An $n-$component face-cubic model on the complete graph

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Collaborators

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Outline

- Brief introduction to lattice models and phase transitions
- Large deviations theory
- An n-component face-cubic model
- Limit theorems for the face-cubic model on the complete graph
- Conclusion

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Ising model

- Graph $G = (V, E)$
- Assign a random variable W_i on i, for $i \in V$
- W_i takes values in a state space $\Sigma = \{1, -1\}$
- Configuration $\omega = \{W_1 = \omega_1, W_2 = \omega_2, \cdots, W_N = \omega_N\} \in \Sigma^N$, where $N = |V|$.
- The Ising model is defined by choosing configurations ω randomly via Gibbs measure

$$\pi(\omega) = \frac{e^{-H(\omega)/T}}{Z_N(T)}\ , \quad \omega \in \Sigma^N$$

- Hamiltonian (energy) $H(\omega)$

$$H(\omega) = -\sum_{ij \in E} \omega_i \cdot \omega_j$$

- Partition sum $Z_N(T)$

$$Z_N(T) = \sum_{\omega \in \Sigma^N} e^{-H(\omega)/T}$$

High and low temperature phases

- Recall Gibbs measure

$$\pi(\omega) = \frac{e^{-H(\omega)/T}}{Z_N(T)}\,, \quad H(\omega) = -\sum_{ij \in E} \omega_i \cdot \omega_j$$

High and low temperature phases

- Recall Gibbs measure

$$\pi(\omega) = \frac{e^{-H(\omega)/T}}{Z_N(T)}, \quad H(\omega) = -\sum_{ij \in E} \omega_i \cdot \omega_j$$

- Relative weight for two configurations ω, ω'

$$\frac{\pi(\omega)}{\pi(\omega')} = e^{-\left(H(\omega) - H(\omega')\right)/T}$$

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High and low temperature phases

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- Relative weight for two configurations ω, ω'

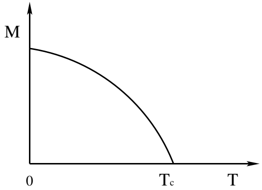
$$\frac{\pi(\omega)}{\pi(\omega')} = e^{-\left(H(\omega)-H(\omega')\right)/T}$$

- If T is low, spins prefer to like their neighbours, which is called ordered phase or low temperature phase.
- If T is high, spins are independent of each other, which is called disordered phase or high temperature phase.
- A critical point at $T = T_c$.

Order parameter

- Order parameter is used to quantitatively characterise phase transitions.
- For Ising model, the order parameter is the magnetisation,

$$M = \left\langle \left| \frac{\sum_{i=1}^{N} W_i}{N} \right| \right\rangle$$



- Critical behaviors
  - If $T \geq T_\mathrm{c}$, $M = 0$
  - If $T \to T_\mathrm{c}^-$, $M \sim (T_\mathrm{c} - T)^\beta$
- The other independent critical exponent is defined from correlation length $\xi \sim |T - T_\mathrm{c}|^{-\nu}$

Phase transitions classification

- Phase transitions are classified by the continuity of the order parameter.
- First order phase transition (discontinuous): ice-liquid-gas transition, phase coexistence.
- Continuous phase transition: ferromagnetic-paramagnetic transition, superconducting transition, Kosterlitz-Thouless transition.

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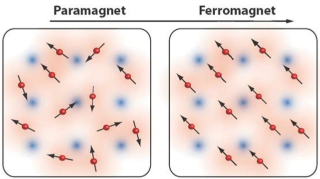

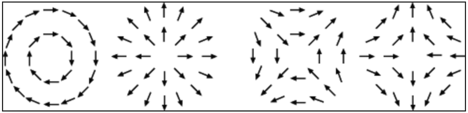

Other Important concepts

- Phase transitions happen only in thermodynamic limit
- Ensemble hypothesis: approximate time average by ensemble average
- Universality class: various continuous phase transitions fall into several universality class, in which all models have the same critical phenomena, and share same critical exponents.

Outline

Cramér's theorem

- Consider a sequence of identically and independently distributed random variables:

$$X_1, X_2, \cdots, X_N$$

- State space $\Sigma = \{a_1, a_2, \cdots, a_m\}$, $a_i \in \mathbb{R}^d$, $d \in \mathbb{N}^+$
- X_i is distributed according to a law μ and $\mathbb{E}(X_i) = \overline{X}$.
- Sample mean $S_N = \frac{1}{N} \sum_{i=1}^{N} X_i$
- Law of large numbers tells $S_N \to \overline{X}$ as $N \to +\infty$.
- What's the probability that $S_N = x$ with x deviating far from $\overline{X}$?

- Cramér's theorem

$$P_N(S_N = x) \sim e^{-NI(x)}\ , \text{ as } N \to +\infty$$

- Logarithmic generating function $\lambda(k)$, for any $k \in \mathbb{R}^d$,

$$\lambda(k) = \log \mathbb{E}[e^{k \cdot S_N}]$$

- Rate function from Legendre-Fenchel transform

$$I(x) = \sup_{k \in \mathbb{R}^d} \{\langle k, x \rangle - \lambda(k)\}$$

- $I(x)$ is convex, non-negative and $\min_x I(x) = 0$
- Set $\{x : I(x) = 0\}$ is called the most probable macroscopic states

Outline

Face-cubic model

- Given $G = (V, E)$.
- Assign a random variable W_i on i, for $i \in V$.
- W_i takes values in a state space Σ.
- State space

$$\begin{aligned}\Sigma = \quad & \{(\pm 1, 0, 0, \cdots, 0), \\ & (0, \pm 1, 0, \cdots, 0), \\ & \vdots \\ & (0, 0, \cdots, 0, \pm 1)\} \quad \subset \mathbb{R}^n\end{aligned}$$

- E.g.
  If $n = 3$, $\Sigma = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$
- Configuration $\omega = \{W_1 = \omega_1, W_2 = \omega_2, \cdots, W_N = \omega_N\} \in \Sigma^N$, where $N = |V|$.

- Choose configurations in Σ^N randomly via Gibbs measure

$$\pi(\omega) = \frac{e^{-\beta H(\omega)}}{Z_N(\beta)} \ , \quad \omega \in \Sigma^N$$

- $\beta = 1/T$
- Hamiltonian (energy) $H(\omega)$

$$H = -\sum_{ij} \langle \omega_i, \omega_j \rangle$$

- Partition sum $Z_N(\beta)$

$$Z_N(\beta) = \sum_{\omega \in \Sigma^N} e^{-\beta H(\omega)}$$

- High temperature, W_i uniformly distributed in Σ.
- Low temperature, W_i prefer to like their neighbors.
- β_c-Critical point

Known results

Square lattice (Nienhuis et al 1982), face-cubic model $\sim$

- O(n) model (n-vector model) for $0 \leq n < 2$
- Ashkin-Teller model for $n = 2$
- First-order transition for $n > 2$

Mean-field (or complete graph) (Kim et al, 1975)

- $n = 1, 2$, continuous (Ising)
- $n > 3$, first-order
- $n = 3$, continuous(tricritical)
- $n = 3$, first-order (Kim and Levy, 1975)

Outline

Probability distribution of S_N under Gibbs measure

- On the complete graph, Hamiltonian

$$\begin{aligned} H(\omega) &= -\frac{1}{2N}\sum_{i,j=1}^{N}\langle\omega_i,\omega_j\rangle \\ &= -\frac{1}{2}NS_N^2(\omega) \end{aligned}$$

- Probability distribution of S_N in n-dimensional cube $\Omega=[-1,1]^n$?
- Assume $P_N^{\beta}(S_N=x)\sim e^{-NI_\beta(x)}$, what is the rate function $I_\beta(x)$?

Derive rate function

- $$P_N^{\beta}(S_N = x) \doteq \frac{1}{Z_N(\beta)} \sum_{\{\omega\in\Sigma^N : S_N(\omega)=x\}} \exp[-\beta H(\omega)]$$
  $$= \frac{1}{Z_N(\beta)} \exp[\beta N x^2/2] P(S_N = x)$$
- Rate function
  $$I_\beta(x) = -\lim_{N\to+\infty} \frac{1}{N} \log P_N^{\beta}(S_N = x)$$
  $$= I(x) - \frac{\beta}{2}x^2 - \min_{x\in\Omega}[I(x) - \beta x^2/2]$$
- Only need to find the global minimum points of
  $$I(x) - \frac{\beta}{2}x^2$$
  in the n-dimensional cube $[-1,1]^n$

- A useful convex duality

$$\min_{x\in\Omega}[I(x) - \frac{\beta}{2}\langle x, x\rangle] = \min_{u\in\mathbb{R}^n}[\frac{1}{2\beta}\langle u, u\rangle - \lambda(u)]$$

- For face-cubic model

$$\lambda(u) = \ln\sum_{i=1}^{n}\cosh(u_i)$$

- Find the global minimum points of

$$G_\beta(u) = \frac{1}{2\beta}\langle u, u\rangle - \ln\sum_{i=1}^{n}\cosh(u_i)\ , \quad \text{with } u \in \mathbb{R}^n$$

Lemma

Let $\overline{\nu}$ be a global minimum point of $G_\beta(u)$, then $\overline{\nu}$ is one of the following $(2n+1)$ vectors.

$$\begin{aligned}
\nu_0 &= (0,0,0,\cdots,0) \\
\nu_1 &= (a,0,0,\cdots,0) \\
\nu_2 &= (0,a,0,\cdots,0) \\
&\vdots \\
\nu_n &= (0,0,\cdots,0,a) \\
\nu_{n+i} &= -\nu_i, i=1,2,\cdots,n
\end{aligned}$$

$0 < a < 1$

$$G_\beta(u=\overline{\nu}) = \frac{1}{2\beta}a^2 - \ln[\cosh(a)+n-1]$$

Theorem

1. *Let* $A \subseteq \mathbb{R}^n$. *For* $1 \leq n \leq 3$,

$$P_N^\beta(S_N \in A) \sim \begin{cases} \delta_{\nu_0}(A) & \text{for } 0 < \beta \leq n \\ \dfrac{1}{2n}\sum_{i=1}^{2n} \delta_{\nu_i}(A) & \text{for } \beta > n \end{cases}$$

as $N \to +\infty$.

2. *For* $n \geq 4$,

$$P_N^\beta(S_N \in A) \sim \begin{cases} \delta_{\nu_0}(A) & \text{for } 0 < \beta < \beta' \\ \lambda_0\delta_{\nu_0}(A) + \lambda_1 \sum_{i=1}^{2n} \delta_{\nu_i}(A) & \text{for } \beta = \beta' \\ \dfrac{1}{2n}\sum_{i=1}^{2n} \delta_{\nu_i}(A) & \text{for } \beta > \beta' \end{cases}$$

as $N \to +\infty$, *with*

$$\lambda_0 = \frac{\kappa_0}{\kappa_0 + 2n\kappa_1}\,, \qquad \lambda_1 = \frac{\kappa_1}{\kappa_0 + 2n\kappa_1}\,,$$

$$\kappa_0 = \left(\det \mathrm{D}^2 G_{\beta_c}(\nu_0)\right)^{-1/2}, \quad \kappa_1 = \left(\det \mathrm{D}^2 G_{\beta_c}(\nu_1)\right)^{-1/2}.$$

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- Rigorously study n-component face-cubic model on the complete graph.
- By large deviations analysis, we derive $P_N^{\beta}(S_N = x) \sim e^{-NI_\beta(x)}$ and explicit form of $I_\beta(x)$.
- For $1 \leq n \leq 3$, continuous phase transition at $\beta_c = n$.
- For $n \geq 4$, first-order phase transition at $\beta_c = \beta'$.

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Many thanks for your attention!