text{ separates } u\&w] \geq \underbrace{2\alpha_i\alpha_{i+1}}_{\alpha'}$$

Upper Bound

Randomly choose a resolving $L \subset V(G)$ with $|L| = C\log n$, for large constant $C > 0$.

Then for a particular pair of vertices u, w
$\mathbf{Pr}[L \text{ does not separate } u\&w] < \alpha^{C\log n} \sim o(\frac{1}{n^2})$ (union bound)
Let $X_C =$ be the number of pairs not separated by L,

$$\mathbf{E}[X_C] < n^2\alpha^{C\log n} \to 0 \Rightarrow \mathbf{Pr}[X_C > 0] \to 0$$

Expected $\beta(G)$ for $\mathcal{G}(n,t)$: Lower Bound

Randomly choose a resolving set $L \subset V(G)$ with $|L| = c \log n$, for small constant $c > 0$.

$\mathbf{Pr}\left[L \text{ does not separate } u\&w\right] \geq \alpha'^{c \log n} \sim \omega(\frac{1}{n^2})$

$\Rightarrow$ If X_c = number pairs not separated by L, then

$$\mathbf{E}\left[X_c\right] > n^2 \alpha'^{c \log n} \to \infty \Rightarrow \mathbf{Pr}\left[X_c > 0\right] = 1 - o(1)$$

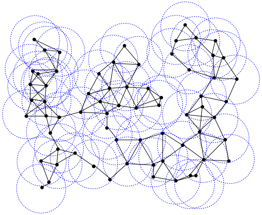
Therefore, $\boxed{\beta(G) = \Theta(\log n)}$.

Prob. 7: Find the constant in $\mathbf{E}\left[\beta(G)\right] = \Theta(\log n)$

For $t = 3$, empirically $\beta(G) = 1.13 \log n$.

Random Geometric Graphs $\mathcal{G}(n, r(n))$

Given a square $Q = [0, \sqrt{n}]^2$ and a real $r(n) > 0$ define a random geometric graph $G \in G(n, r)$ by scattering n expected vertices V on Q according to a Poisson distribution with intensity 1, and for any $u, v \in V$, $(u, v) \in E$ iff $d_E(u, v) \leq r$.



It is known: (1) The giant component appears at $r_t = \Theta(1)$.
(2) There is a sharp connectivity threshold at $r_c = \Theta\sqrt{\log n}$.
(3) For $v \in V$, the expected degree $d(v) = \pi \log n$.

M. Penrose: *Random Graphs*. Oxford (2002)

Expected metric dimension on $\mathcal{G}(n, r(n))$

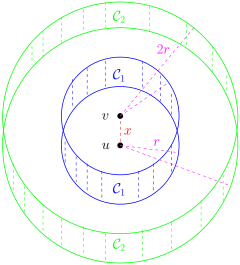
Given $G \in \mathcal{G}(n, r(n))$ what can we say about $\mathbf{E}[\beta]$?

If $r_t = O(1) \Rightarrow \beta(G) = \Theta(n)$

Given $v, u \in V(G)$ how can they be separated?

$\mathbf{E}[\beta(G)]$ for $r_c = c\sqrt{\log n}$

Let $G \in \mathcal{G}(n, r_c)$ and let $u, v \in V(G)^2$ with $d_E(u,v) = x$
Define the crowns:
$\mathcal{C}_i(u,v) := \{w \in V(G) : \ d_E(u,w) = i \text{ and } d_E(v,w) = i+1\}$



LB: Compute the number of pairs for which $\mathcal{C}_1 = \emptyset$.
Area of $\mathcal{C}_1 = 4xr_c \Rightarrow \mathbf{Pr}[\mathcal{C}_1 = \emptyset] = e^{-4xr_c n}$
Number of (u,v) with $\mathcal{C}_1 = \emptyset$ is $2\pi n^2 \int_0^r xe^{-4xr_c n} dx = \frac{n}{\log n}$

$\mathbf{E}[\beta(G)]$ for $r_c = c\sqrt{\log n}$

UB: Let $x_0 = \frac{c}{\sqrt{n(\log n)^{1/3}}}$

Divide the pairs (u, v) in two groups : those with $x \leq x_0$ and the remaining ones.

For the first group, $\mathbf{E}[|(u,v) \leq x_0|] = O(\frac{n}{(\log n)^{1/3}})$

For the second group choose a random resolving $L \subseteq V(G)$, with $|L| = \frac{n}{(\log n)^{1/3}}$,

If $d(u,v) > x$ there are sufficiently large numbers of crowns each with enough vertices assure us each $\mathcal{C}_i(u,v)$ intersects L.

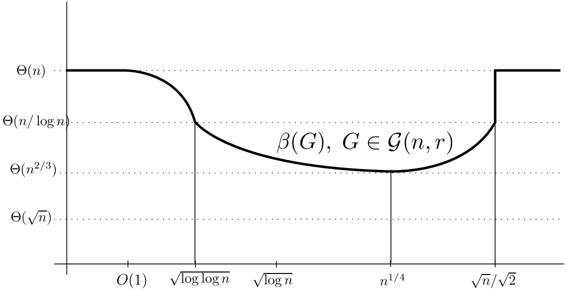
Therefore at $r_c = \Theta(\sqrt{\frac{\log n}{n}})$:

$$\frac{n}{\log n} \leq \beta(G) \leq \frac{n}{\log^{1/3} n}$$

Expected metric dimension on $\mathcal{G}(n, r(n))$

What we know and don't know:

- If $r = O(1) \Rightarrow \beta(G) = \Theta(n)$ ✓
- If $1 << r << \sqrt{\log\log n} \Rightarrow \beta(G) = \Theta(ne^{-\pi r^2})$ ✓
- If $r = C\sqrt{\log n} \Rightarrow \frac{n}{\log n} \leq \beta(G) \leq \frac{n}{\log^{1/3} n}$?
- If $\log n \leq r \leq (n \log^3 n)^{1/4} \Rightarrow \frac{n}{r^2} \leq \beta(G) \leq \frac{n \log^2 n}{r^2}$?
- If $(n \log^{1/3} n)^{1/4} \leq r \leq \frac{\sqrt{n}}{4} \Rightarrow \beta(G) = \Theta(r^{2/3} n^{1/3})$?
- If $r \geq \frac{\sqrt{n}}{\sqrt{2}} \Rightarrow \beta(G) = \Theta(n)$?



Thank you for your attention