Heffter Arrays: Biembeddings of Cycle Systems on Surfaces

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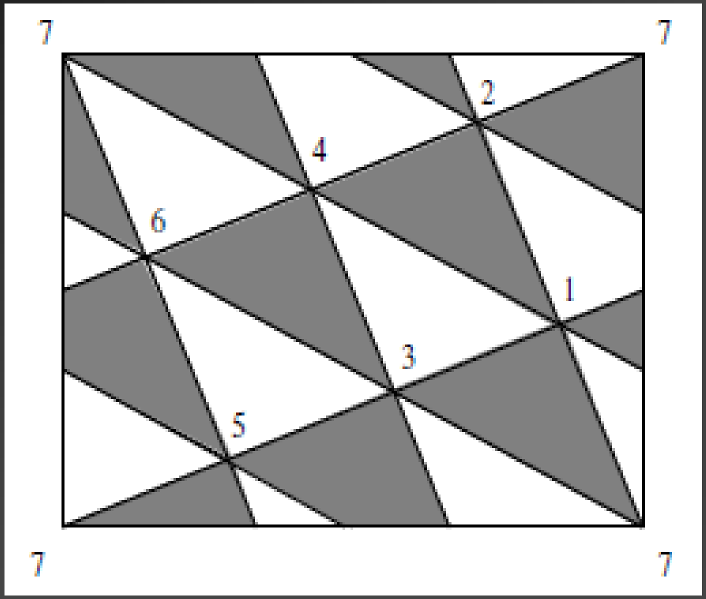
Our goal is to embed the complete graph K_n on a surface (orientable closed 2-manifold) so that each face is either an *s*-cycle or a *t*-cycle and each edge bounds exactly one face of each size.

A famous example

This is K_7 embedded on the torus.



Here we decompose K_7 into two sets of 3-cycles (black and white) and each edge borders exactly one black and one white triangle.

Each face is a triangle and there are 14 faces.

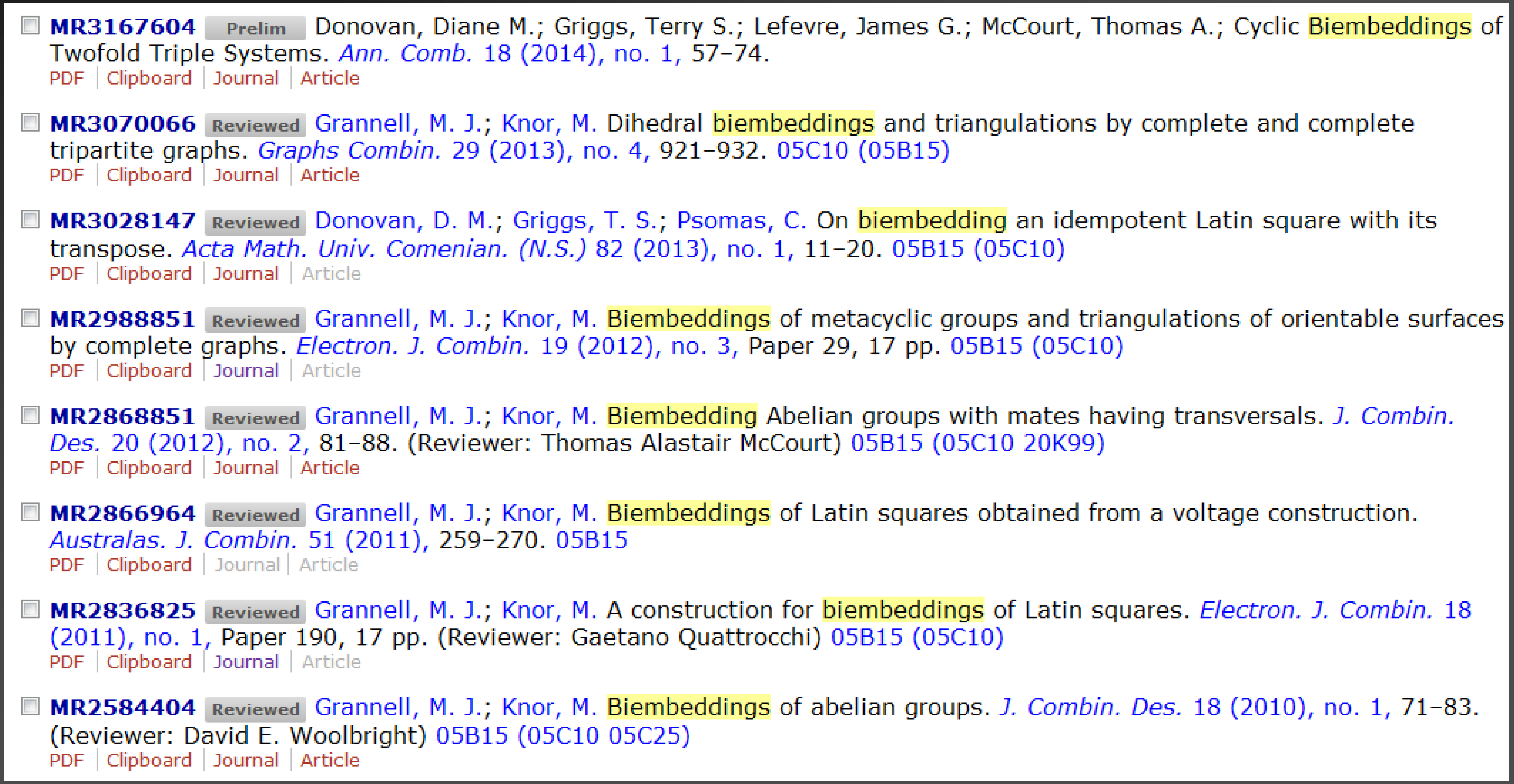
There are 7 vertices and $\binom{7}{2}$ edges so

$$v - e + f = 7 - 21 + 14 = 0$$

So $0 = 2 - (2 \times g)$ (where g is the genus of the surface).

So the genus is 1 and thus we see (again) that this embedding is on the torus.

What else is known??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Heffter systems

A Heffter *k*-system of order *n* is a collection of disjoint *k*-subsets S_j of $\mathbb{Z}_n \backslash \{0\}$ (*n* odd) satisfying:

1) For each subset *S*, $\sum_{a \in S} a = 0$. *(the elements sum to 0)*

2) *x is in a subset if and only if –x is not in any subset.*

Example: a Heffter 4-system in $\mathbb{Z}_{25} \backslash \{0\}$

{1, -2, 11,-10}, {7,-4,9,-12}, {-8,6,5,-3}

Orthogonal Heffter systems

A Heffter *s*-system *S* and a Heffter *t*-system *T* on $\mathbb{Z}_{2st+1} \setminus \{0\}$ are orthogonal if each subset in *S* intersects each subset in *T* in exactly one symbol.

Example: The rows form a Heffter 4-system and the columns form a Heffter 3-system (both in $\mathbb{Z}_{25}$).

$$\begin{pmatrix} 1 & -2 & 11 & -10 \\ 7 & -4 & 9 & -12 \\ -8 & 6 & 5 & -3 \end{pmatrix}$$

Tight Heffter Arrays $H(s,t)$

A tight Heffter array $H(s,t)$ is an $s \times t$ rectangular array with entries $a_{i,j}$ satisfying

1) $\{ | a_{i,j} | \} = \{1,2,\ldots,st\}$, that is, we use the first st numbers up to sign and
2) every row and column sum is 0 (termed an integer Heffter array),

or if that is not possible, relax to sums to 0 modulo $2st + 1$.

Example: An H(3,4)

$$\begin{pmatrix} 1 & -2 & 11 & -10 \\ 7 & -4 & 9 & -12 \\ -8 & 6 & 5 & -3 \end{pmatrix}$$

The name "Heffter array" comes from a relation to solutions to Heffter's difference problems that will be explained shortly.

Tight refers to the fact that each cell is filled – we will have other examples where this is not the case.

So a tight $s \times t$ Heffter array is equivalent to a Heffter s-system *S* and an orthogonal Heffter t- system *T* both on the symbols of $\mathbb{Z}_{2st+1} \setminus \{0\}$.

How to make the embedding

by example

$$\begin{pmatrix} 1 & -2 & 11 & -10 \\ 7 & -4 & 9 & -12 \\ -8 & 6 & 5 & -3 \end{pmatrix}$$

Starting with an H(3,4)

We will embed K_{25} ($25 = 2 \times 3 \times 4 + 1$) on a surface such that each face is either a triangle or a 4-cycle and each edge borders exactly one triangle and one 4-cycle.

First generate the 3-cycles by developing the columns in Z_{25}

From first column we get the 3-cycles	The second column gives the 3-cycles	The third column gives the 3-cycles	The fourth column gives the 3-cycles
(0,1,8)	(0,23,19),	(0,11,20),	(0,15,3),
(1,2,9)	(1,24,20)	(1,12,21)	(1,12,21)
(2,3,10)	(2,0,21)	(2,13,22)	(2,13,22)
…	…	…	…
(24,0,7)	(24,22,18)	(24,10,19)	(24,14,2)

Note that this is a "difference construction" and hence since each difference from Z_{25} is used exactly once, we have developed all of the edges of K_{25} and each edge is in exactly one 3-cycle. So the rows generate a cyclic 3-cycle system (a cyclic Steiner triple system).

Now do the same with the rows to get all the 4-cycles.
Note that each edge is on exactly one 4-cycle, too.

$$\begin{pmatrix} 1 & -2 & 11 & -10 \\ 7 & -4 & 9 & -12 \\ -8 & 6 & 5 & -3 \end{pmatrix}$$

The first row generates the 4-cycles
(0,1,24,10)
(1,2, 0, 11)
(2,3, 1, 12)
…

(24,0,23,9)

The second row generates the 4-cycles
(0,7,3,12)
(1,8,4,13)
(2,9,5,14)
…

(24,6,2,11)

The third row generates the 4-cycles
(0,17,23,3)
(1,18,24,4)
(2,19,25,5)
…

(24,16,22,2)

Similar to how the columns generate a cyclic 3-cycle system, we see that the rows generate a cyclic 4-cycle system.

From the construction, we have that a pair of edges that are in a triangle together in the 3-cycle system are not in a 4-cycle together and vice versa.

One final condition (partial sum condition)

For an H(s,t) to give the cycle systems it must also be the case that the partial sums of each row and each column are all distinct (modulo 2st+1).

$$\begin{pmatrix} 1 & -2 & 11 & -10 \\ 7 & -4 & 9 & -12 \\ -8 & 6 & 5 & -3 \end{pmatrix}$$

In this example,

- the partial sums of row 1 are 1,-1,10, 0,
- the partial sums of row 2 are 7,3,12, 0
- the partial sums of row 3 are -8,-2, 3, 0
- The columns are all ok too.

More on this later.

Why have this condition?

It is key that when each row and column is developed modulo 2st+1, that it generates a simple cycle and not a closed walk.

The condition that the sum is zero implies that it is a closed walk, while the partial sum condition guarantees that it is a simple closed walk (a cycle).

Theorem: An H(*s*,*t*) (*s*,*t* not both even) creates an embedding of K_{2st+1} on an orientable surface provided the rows and columns can all be ordered with all partial sums distinct.

To summarize (from design theory):

If there exists a Heffter array(s,t), and the rows and columns can be ordered so that all the partial sums are distinct, then there exists a cyclic s-cycle system S and a cyclic t-cycle system T, both on 2st+1 points.

Furthermore, if two edges are together in an *s*-cycle of S, they are not together in any *t*-cycle in T (and vice versa).

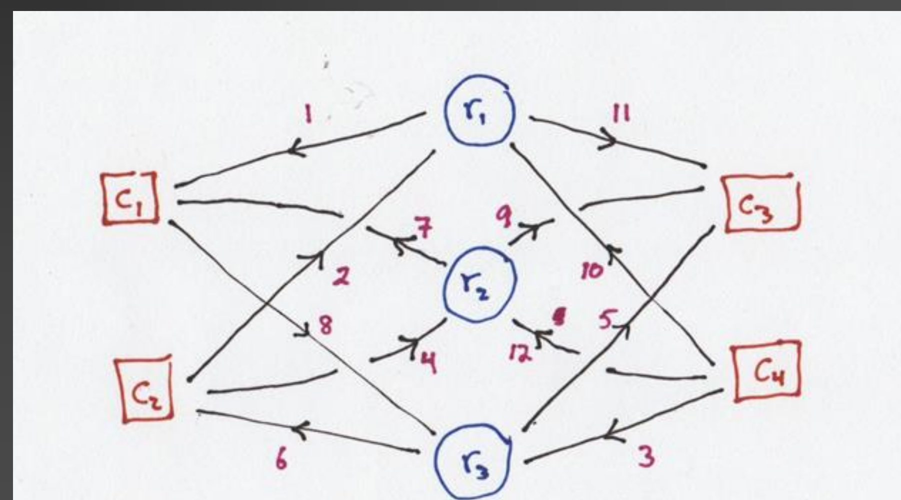
A Heffter *s*-system is orthogonal to a Heffter *t*-system if each set in the *s*-system intersects each set in the *t*-system exactly once. The existence is equivalent to an H(*s*,*t*).

Back to embeddings

Need to show that each vertex is ok. We use current graphs.

$$\begin{pmatrix} 1 & -2 & 11 & -10 \\ 7 & -4 & 9 & -12 \\ -8 & 6 & 5 & -3 \end{pmatrix}$$

Gives the current graph



This can be embedded on a surface with only one face.

Use the ordering of edges from that face to get the ordering of edges around each vertex.

The embedding theorem

Theorem: An H(s,t) (s,t not both even) creates an embedding of K_{2st+1} on an orientable surface with each face either an s-cycle or a t-cycle and each edge bordering exactly one s-cycle and one t-cycle provided the rows and the columns can be ordered with all partial sums distinct.

Current Graphs

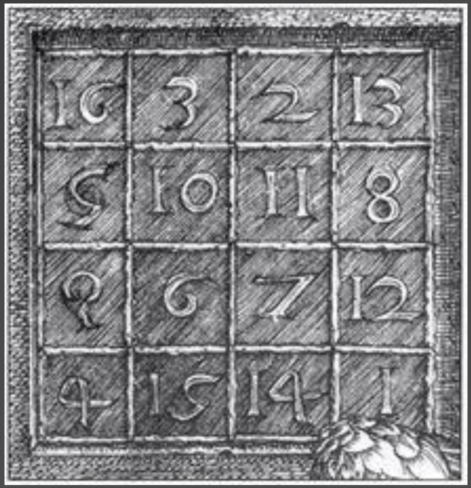
Graph G, each directed edge elt from grap Γ

1) $\alpha(e^{-1}) = \alpha(e)^{-1}$
2) all non-zero exactly once
3) @ each vertex net current in = net current out

Used in The Map Color Theorem to construct embeddings of complete graphs using lifts of embedded current graphs & branched coverings of Riemann Surfaces

Magic Squares: (a slight digression)

- $n \times n$ array with entries $1\ldots n^2$ such that the row and column sums are all the same number $n(n^2+1)/2$, called the magic constant. Usually required the two long diagonal sums are also this magic constant
- Two early magic squares: an iron plate from the Yuan Dynasty (1271-1368) and a detail from Melencolia I by Albrecht Durer (1541)



Construction of magic squares

- Oldest reference is from the 4th century BCE China, but legend dates it back to 23rd century BCE
- In the 13th century Islamic mathematicians gave several construction techniques
- Related to orthogonal Latin Squares
- Known to exist for all *n*
- For details see Section VI.34 “Magic Squares” by Joseph Kudrle and Sarah Menard in (where else) *The Handbook of Combinatorial Designs (Vol 2)*.
- ***So in some sense Heffter Arrays are “signed magic rectangles.”***

Necessary conditions for the existence of an H(s,t) (especially for integer sums)

- $s,t \geq 3$ (or else you get either 0 or both x, $-x$ in the array)
- Lemma: If an integer $s \times t$ Heffter array exists, then $s \times t \equiv 0,3 \pmod 4$
- Proof: Reduce the entries in the array modulo 2. Each row and column sums to 0 so it contains an even number of odd numbers. Hence the number of odds from 1,…,st must be even, giving the parity condition.

An H(4,4)

$$\begin{pmatrix} 1 & -2 & -3 & 4 \\ -5 & 6 & 7 & -8 \\ -9 & 10 & 11 & -12 \\ 13 & -14 & -15 & 16 \end{pmatrix}$$

Main Result # 1: Integer solutions

Theorem: There is an $s \times t$ integer Heffter array whenever $s,t \geq 3$ and $s \times t \equiv 0,3 \bmod 4$

The proof is constructive relying on a combination of difference techniques when s or t is small and recursive constructions for larger values.

Different constructions are used depending on congruence conditions.

Main result #2: Modulo solutions

Theorem: There is an $s \times t$ Heffter array modulo $2st+1$ for all $s,t \geq 3$.

The proof is similar to that of the first main result relying on a combination of difference and recursive constructions.

Conjecture: For all s,t there is a Heffter array where all but one row and one column sum to 0 in the integers.

(We may have this)

The easy case: s,t ≡ 0 (mod 4)

- Consider the 4 x 4 square *A* shown on the right with $\{ |a_{i,j}| \} = 1,\ldots,16$
- Add 16 to the magnitude of each entry, i.e.,
  $b_{i,j} = a_{i,j} + 16$ if $a_{i,j}$ is positive,
  $b_{i,j} = a_{i,j} - 16$ if $a_{i,j}$ is negative.
- Call the new array $B = A \pm 16$
- Since there are the same number of positive and negative entries in each row and column, the result *B* has row and columns sums equal to 0 and has entries ±17…32 .

$$\begin{pmatrix} 1 & -2 & -3 & 4 \\ -5 & 6 & 7 & -8 \\ -9 & 10 & 11 & -12 \\ 13 & -14 & -15 & 16 \end{pmatrix}$$

Define a *shiftable Heffter array* if all rows and columns have the same number of positive and negative entries. The H(4,4) shown is shiftable.

Fitting 4 × 4's to make an $s \times t$

Let's make an H(8,12).

Start with the shiftable H(4,4), A = $\begin{pmatrix} 1 & -2 & -3 & 4 \\ -5 & 6 & 7 & -8 \\ -9 & 10 & 11 & -12 \\ 13 & -14 & -15 & 16 \end{pmatrix}$

Now make the 8 × 12 array:

A	***A* ± 16**	***A* ± 32**
***A* ± 48**	***A* ± 64**	***A* ± 80**

or

1	-2	-3	4	17	-18	-19	20	33	-34	-35	36
-5	6	7	-8	-21	22	23	-24	-37	38	39	-40
-9	10	11	-12	-25	26	27	-28	-42	42	43	-44
13	-14	-15	16	29	-30	-31	32	45	-46	-47	48
49	-50	-51	52	65	-66	-67	68	81	-82	-83	84
-53	54	55	-56	-69	70	71	-72	-85	86	87	-88
-57	58	59	-60	-73	74	75	-76	-89	90	91	-92
61	-62	-63	64	77	-78	-79	80	93	-94	-95	96

Note that the table entries are 1 – 96 (in absolute value) and each row and column adds to 0.

Hence it is an integer H(8,12)

Next easiest: $s \equiv 0$, $t \equiv 2 \bmod 4$

The proof is similar. The array below is a shiftable 4×6 Heffter array.

1	-2	3	-4	11	-9
-7	8	-12	10	-5	6
-13	14	-15	16	-23	21
19	-20	24	-22	17	-18

We can piece together shifts of this 4×6 array and 4×4 arrays to cover the above congruence classes.

An example: An H(12,14)

Begin with A =

1	-2	3	-4	11	-9
-7	8	-12	10	-5	6
-13	14	-15	16	-23	21
19	-20	24	-22	17	-18

a shiftable H(4,6)

and B =

1	-2	-3	4
-5	6	7	-8
-9	10	11	-12
13	-14	-15	16

our shiftable H(4,4)

To get

	6	4	4
4	**A**	**B ± 72**	**B ±120**
4	A ± **24**	B ± **88**	B ±**136**
4	A ± **48**	B ±**104**	B ±**152**

This array has row and column sums equal to 0 and contains the symbols 1 .. 168 (in absolute value)

Hence it is an H(12,14)

Now assume $s,t \equiv 2 \bmod 4$

There is no shiftable H(6,6).

We start with the non-shiftable 6 x 6 aHeffter array, A, shown at the right. Use this to cover entries $|a_{i,j}| = 1,\ldots,36$.

1	2	3	4	5	-15
6	10	11	12	-13	-26
7	14	18	19	-22	-36
8	16	21	-32	-33	20
9	-17	-24	-30	28	34
-31	-25	-29	27	35	23

Pack with shiftable 4 × 6 and 4 × 4 arrays to fill out the s x t square.

We give an example.

Example: An H(18,14)

	6	4	4
6	**A**	**6 × 4 shiftable**	**6 × 4 shiftable**
4	**4 × 6 shiftable**	**4 × 4 shiftable**	**4 × 4 shiftable**
4	**4 × 6 shiftable**	**4 × 4 shiftable**	**4 × 4 shiftable**
4	**4 × 6 shiftable**	**4 × 4 shiftable**	**4 × 4 shiftable**

So we have constructed a tight integer Heffter array H(s,t) for all even values of $s,t \geq 4$.

Next we tackle ones with odd side.

$3 \times t$ Heffter arrays

Possible over the integers if $t \equiv 0,1 \bmod 4$.

Over the integers modulo $6t+1$, otherwise.

6t+1 sure looks familiar (think STS) – this is where the term Heffter arrays came from. It relates to Heffter's first difference problem from 1897.

Heffter's first difference problem

Can one partition the set $\{1,2,\ldots 3n\}$ into n triples $\{a,b,c\}$ such that

$$a + b = c$$

or $\quad a + b + c \equiv 0 \pmod{6n+1}$?

The answer is yes for all $n \geq 1$. Proved by Peltesohn (1939). Closely relates to Skolem sequences.

A solution to Heffter's first difference problem can be used to form a cyclic Steiner triple system (STS) of order 6t+1.

For each triple $\{a_i , b_i , c_i \}$ in the solution to Heffter's, construct the new triple

$$(0,\ a_i ,\ a_i + b_i).$$

The collection of these triples gives the base blocks for a cyclic STS(6n+1). Look familiar?

A Heffer array (3,t) is an arrangement of the triples solving Heffters first difference problem (on the set {±1, ±2,… ±3*t*}) into a 3 × *t* array such that the triples are in the columns and each row sum is 0 (mod 6*t*+1).

Wow!

Below is an integer H(3,9)

7	**12**	**18**	**6**	**3**	**-5**	**-13**	**-27**	**-1**
10	**9**	**-14**	**-26**	**22**	**16**	**-2**	**8**	**-23**
-17	**-21**	**-4**	**20**	**-25**	**-11**	**-15**	**19**	**24**

Note that a column containing a,b and c has a + b = -c or a + b = c.

The rows add to 0.

Theorem: Heffter Arrays H(3,*t*) exist for all $t \geq 3$.

Wow!

Here's an integer H(3,12) coming from a solution to Heffter's difference problem

$$\begin{pmatrix} 13 & 16 & 34 & -5 & 7 & -35 & -18 & -1 & -23 & -12 & 21 & 3 \\ 6 & 11 & -8 & 33 & -22 & 10 & -2 & -30 & 14 & -24 & -17 & 29 \\ -19 & -27 & -26 & -28 & 15 & 25 & 20 & 31 & 9 & 36 & -4 & -32 \end{pmatrix}$$

Construction method

Use the known solutions to Heffter's difference problem (coming from Skolem sequences) and do a pretty long computer search to sort out an appropriate row pattern.

Got a program for that!

Run program for $3 \times n$

Sample code for finding 3 x t, $t \equiv 5 \bmod 8$

```
(* The case t = 8m + 5, any nonnegative m *)

heffter5[m_] := Module[{partial} (* the local variables *),
  (* start with the sporadic blocks *)

  partial :=
                  {{8*m + 6, -16*m - 9, 8*m + 3},
                   {10*m + 7, 8*m + 5, -18*m - 12},
                  {-16*m - 10, 4*m + 2, 12*m + 8},
                  {-4*m - 4, -18*m - 11, 22*m + 15},
                  {4*m + 1, 18*m + 13, -22*m - 14}};
  (* and add in the 4 infinite classes *)
  Do[partial = Join[partial,
                 {{-(8*m - 2*r + 1)*(-1)^r , (16*m - r + 8)*(-1)^r, -(8*m + r + 7)*(-1)^r}},
                 {{-(14*m - r + 8)*(-1)^r, 2*(2*m - r)*(-1)^r , (10*m + r + 8)*(-1)^r}},
                 {{(16*m + r + 11)*(-1)^r, 2*(4*m - r + 2)*(-1)^r, -(24*m - r + 15)*(-1)^r}},
                 {{(4*m - 2*r - 1)*(-1)^r , (18*m + r + 14)*(-1)^r, -(22*m - r + 13)*(-1)^r}}]
                 , {r, 0, 2 m - 1}];
  partial ] (* end \
module *)
```

Just showing off

150	101	63	48	25	49	-47	65	-137	-46	45	-66	136	44	-43	67	-135	-42	41	-68	134	40	-39	69	-133
149	-51	-64	76	100	77	99	23	24	-102	-98	-21	-22	103	97	19	20	-104	-96	-17	-18	105	95	15	16
2	-50	1	-124	-125	-126	-52	-88	113	148	53	87	-114	-147	-54	-86	115	146	55	85	-116	-145	-56	-84	117

-38	37	-70	132	36	-35	71	-131	-34	33	-72	130	32	-31	73	-129	-30	29	-74	128	28	-27	75	-127	-26
-106	-94	-13	-14	107	93	11	12	-108	-92	-9	-10	109	91	7	8	-110	-90	-5	-6	111	89	3	4	-112
144	57	83	-118	-143	-58	-82	119	142	59	81	-120	-141	-60	-80	121	140	61	79	-122	-139	-62	-78	123	138

This is an H(3,50) -- check it ☺

Can get all H(s,t) with $s \equiv 3$ and $t \equiv 0 \bmod 4$

Similar method as before. Place the H(3,t) on top and fill in with shiftable H(4,4).

Example: An H(11,12)

3	**H(3,12)**		
4	H(4,4) shiftable	H(4,4) shiftable	H(4,4) shiftable
4	H(4,4) shiftable	H(4,4) shiftable	H(4,4) shiftable
	4	4	4

The $5 \times t$ case

Theorem: There exist $5 \times t$ Heffter arrays for all $t \geq 3$.

Proof: An 8-part difference construction depending on t mod 8 with some small sporadics. Even more searching on the computer for general patterns.

The beginning of a proof:

-------- Original Message --------

Subject: 5xn
Date: Mon, 4 Nov 2013 10:23:33 -0800 (PST)
From: Tom Boothby <tboothby@sfu.ca>
To: Dan Archdeacon <darchdea@uvm.edu>

Dan,

I have a potential construction for 5x4n. As always, I'll share some sketchy details and rough them out over the next few days, hopefully into a proper construction.

Recall that a k-near Skolem sequence has pairs of all lengths except k. So a 1-near Skolem sequence of length 2n has pairs of lengths (2,3,4,...,n+1).

Take your favorite 1-near Skolem sequence of length 8n. For example, with n=4,

34532425

now starting with n+3, fill in a new row:

```
3  4  5  3  2  4  2  5
7  8  9 10 11 12 13 14
```

now, add 1 to everybody in the top row:

```
4  5  6  4  3  5  3  6
7  8  9 10 11 12 13 14
```

and proceed to make a 3xn array as before,

```
 3  4  5  6
11  7  8  9
13 10 12 14
```

and notice that we've partitioned (3,4,...,3n+2) into triples of the form (a,b,c) with a+b-c = 1. The remaining symbols are (1,2,3n+3,3n+4,...,5n)... so pair them off at distance 1 to finish the rows

```
 3  4  5  6
11  7  8  9
13 10 12 14
 1 15 17 19
 2 16 18 20
```

with signs...

```
  3   4   5   6
 11   7   8   9
-13 -10 -12 -14
  1  15  17  19
 -2 -16 -18 -20
```

Turned into a constructive proof

Here is a solution to the case where $n = 8m+7$. Each 5-tuple (a,b,c,d,e) adds to 0 and all other conditions are satisfied too (note a+b+c = ±1).

Each 5 tuple below generates *m* columns in the H(5,*n*).

```
Do[partial = Join[partial,
    {{-4*r+4*m+1, 2*r+18*m+21, 2*r-22*m-21, 2*r+24*m+36, -2*r-24*m-37}
 {4*r-8*m-4, -2*r-16*m-19, -2*r+24*m+22, -2*r-26*m-36, 2*r+26*m+37},
 {-4*r+4*m+2, 2*r+10*m+13, 2*r-14*m-14, 2*r+28*m+36, -2*r-28*m-37},
 {4*r-8*m-5, -2*r-8*m-11, -2*r+16*m+15, -2*r-30*m-36, 2*r+30*m+37},
 {4*r-4*m+1, -2*r-18*m-22, -2*r+22*m+20, -2*r-32*m-36, 2*r+32*m+37},
 {-4*r+8*m+2, 2*r+16*m+20, 2*r-24*m-21, 2*r+34*m+36, -2*r-34*m-37},
 {4*r-4*m, -2*r-10*m-14, -2*r+14*m+13, -2*r-36*m-36, 2*r+36*m+37},
 {-4*r+8*m+3, 2*r+8*m+12, 2*r-16*m-14, 2*r+38*m+36, -2*r-38*m-37}}
        ]
    , {r, 0, m-1}];
```

A few more special columns have to be added.

```
   {{8*m+9, 10*m+11, -18*m-19, 1, -2},
 {-4*m-5, 16*m+17, -12*m-13, -24*m-24, 24*m+25},
 {24*m+27, -24*m-26, -8*m-8, 20*m+21, -12*m-14},
 {-4*m-3, 22*m+22, 24*m+29, -18*m-20, -24*m-28},
 {-24*m-31, -24*m-23, 8*m+6, 16*m+18, 24*m+30},
 {-24*m-32, 24*m+33, 14*m+15, -10*m-12, -4*m-4},
 {24*m+35, -24*m-34, -8*m-10, 16*m+16, -8*m-7}};
```

H(*s*,*t*) with s ≡ 1 and t ≡ 0 mod 4

Theorem: There exist an integer *s* × *t* Heffter array H(s,t) for all s ≡ 1, t ≡ 0 mod 4 (s ≥ 5).

Proof: Again place the H(5,*t*) on top and fill in with shiftable H(4,4). Use induction on *s* in steps of size 4 by adding shiftable 4 × 4 arrays.

Last case for integer Heffter arrays: $s \equiv 1 \bmod 4$, $t \equiv 3 \bmod 4$

Use an “ell” construction with a border of 5 rows and 3 columns and fill in with shiftable H(4,4).

3

5

<table>
<tr><td rowspan="6">A cool pattern here</td><td colspan="6">that satisfies the needed properties</td></tr>
<tr><td>H(4,4) shiftable</td><td>H(4,4) shiftable</td><td>H(4,4) shiftable</td><td>H(4,4) shiftable</td><td>H(4,4) shiftable</td><td>H(4,4) shiftable</td></tr>
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The non-integer (modular) congruence classes

The remaining congruence classes can not have all rows and columns adding to 0, some must add to 0 mod 2st+1.

These cases are:

- $s \equiv t \equiv 1 \bmod 4$, 5 rows and 5 col. border with shiftable H(4,4)
- $s \equiv 1, t \equiv 2 \bmod 4$ H(5,s) above shiftable H(4,6) and H(4,4)
- $s \equiv 3, t \equiv 2 \bmod 4$ H(3,s) above shiftable H(4,6) and H(4,4)
- $s \equiv 3, t \equiv 3 \bmod 4$ 3 rows and 3 col. border with shiftable H(4,4)

A computer program

Tom Boothby wrote a program in python which finds tight H(*s*,*t*) for all *s*,*t* $\geq$ 3.

A demo

The Theorems:

Theorem 1: There is an $s \times t$ integer Heffter array whenever, $s,t \geq 3$ and $s \times t \equiv 0,3 \bmod 4$

Theorem 2: There is an $s \times t$ Heffter array modulo $2st+1$ for all $s,t \geq 3$.

Conjecture: For all s,t there is a Heffter array $H(s,t)$ where all but one row and all but one column sum to 0 in the integers.

Remember embeddings

The last condition was the partial sum condition: *It must also be the case that the partial sums of each row and column are all distinct.*

Do our Heffter arrays satisfy this??

NO!!

Not even close as each shiftable H(4,4) and H(4,6) adds to 0.

Knowing the exact structure of the solutions we think that we can indeed find orderings of the row and columns that satisfy the partial sum condition.

But the question leads to some very interesting general conjectures about sequencing subsets of $\mathbb{Z}_n$.

We'd love a big theorem here.

Partial sums in cyclic groups

Let $A \subseteq \mathbb{Z}_n \setminus\{0\}$ with $|A| = k$. Let $(a_1, a_2, \ldots a_k)$ be an ordering of the elements of *A*. The partial sums are $s_j = \sum_{i=1}^{j} a_i$ (arithmetic all in $\mathbb{Z}_n$).

Say that *A* is sequenceable if it can be ordered so that the partial sums are distinct.

Example: *n* = 10, *k* = 6, *A* = {1,2,4,6,7,8}.

Note that (1,2,4,6,7,8) is not a sequencing of A, but (1,2,4,7,6,8) has partial sums (1,3,7,4,0,8) and hence *A* is sequenceable.

Sequenceable groups

Well studied, but no one seems to have looked at sequencing arbitrary subsets of groups.

A *sequencing* of a group G of order n is an ordering $a_0 = e, a_1, a_2, \ldots, a_{n-1}$ of the elements of G such that all of the partial products $b_0 = a_0 = e$, $b_1 = a_0 a_1$, $b_2 = a_0 a_1 a_2$ $, \ldots, b_{n-1} = a_0 a_1 a_2 \cdots a_{n-1}$ are distinct. A group is *sequenceable* if it possesses a sequencing. The sequence $(b_0, b_1, \ldots, b_{n-1})$ is a *basic directed terrace.*

Since the first element is *e*, the entire group can't add to 0 since that would be both the first and last partial sum. So there must be an element of order 2. The following covers all abelian groups.

Theorem (B. Gordon, 1961): An abelian group is sequenceable if and only if it has a unique element of order 2.

It is conjectured that all nonabelian groups of order at least 10 are sequenceable (none less than 10 are).

This conjecture has been proven in the following cases:

1. All nonabelian groups of order n, $10 \leq n \leq 32$.
2. A_5, S_5, and all dihedral groups D_n of order at least 10.
3. Solvable groups with a unique element of order 2.
4. Some nonsolvable groups with a unique element of order 2.
5. Some groups of order pq, p, and q odd primes.

An *R-sequencing* of a group G of order n is an ordering $a_0 = e, a_1, a_2, \ldots, a_{n-1}$ of the elements of G such that the partial products $b_0 = a_0 = e$, $b_1 = a_0 a_1$, $b_2 = a_0 a_1 a_2$, …, $b_{n-2} = a_0 a_1 a_2 \cdots a_{n-2}$ are distinct and $a_0 a_1 a_2 \cdots a_{n-1} = e$. A group is *R-sequenceable* if it possesses an R-sequencing.

Theorem The following groups are R-sequenceable.

1. $\mathbb{Z}_n$, n odd.
2. Abelian groups of order n with $gcd(n, 6) = 1$.
3. D_n, n even.
4. Q_{2n} if $n + 1$ is a prime of the form $4k + 1$, for which -2 is a primitive root.
5. Nonabelian groups of order pq, $p < q$ odd primes, with 2 a primitive root of p.

A nice reference for this (besides the Handbook) is
Matt Ollis, Sequenceable Groups and Related Topics, *The Electronic Journal of Combinatorics* 20(2) (2013)

We present some conjectures (and results) on sequencing arbitrary subsets of $\mathbb{Z}_n \setminus \{0\}$. Any of these would satisfy our condition for the Heffter array to give an embedding *s*-cycles and *t*-cycles.

Conjecture 1

For any $A \subseteq \mathbb{Z}_n \setminus \{0\}$, A is sequenceable.

We have checked this conjecture for every subset of $\mathbb{Z}_n \setminus \{0\}$ up to $n = 25$.

It's true ☺

We also prove it true for $k \leq 5$ (all n).

We now give successively weaker conjectures that would still solve our problem for Heffter arrays.

Conjecture 2

Conjecture 1 holds with the additional condition that $\sum_{a \in A} a = 0$.

This is the row and column sum of a Heffter array.

Conjecture 3

Conjecture 2 (or Conjecture 1) holds with the additional condition that $a_i \neq -a_j$ for any two elements of *A* and $k \leq (n-1)/2$.

Again this is true for all Heffter arrays.

Conjecture 4

Conjecture 3 is true when n is odd and $k \leq (n-1)/6$.

This is the maximum number of symbols that can be in any row or column of a Heffter array. So this would also solve the sequencing problem for Heffter arrays.

As a start: find a c such that if $k \leq cn$, then Conjecture 1 holds.

A probabilistic result

Theorem: Let *A* be a randomly chosen *k*-subset of $\mathbb{Z}_n \setminus \{0\}$. Then the probability that *A* can not be sequenced is at most $\binom{k}{2} \times \frac{2}{n}$.

Basically there are $\binom{k}{2}$ "runs" and each has probability of at most $\frac{2}{n}$ that it's sum is 0.

It follows that if $k \approx \sqrt{n/2}$, then the probability that a randomly chosen k −subsest of $\mathbb{Z}_n \setminus \{0\}$ is sequenceable is at least ½.

Other Heffter arrays

A Heffter s-system S and a Heffter t-system T are weakly orthogonal if any subset of S and any subset in T intersect in a unique element up to sign.

Example:

$$\begin{pmatrix} 1 & -7 & 12 & -6 \\ 2 & -4 & \mp 8 & \pm 10 \\ -3 & \mp 11 & 5 & \pm 9 \end{pmatrix}$$

Mod 25

Use upper sign in row sums, lower sign in column sums

Theorem: This gives an embedding of the complete graph with s-cycles and t-cycles on a nonorientable surface.

Other Heffter Arrays

Say that a Heffter s-system S and a Heffter t-system T are sub-orthogonal if any subset of S and any subset in T intersect in at most one element. (So not tight).

Example:

$$\begin{pmatrix} 1 & -5 & 24 & -20 & & & & \\ 10 & -9 & \mp 16 & & \pm 15 & & & \\ -11 & \mp 14 & 8 & & & 17 & & \\ & & & & 3 & \mp 6 & 22 & -19 \\ & & & 13 & 12 & & -4 & \mp 21 \\ & & & 7 & & -23 & \pm 18 & -2 \end{pmatrix}.$$

Mod 49

This array embeds K_{49} in a nonorientable surface where each edge bounds a 3-cycle and a 4-cycle.

Found last week! (by Diane Donovan)

D =

	17	-8	-14	5
1		18	-9	-10
-6	2		19	-15
-11	-12	3		20
16	-7	-13	4	

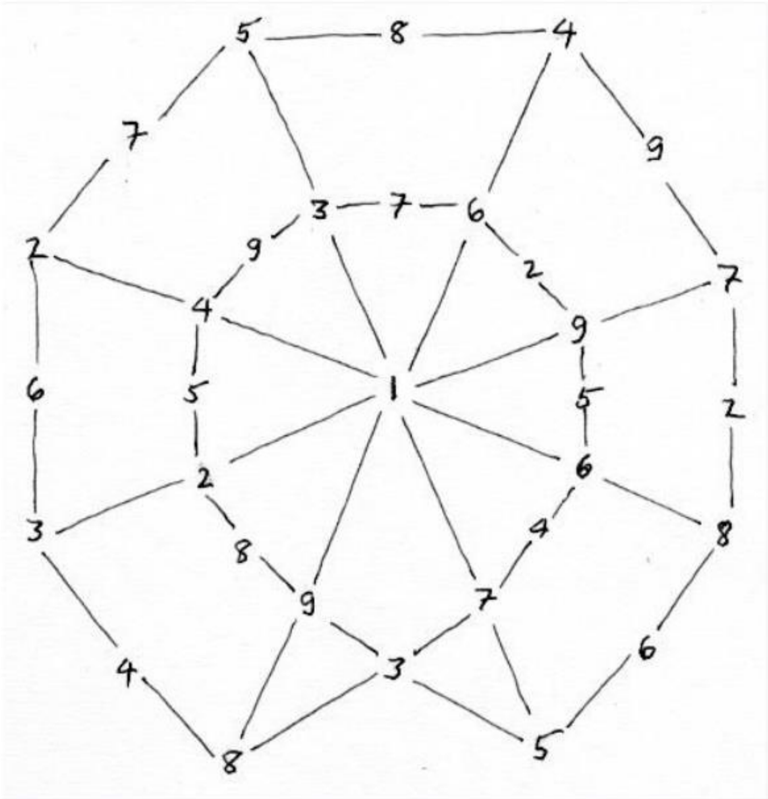
It is a shiftable 5×5 Heffter array with 4 filled cells in each row and each column. There are two positive and two negative numbers in each row and column.

We can use this to get an $n \times n$ Heffter array with $4k$ filled cells in each row and each column for all k *and all* $n \geq 4k$.

A 9 × 9 Heffter Array with 8 filled cells in each row and each column

	17	-8	-14	21	-22	-23	24	5
1		18	-9	-25	26	27	-28	-10
-6	2		19	-29	30	31	-32	-15
-11	-12	3		33	-34	-35	36	20
37	-38	-39	40		69	-60	-66	57
-41	42	43	-44	53		70	-61	-62
-45	46	47	-48	-58	54		71	-67
49	-50	-51	52	-63	-64	55		72
16	-7	-13	4	68	-59	-65	56	

A bonus picture (courtesy of Tom Johnson)



(9,3,2) design #36

Thanks!!