Minors and Tutte invariants for alternating dimaps

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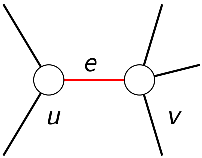
Work done partly at: Isaac Newton Institute for Mathematical Sciences (Combinatorics and Statistical Mechanics Programme), Cambridge, 2008; University of Melbourne (sabbatical), 2011; and Queen Mary, University of London, 2011.

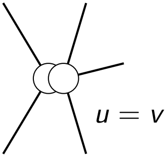
20 March 2014

Contraction and Deletion

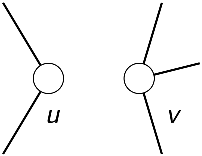
G



G/e



$G \setminus e$



Minors

H is a **minor** of G if it can be obtained from G by some sequence of deletions and/or contractions.

The order doesn't matter. Deletion and contraction **commute**:

$$
\begin{aligned}
G/e/f &= G/f/e \\
G \setminus e \setminus f &= G \setminus f \setminus e \\
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Importance of minors:

- excluded minor characterisations
  - planar graphs (Kuratowski, 1930; Wagner, 1937)
  - graphs, among matroids (Tutte, PhD thesis, 1948)
  - Robertson-Seymour Theorem (1985–2004)
- counting
  - Tutte-Whitney polynomial family

Duality and minors

Classical duality for embedded graphs:

$$\begin{array}{rcl} G & \longleftrightarrow & G^* \\ \text{vertices} & \longleftrightarrow & \text{faces} \end{array}$$

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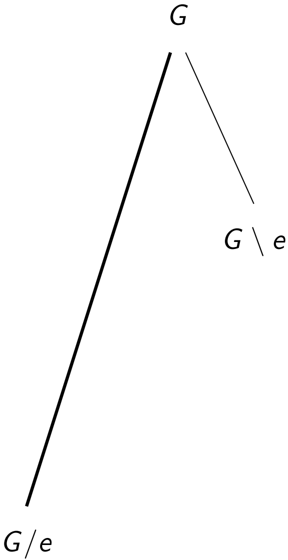
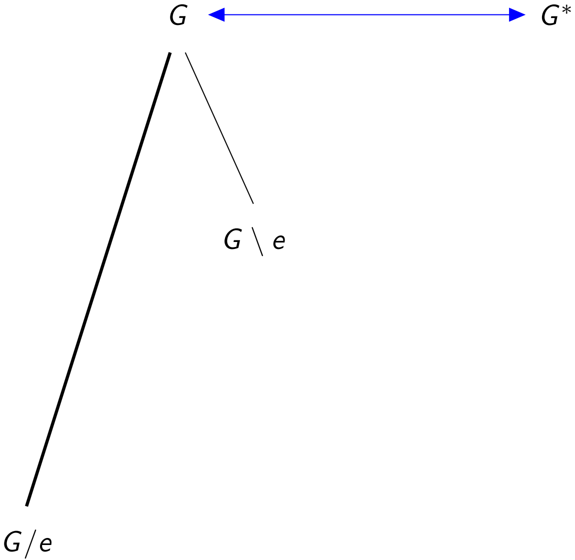
$$\text{contraction} \quad \longleftrightarrow \quad \text{deletion}$$

$$\begin{array}{rcl} (G/e)^* & = & G^* \setminus e \\ (G \setminus e)^* & = & G^*/e \end{array}$$

Duality and minors



Duality and minors



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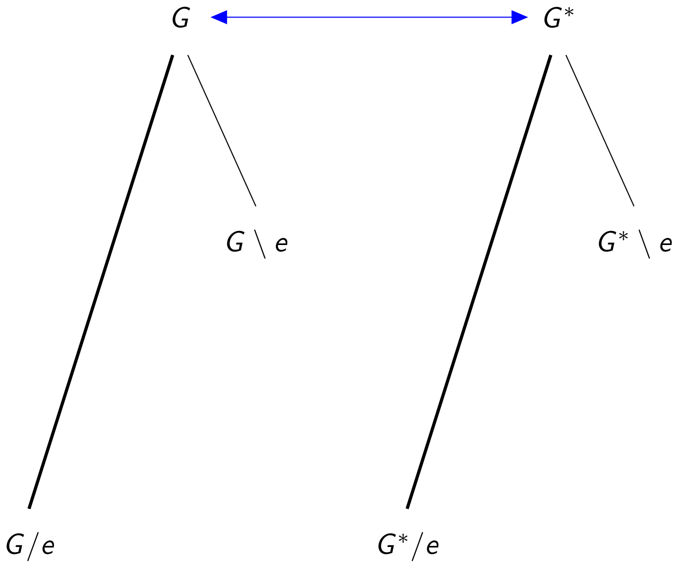

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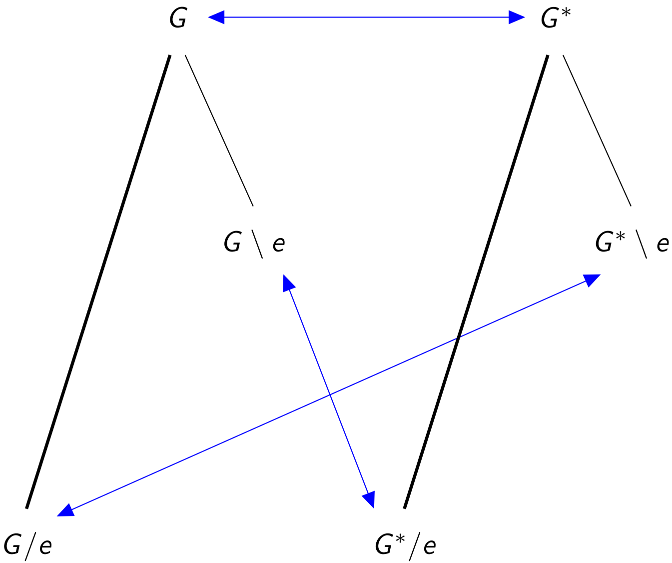
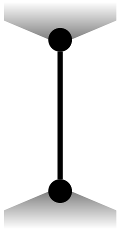


Loops and coloops

loop

coloop = bridge = isthmus



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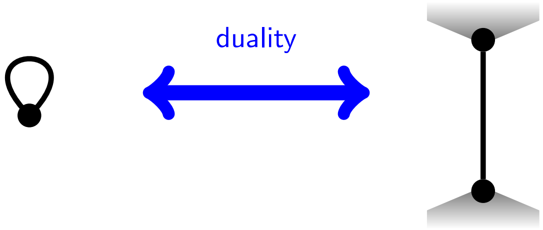
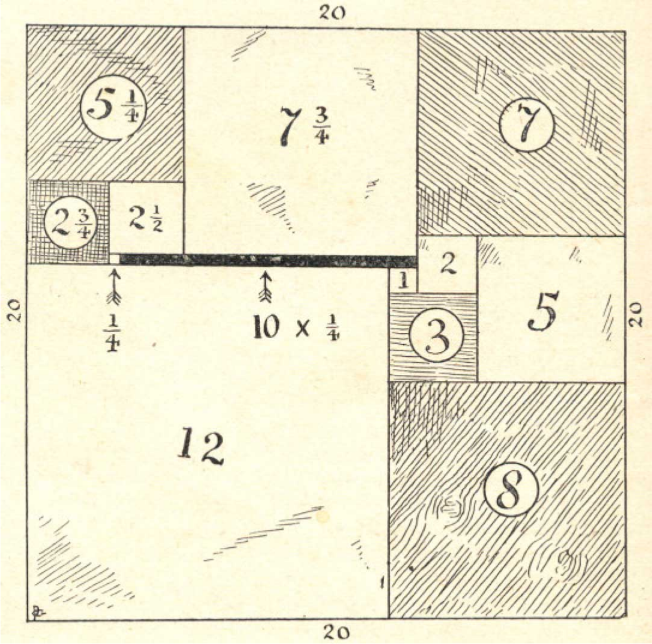


History

WHEN YOUNG MEN SUED FOR THE HAND OF LADY ISABEL, SIR HUGH PROMISED HIS CONSENT TO THE ONE WHO WOULD TELL HIM THE DIMENSIONS OF THE TOP OF THE CASKET.

H. E. Dudeney, Puzzling Times at Solvamhall Castle: Lady Isabel's Casket, *London Magazine* **7** (42) (Jan 1902) 584

History



London Magazine
8 (43) (Feb 1902) 56

History

THE

CANTERBURY PUZZLES

AND OTHER CURIOUS PROBLEMS

BY

HENRY ERNEST DUDENEY

AUTHOR OF

"AMUSEMENTS IN MATHEMATICS," ETC.

First published by Heinemann, London, 1907. Above is from 4th edn, Nelson, 1932.

History

THE DISSECTION OF RECTANGLES INTO SQUARES

By R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte

Introduction. We consider the problem of dividing a rectangle into a finite number of non-overlapping squares, no two of which are equal. A dissection of a rectangle R into a finite number n of non-overlapping squares is called a *squaring* of R of *order* n; and the n squares are the *elements* of the dissection. The term "elements" is also used for the lengths of the sides of the elements. If there is more than one element and the elements are all unequal, the squaring is called *perfect*, and R is a *perfect rectangle*. (We use R to denote both a rectangle and a particular squaring of it.) Examples of perfect rectangles have been published in the literature.[1]

Our main results are:

Every squared rectangle has commensurable sides and elements.[2] (This is (2.14) below.)

Conversely, every rectangle with commensurable sides is perfectible in an infinity of essentially different ways. (This is (9.45) below.) (**Added in proof.** Another proof of this theorem has since been published by R. Sprague: Journal für Mathematik, vol. 182(1940), pp. 60–64; Mathematische Zeitschrift, vol. 46(1940), pp. 460–471.)

In particular, we give in §8.3 a perfect dissection of a square into 26 elements.[3]

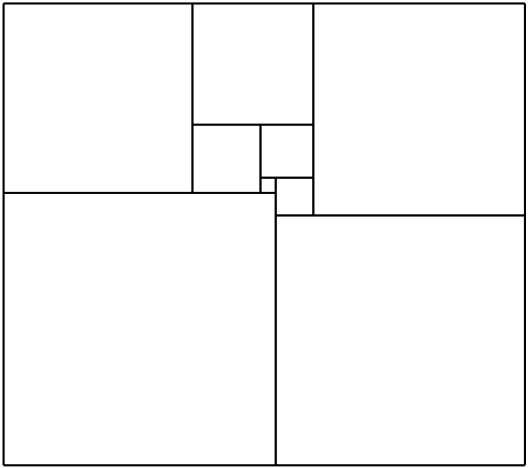
There are no perfect rectangles of order less than 9, and exactly two of order 9.[4] (This is (5.23) below.)

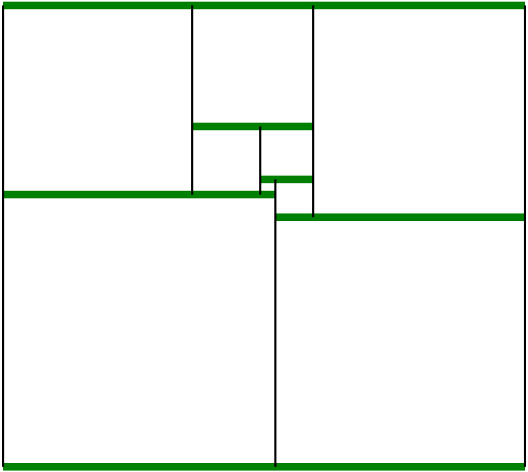
Duke Math. J. **7** (1940) 312–340.

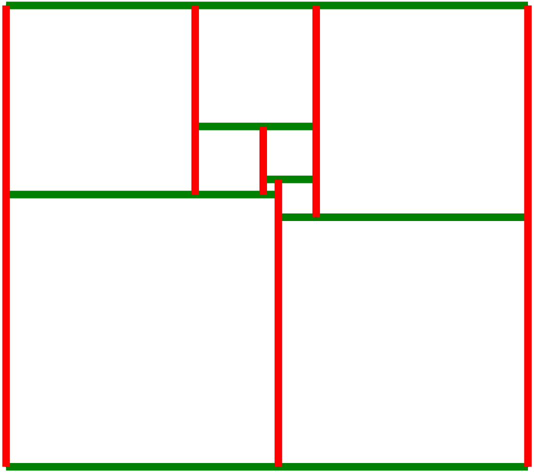
History

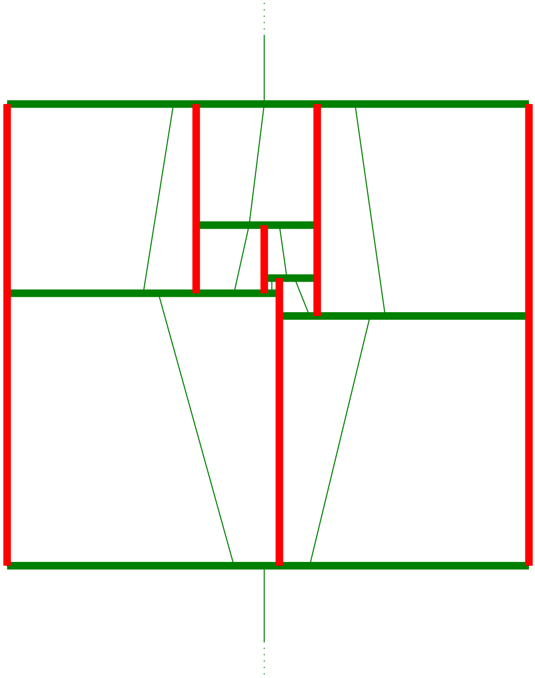
from a design for a proposed memorial to Tutte in Newmarket, UK.

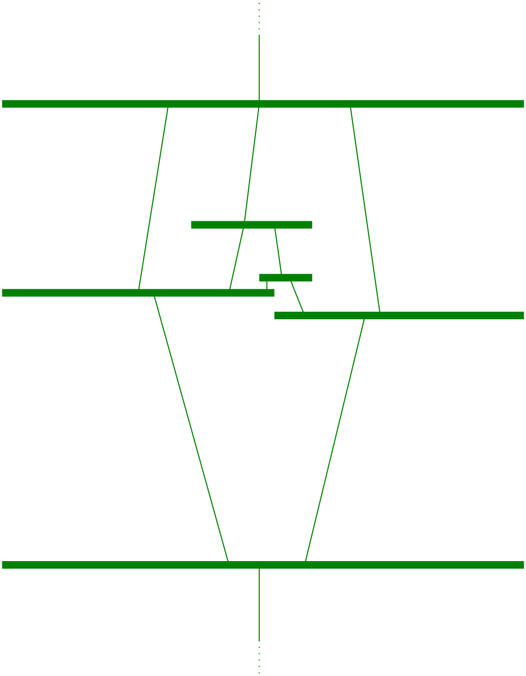
`https://www.facebook.com/billtutte`

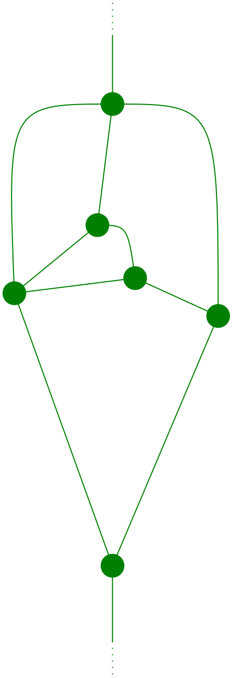


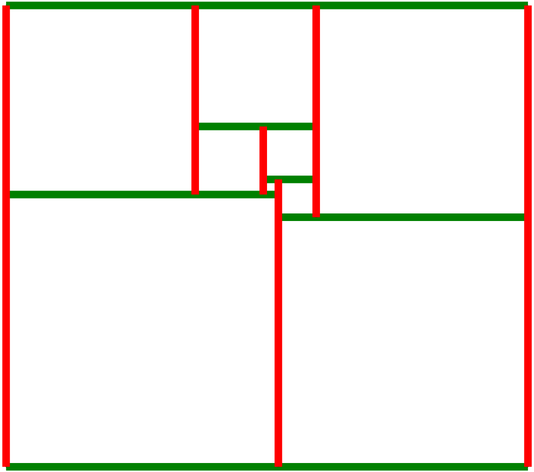


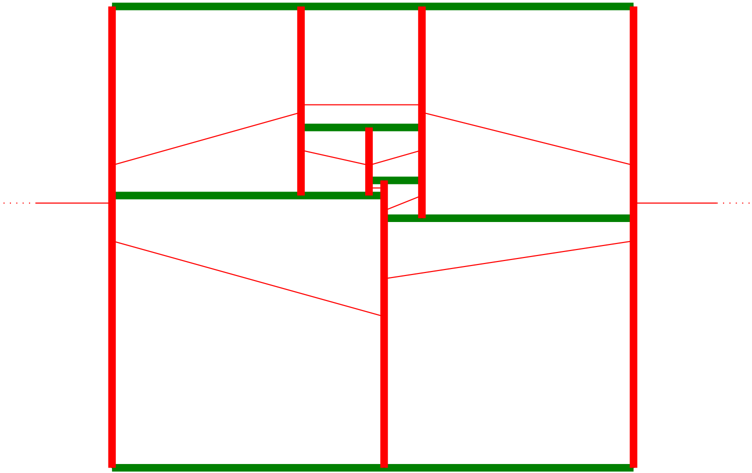


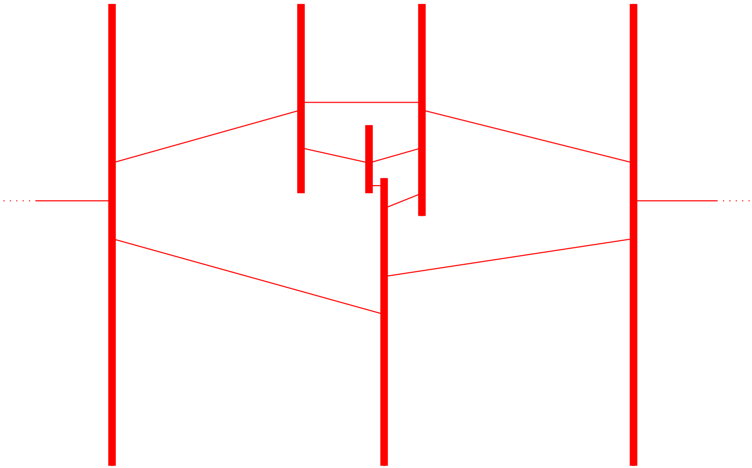


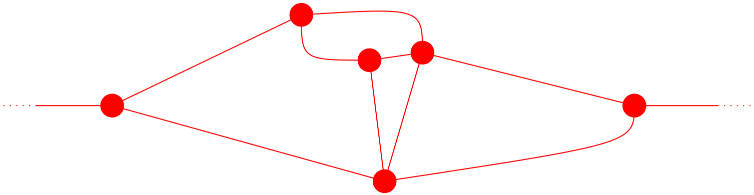


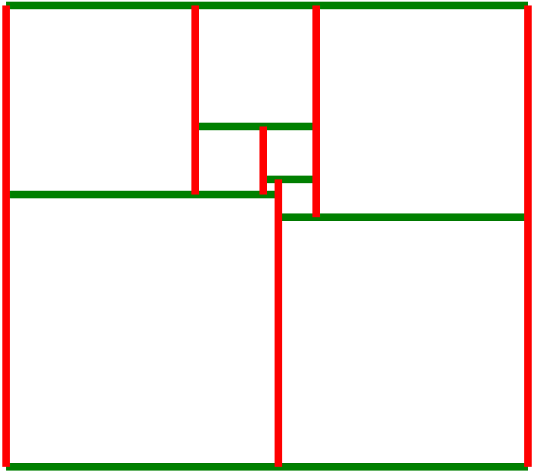


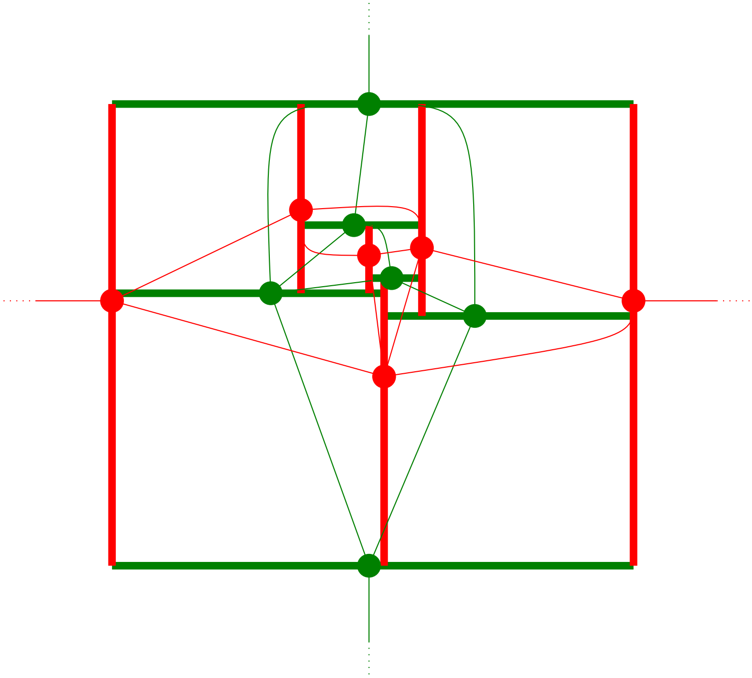


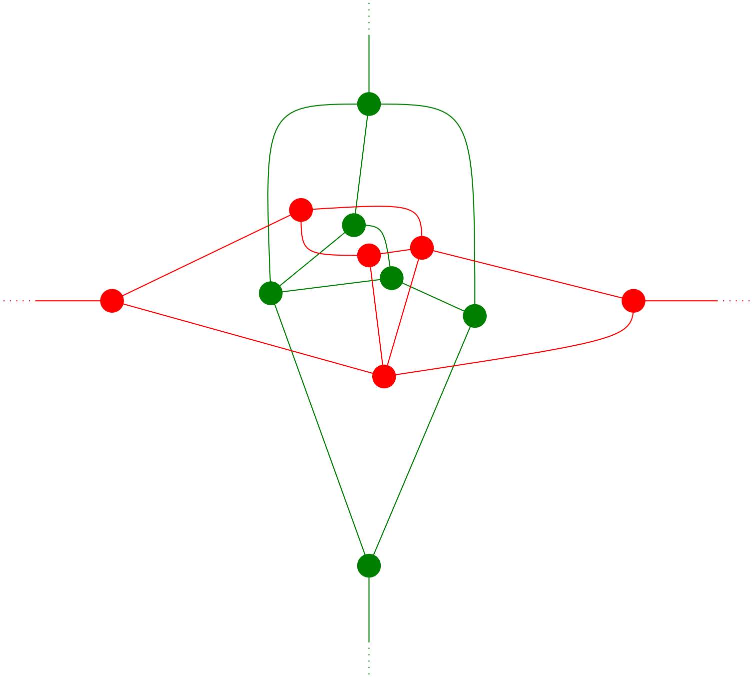












History

THE DISSECTION OF EQUILATERAL TRIANGLES INTO EQUILATERAL TRIANGLES

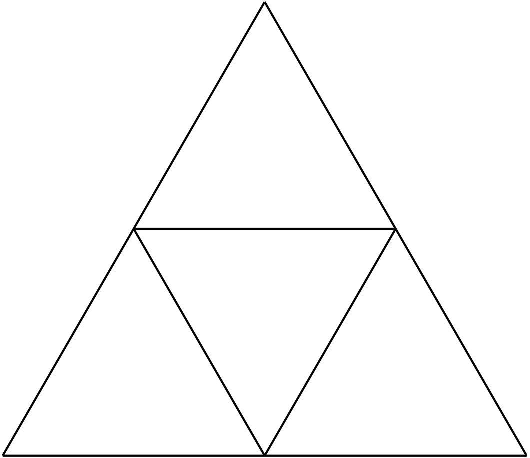
BY W. T. TUTTE

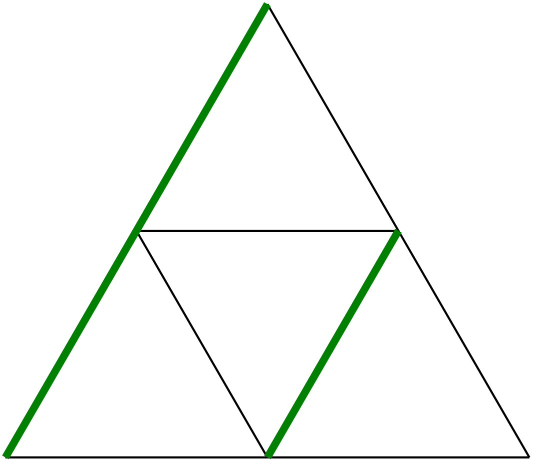
Received 10 December 1947

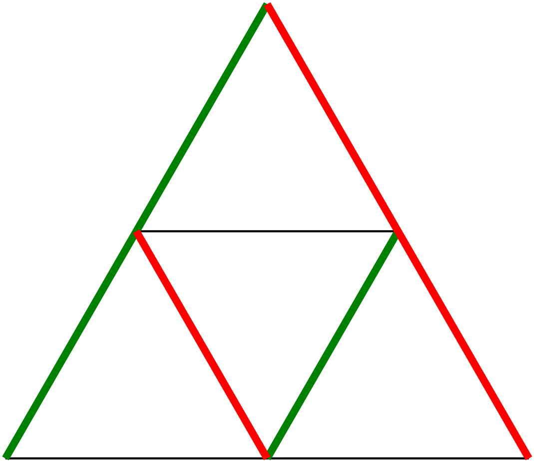
1. INTRODUCTION

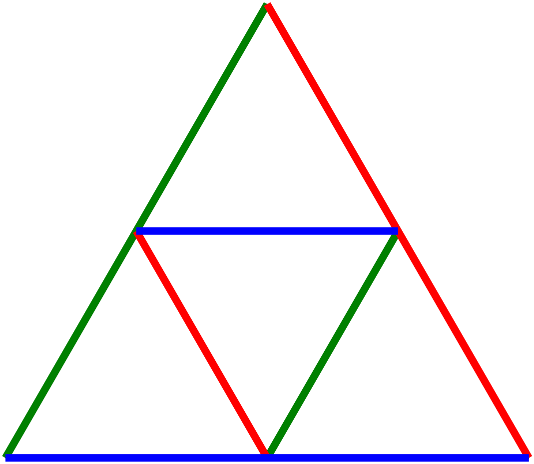
In a previous joint paper ('The dissection of rectangles into squares', by R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, *Duke Math. J.* 7 (1940), 312–40), hereafter referred to as (A) for brevity, it was shown that it is possible to dissect a square into smaller unequal squares in an infinite number of ways. The basis of the theory was the association with any rectangle or square dissected into squares of an electrical network obeying Kirchhoff's laws. The present paper is concerned with the similar problem of dissecting a figure into equilateral triangles. We make use of an analogue of the electrical network in which the 'currents' obey laws similar to but not identical with those of Kirchhoff. As a generalization of topological duality in the sphere we find that these networks occur in triplets of 'trial networks' N^1, N^2, N^3. We find that it is impossible to dissect a triangle into unequal equilateral triangles but that a dissection is possible into triangles and rhombuses so that no two of these figures have equal sides. Most of the theorems of paper (A) are special cases of those proved below.

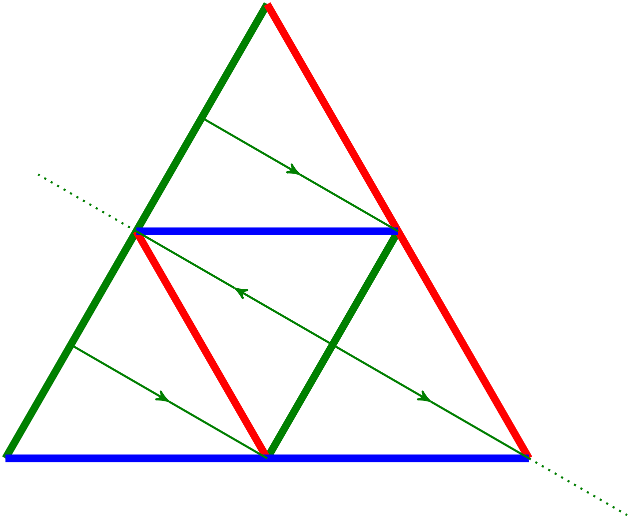
Proc. Cambridge Philos. Soc. **44** (1948) 463–482.

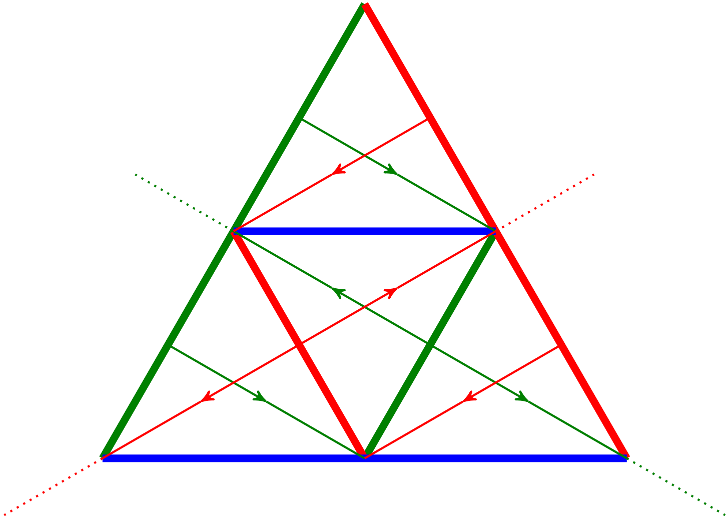


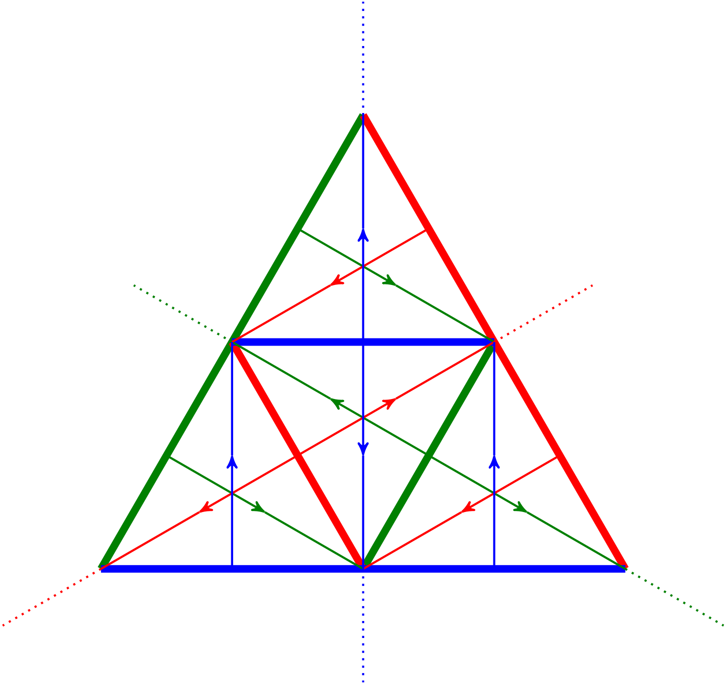


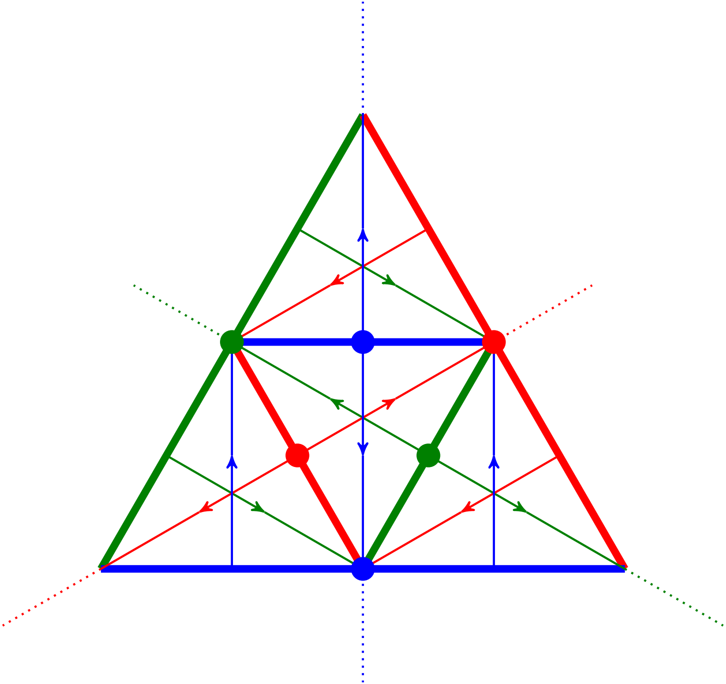


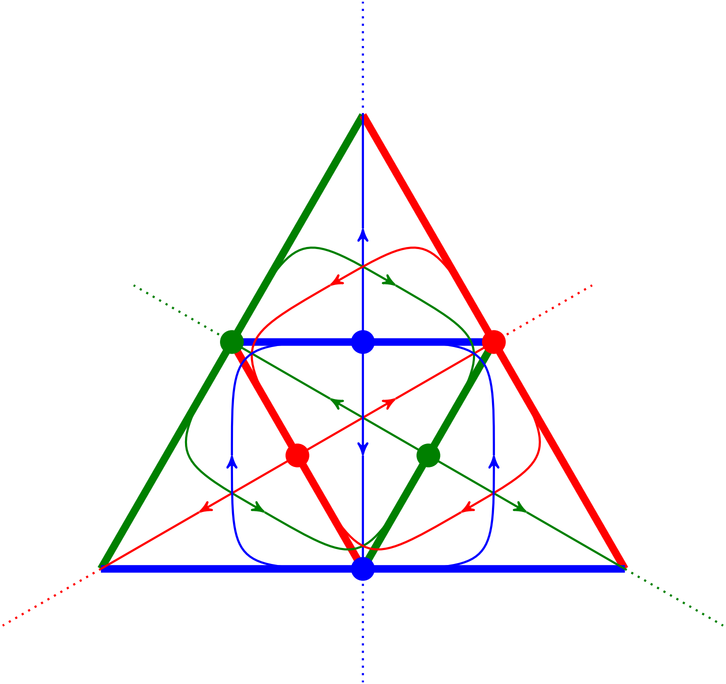


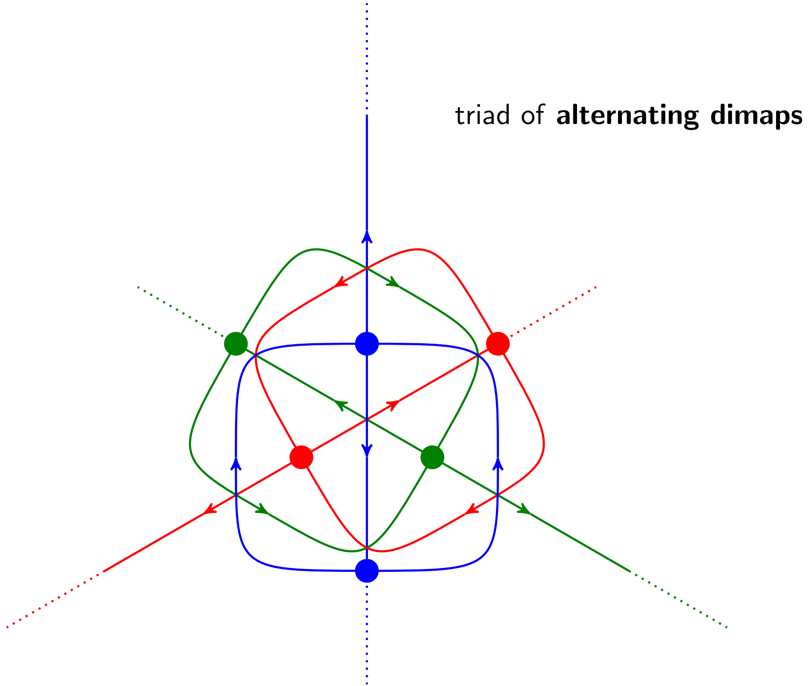


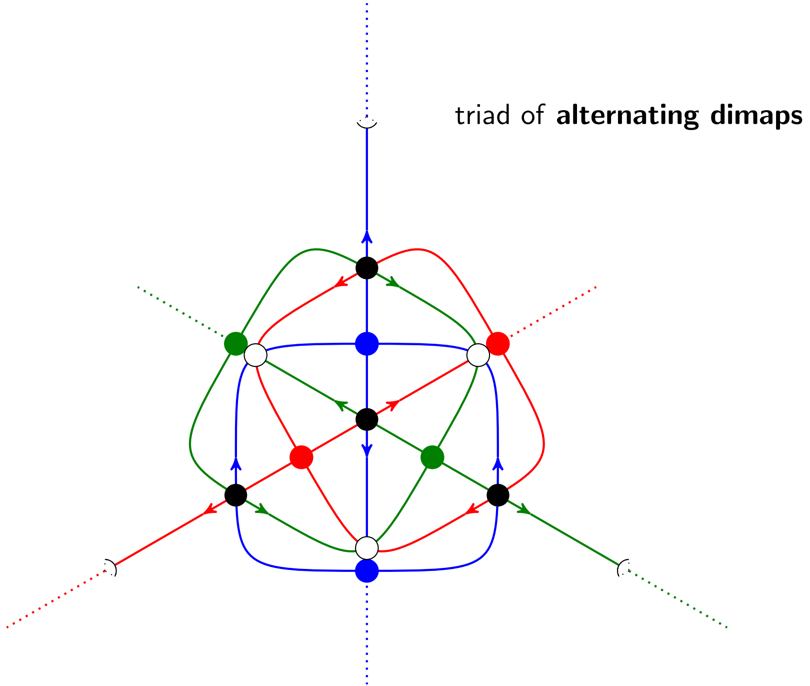









triad of **alternating dimaps**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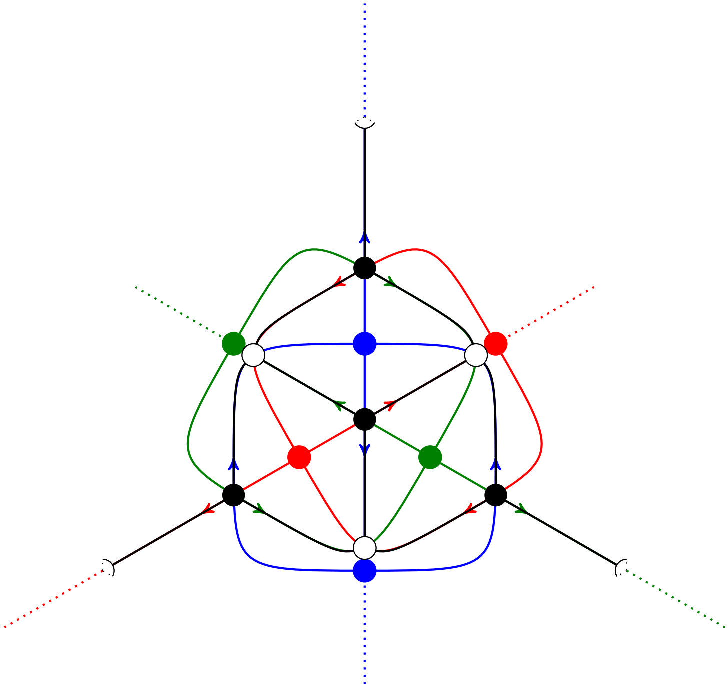
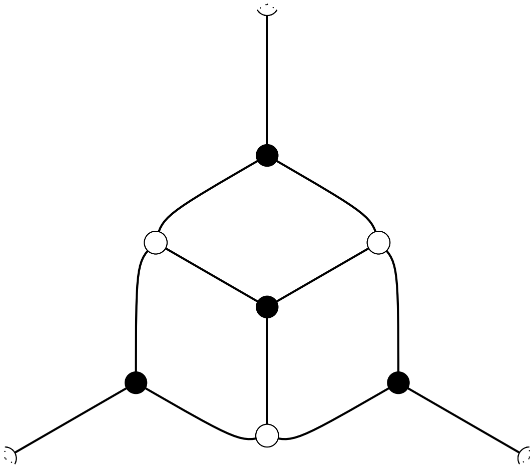
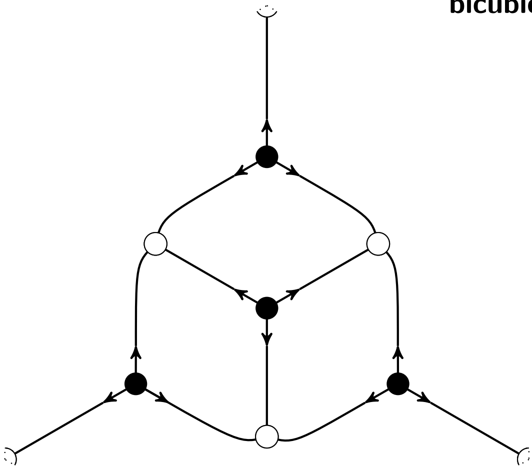




bicubic map



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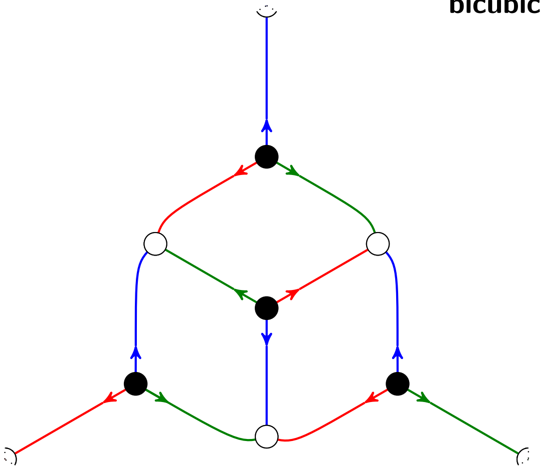

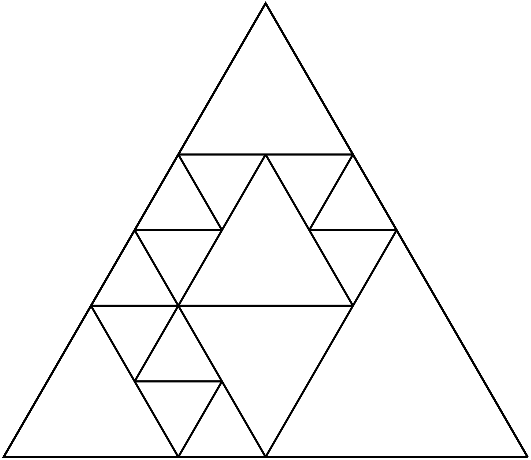

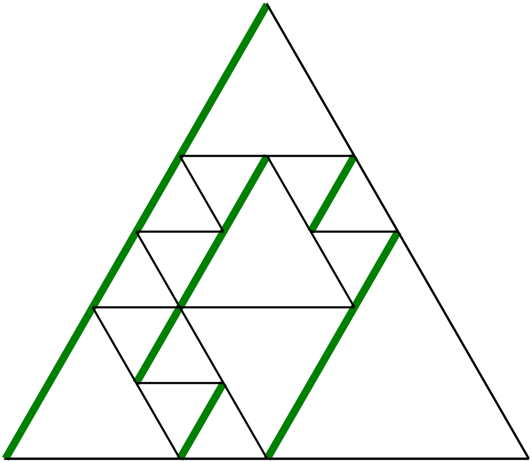

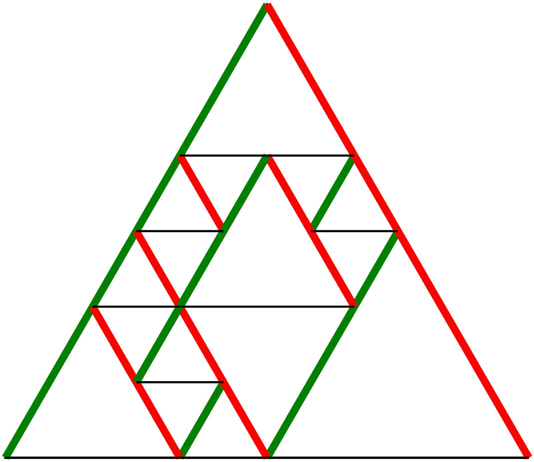

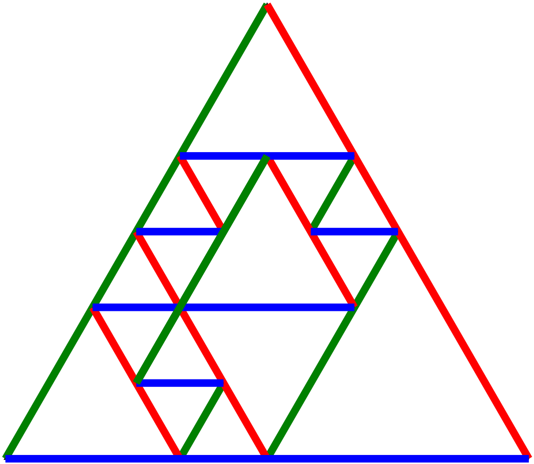

Alternating dimaps

Alternating dimap (Tutte, 1948):

- directed graph without isolated vertices,
- 2-cell embedded in a disjoint union of orientable 2-manifolds,
- each vertex has even degree,
- $\forall v$: edges incident with v are directed alternately into, and out of, v (as you go around v).

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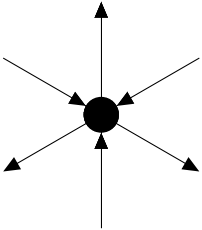
So vertices look like this:



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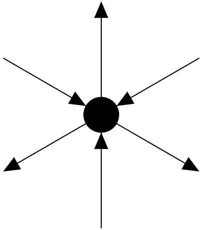

Genus $\gamma(G)$ of an alternating dimap G:

$$V - E + F = 2(k(G) - \gamma(G))$$

Alternating dimaps

Three special partitions of $E(G)$:

- *clockwise faces*
- *anticlockwise faces*
- *in-stars*

(An *in-star* is the set of all edges going into some vertex.)

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Alternating dimaps

Three special partitions of $E(G)$:

- *clockwise faces* σ_c
- *anticlockwise faces* σ_a
- *in-stars* σ_i

(An *in-star* is the set of all edges going into some vertex.)
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Alternating dimaps

Three special partitions of $E(G)$:

- *clockwise faces* σ_c
- *anticlockwise faces* σ_a
- *in-stars* σ_i

(An *in-star* is the set of all edges going into some vertex.)
Each defines a permutation of $E(G)$. These permutations satisfy

$$\sigma_i \sigma_c \sigma_a = 1$$

Triality (Trinity)

Construction of trial map:

clockwise faces $\longrightarrow$ vertices $\longrightarrow$ anticlockwise faces $\longrightarrow$ clockwise faces

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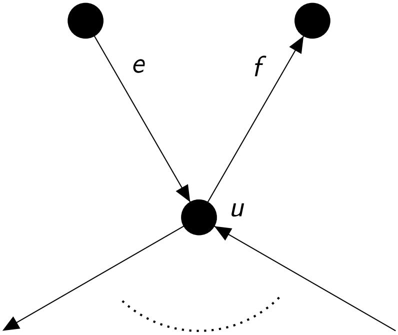

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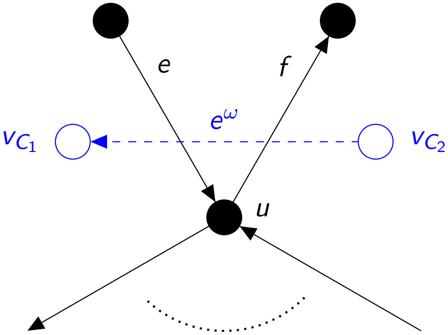
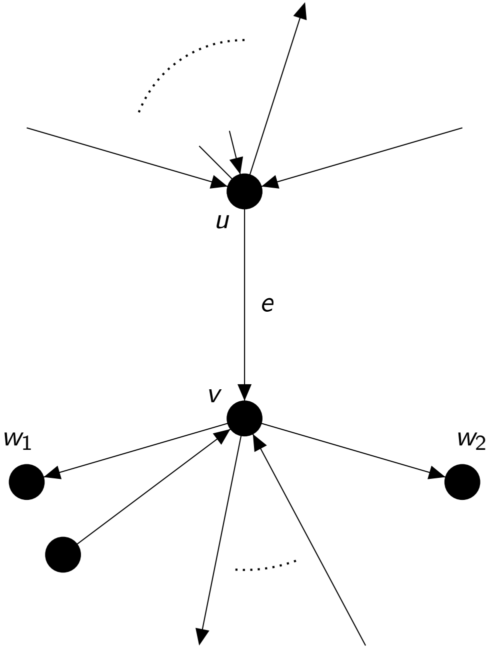


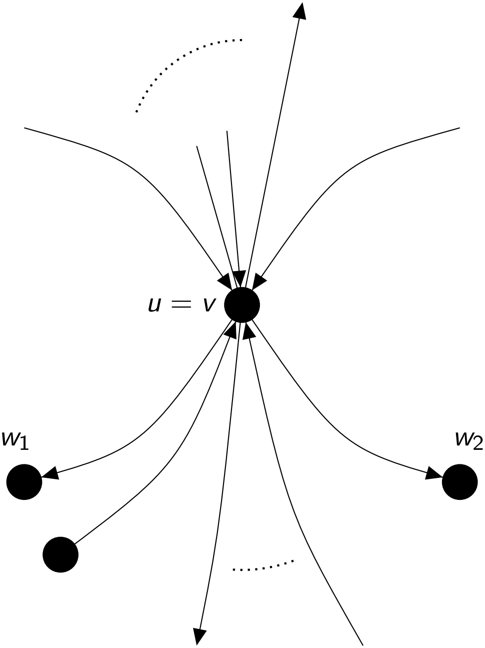
Minor operations

G



Minor operations

$G[1]e$



Minor operations

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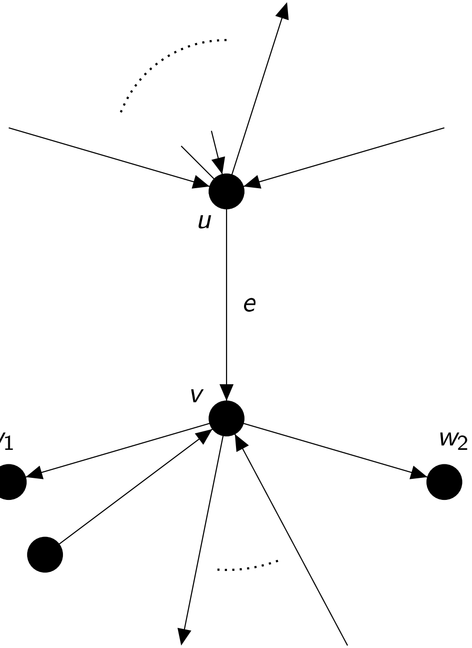
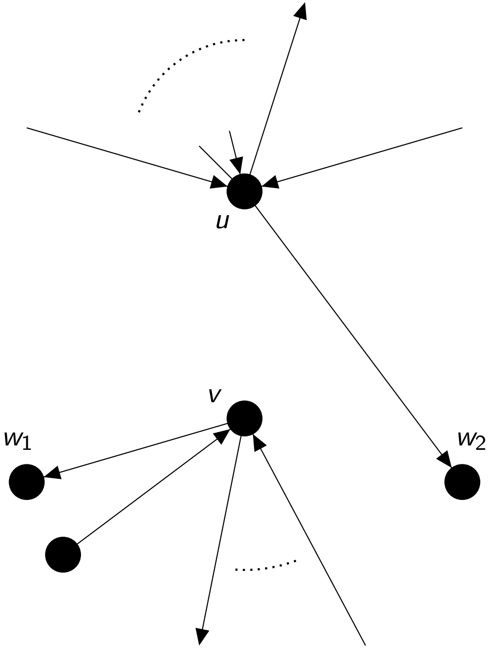


Minor operations

$G[\omega]e$



Minor operations

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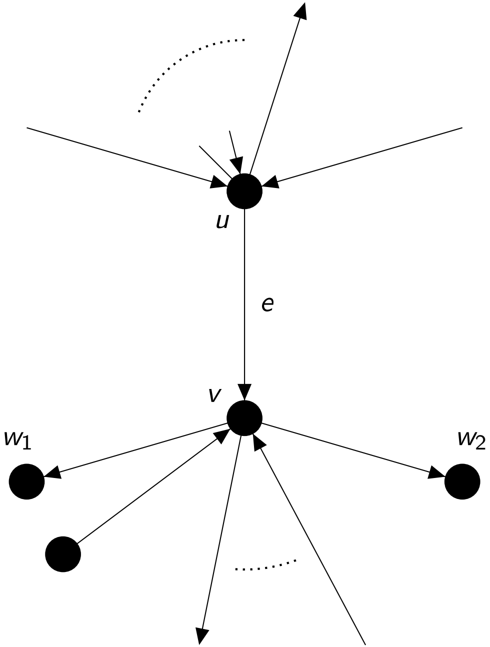
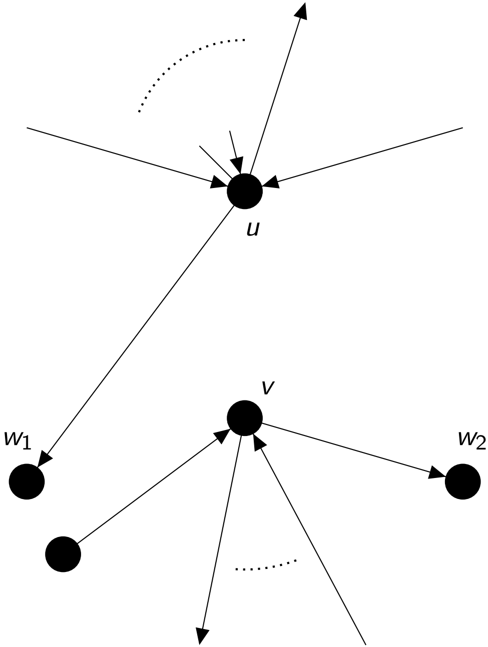


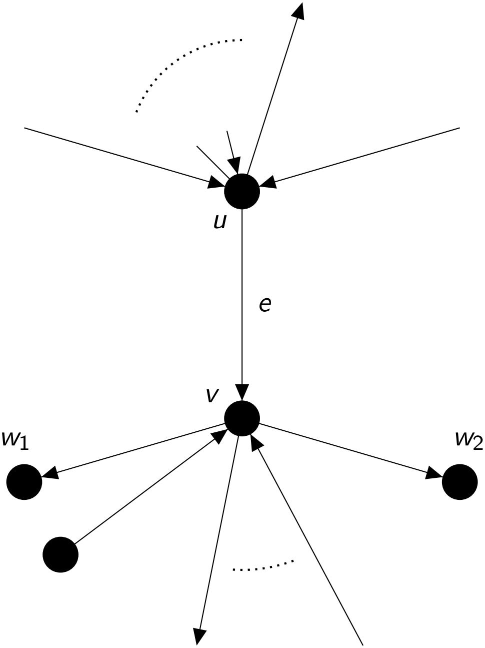
Minor operations

$G[\omega^2]e$



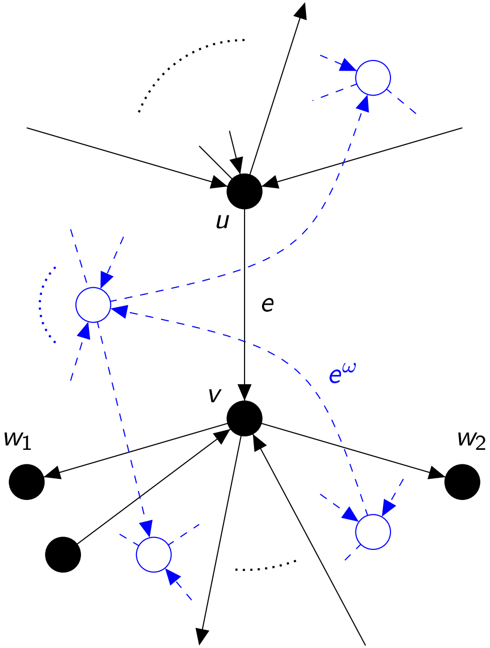
Minor operations

G



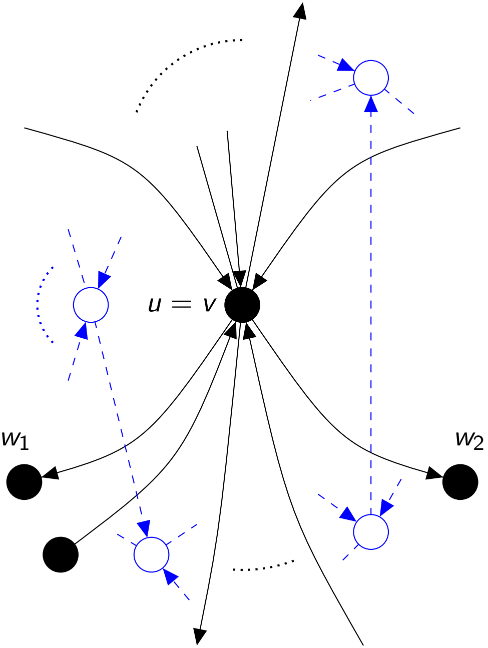
Minor operations

G



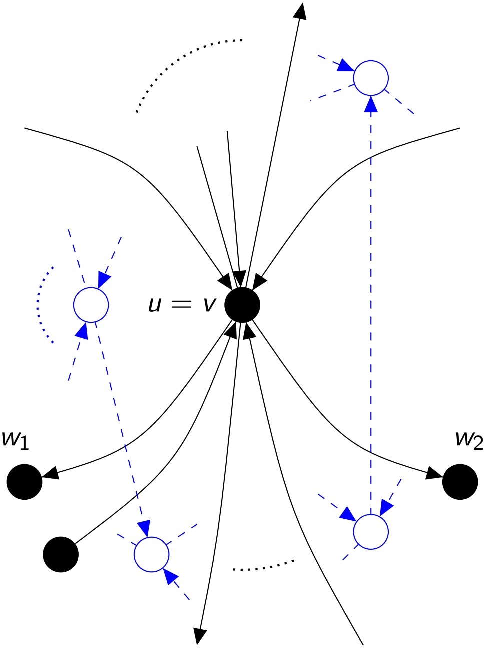
Minor operations

$G[1]e$



Minor operations

$(G[1]e)^\omega = G^\omega[\omega^2]e^\omega$



Minor operations

$$\begin{aligned} G^{\omega}[1]e^{\omega} &= (G[\omega]e)^{\omega}, \\ G^{\omega}[\omega]e^{\omega} &= (G[\omega^2]e)^{\omega}, \\ G^{\omega}[\omega^2]e^{\omega} &= (G[1]e)^{\omega}, \end{aligned}$$

$$\begin{aligned} G^{\omega^2}[1]e^{\omega^2} &= (G[\omega^2]e)^{\omega^2}, \\ G^{\omega^2}[\omega]e^{\omega^2} &= (G[1]e)^{\omega^2}, \\ G^{\omega^2}[\omega^2]e^{\omega^2} &= (G[\omega]e)^{\omega^2}. \end{aligned}$$

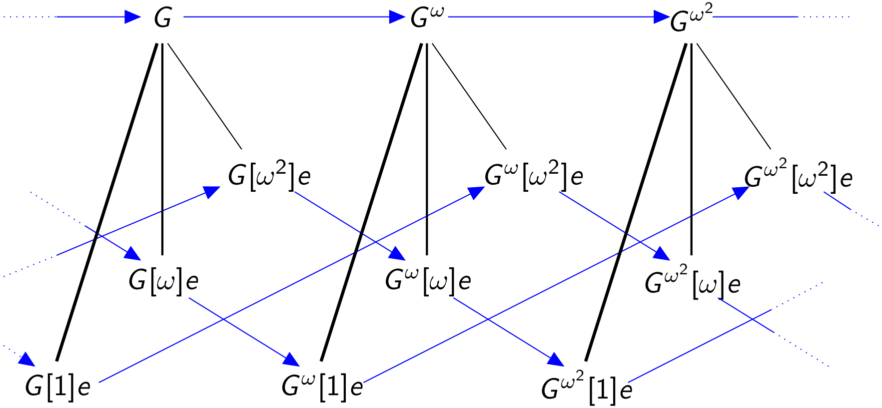
Theorem

If $e \in E(G)$ and $\mu, \nu \in \{1, \omega, \omega^2\}$ then

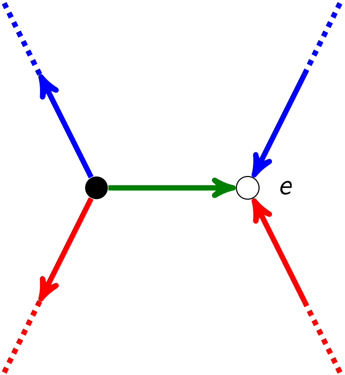
$$G^{\mu}[\nu]e^{\omega} = (G[\mu\nu]e)^{\mu}.$$

Same pattern as established for other generalised minor operations (GF, 2008/2013. . .).

Minor operations



Minors: bicubic maps



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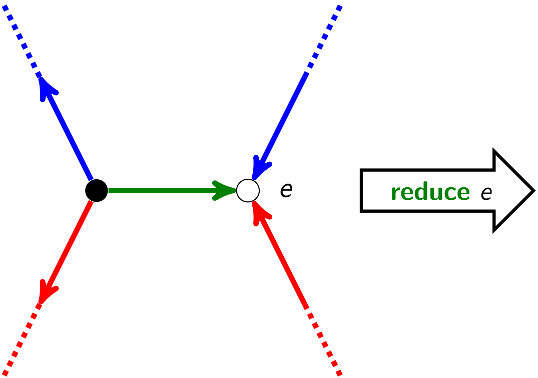

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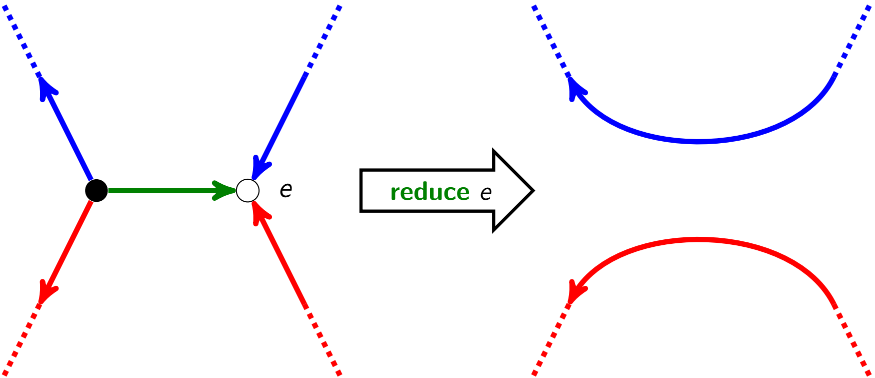

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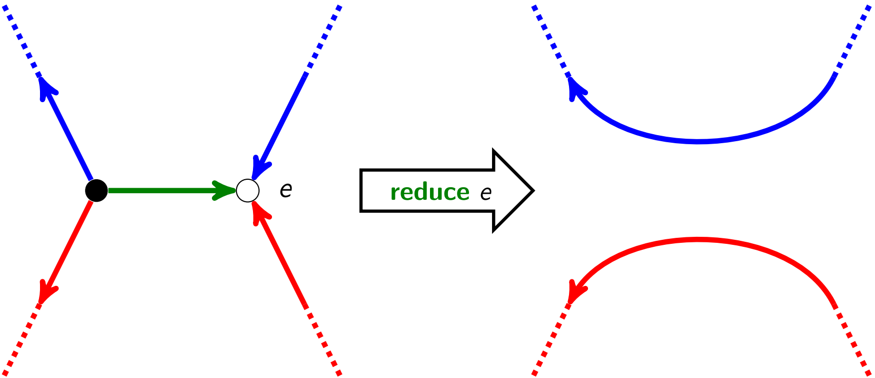

Tutte, *Philips Res. Repts* **30** (1975) 205–219.

Relationships

triangulated triangle

$\updownarrow$

alternating dimaps

$\updownarrow$

bicubic map (reduction: Tutte 1975)

$\updownarrow$ duality

Eulerian triangulation

Relationships

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Eulerian triangulation (reduction, in inverse form . . . : Batagelj, 1989)

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bicubic map (reduction: Tutte 1975)

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Eulerian triangulation (reduction, in inverse form . . . : Batagelj, 1989)

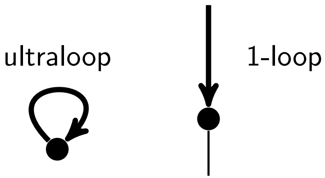
$\updownarrow$ (Cavenagh & Lisoněck, 2008)

spherical latin bitrade

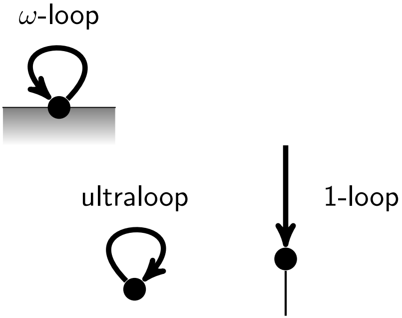
Ultraloops, triloops, semiloops

ultraloop

Ultraloops, triloops, semiloops



Ultraloops, triloops, semiloops



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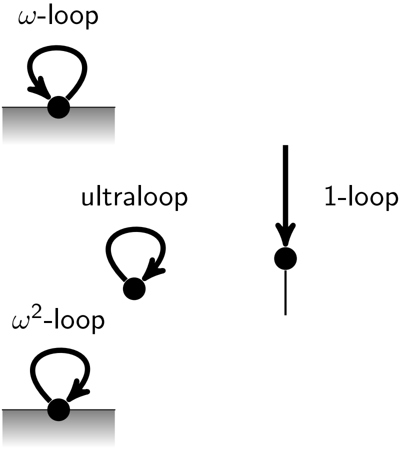

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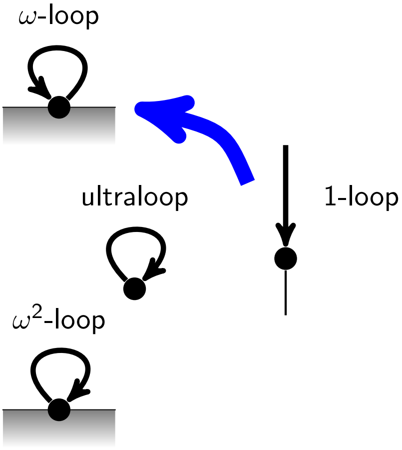

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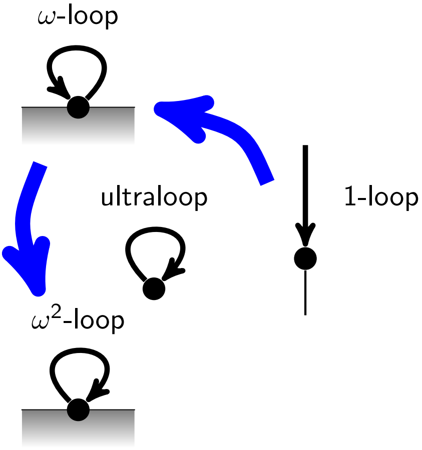

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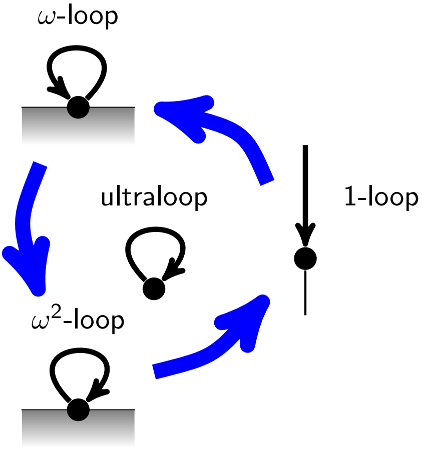

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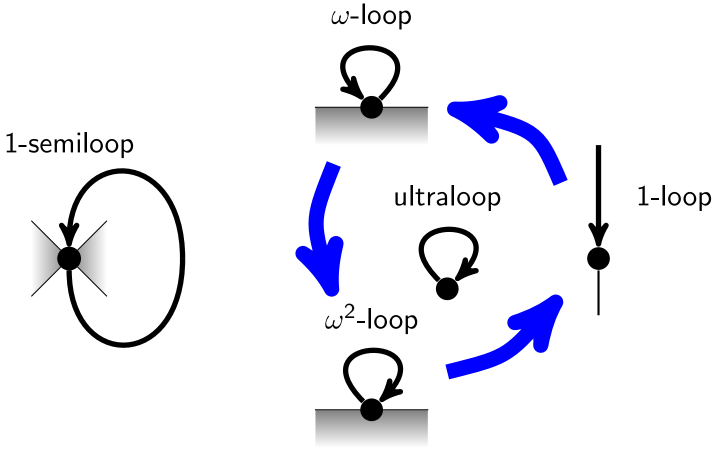

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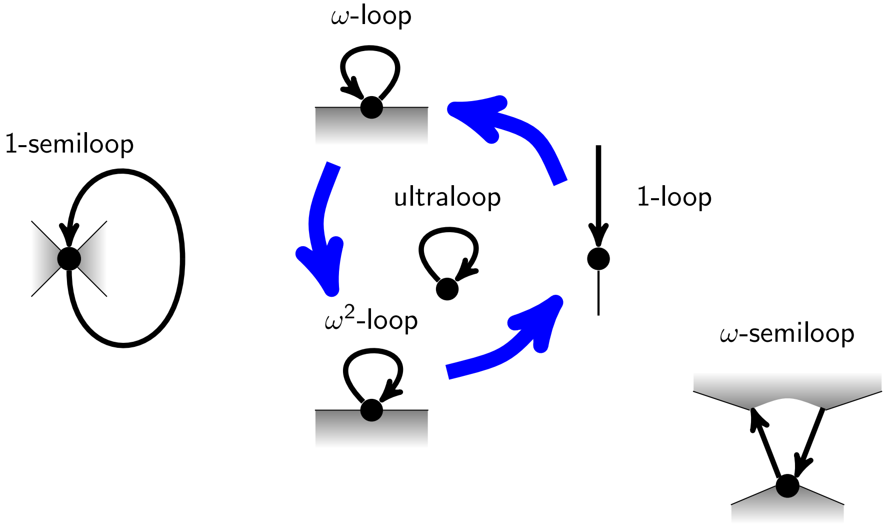

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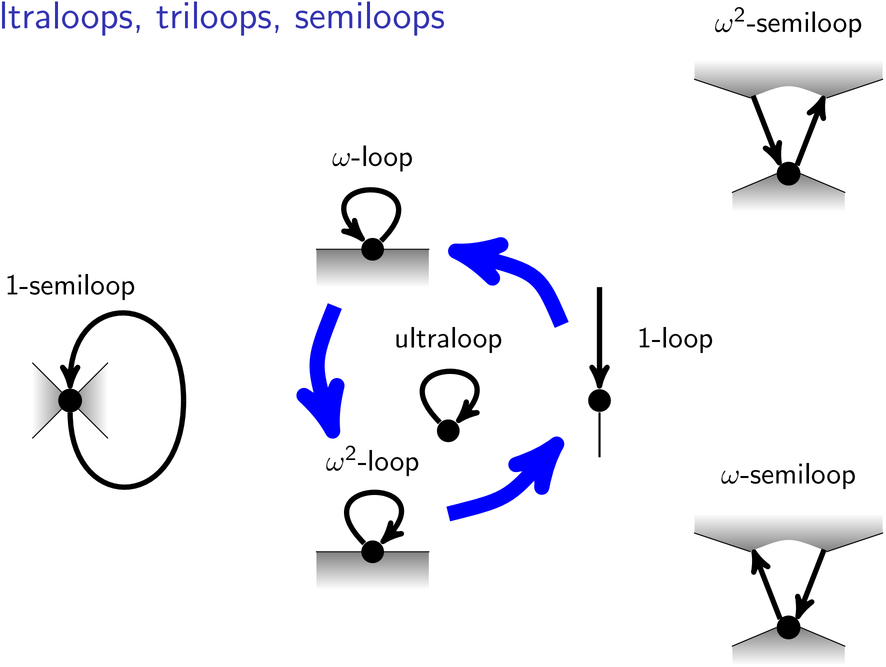

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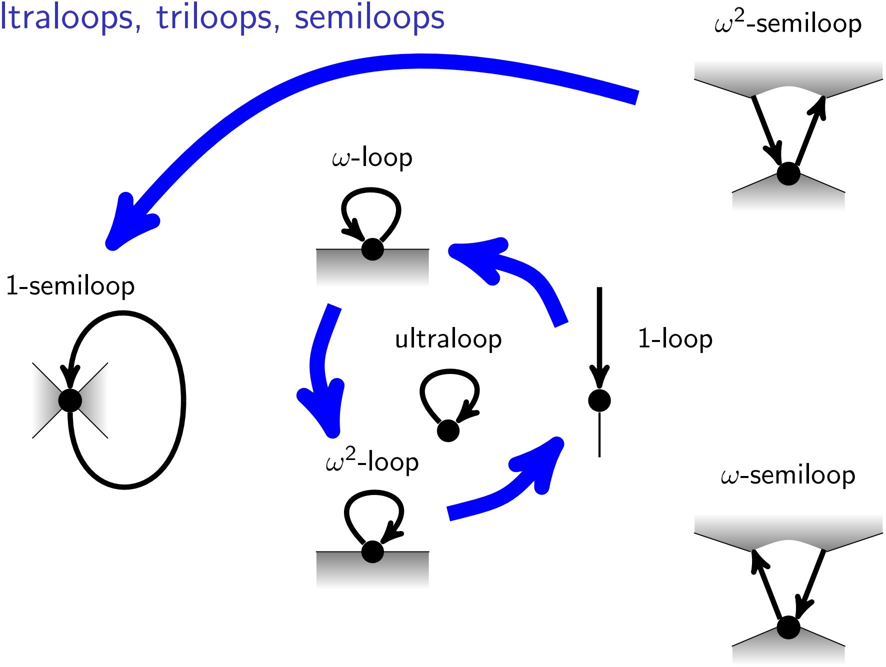

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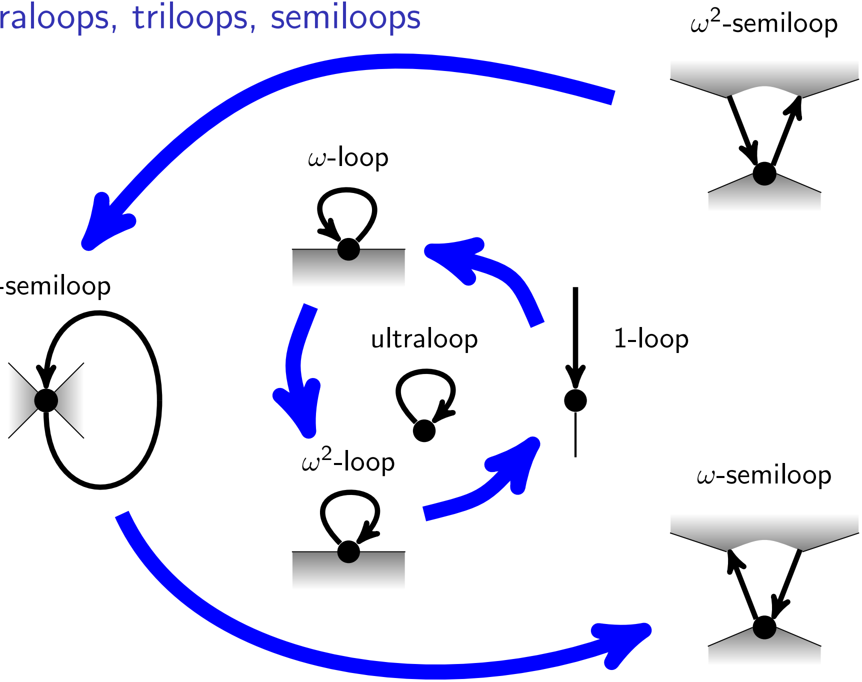

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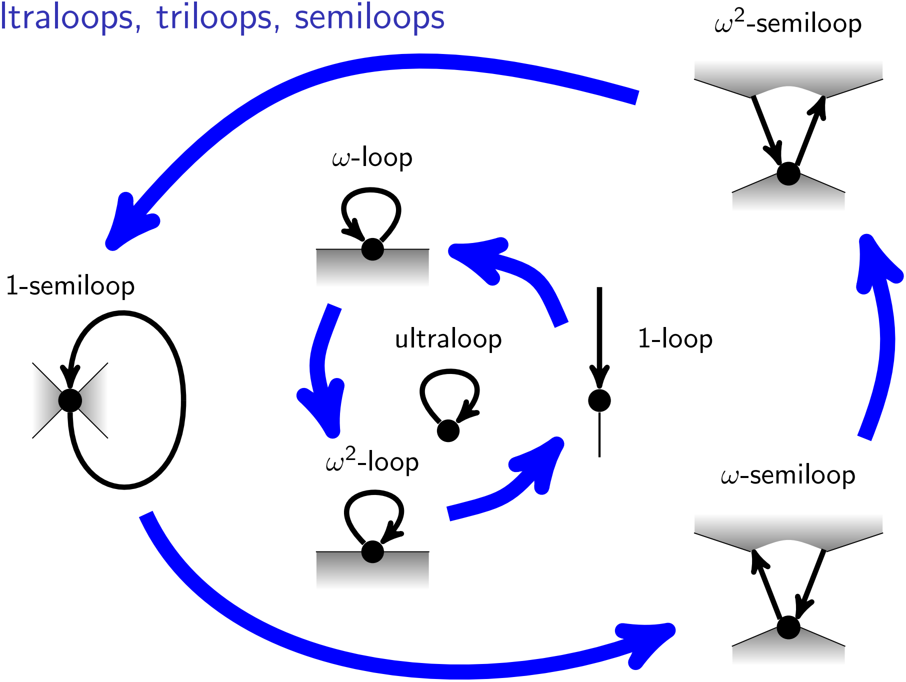
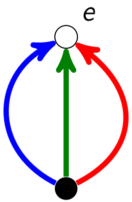


Ultraloops, triloops, semiloops: the bicubic map

trihedron
(ultraloop)



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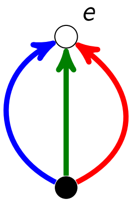

digon
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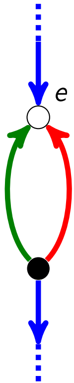

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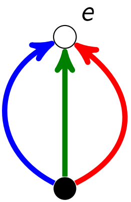

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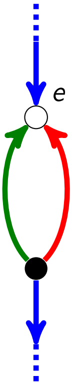
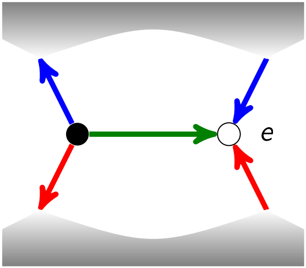


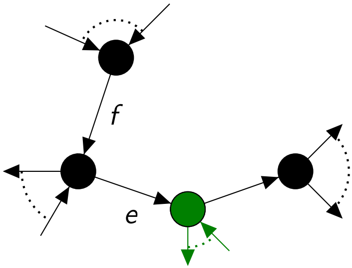
(semiloop)



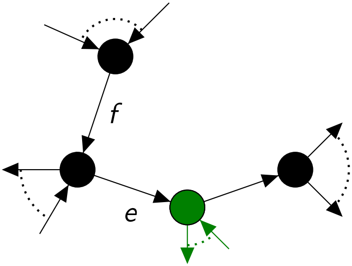
Non-commutativity

Some bad news: sometimes,

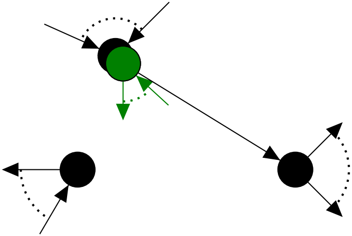
$$G[\mu]e[\nu]f \neq G[\nu]f[\mu]e$$



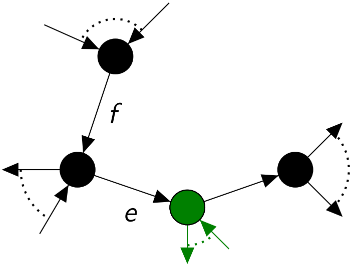
G



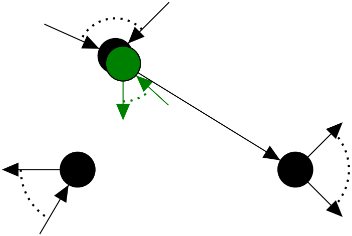
G



$G[\omega]f[1]e$



G



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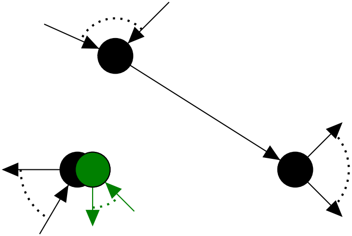

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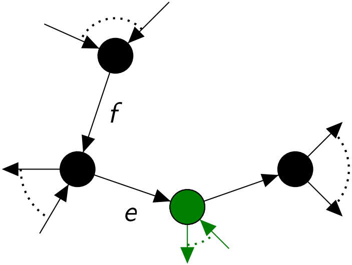

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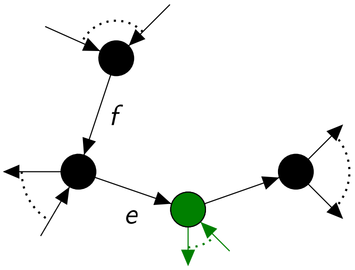


$$G[\omega]f[1]e \neq G[1]e[\omega]f$$

Theorem

Except for the above situation and its trials, reductions commute.

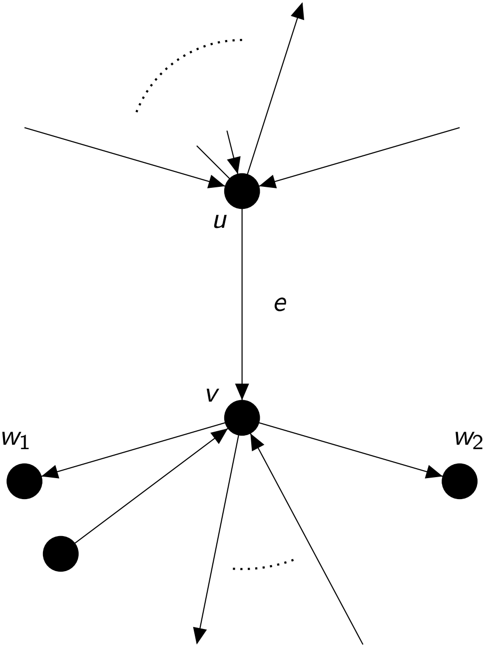
$$G[\mu]f[\nu]e = G[\nu]e[\mu]f$$

Corollary

If $\mu = \nu$, or one of e, f is a triloop, then reductions commute.

Trimedial graph

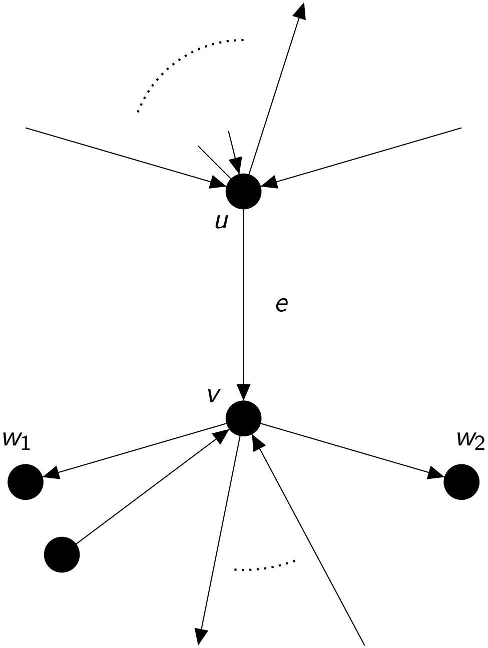
G



Trimedial graph

G

$\text{tri}(G)$



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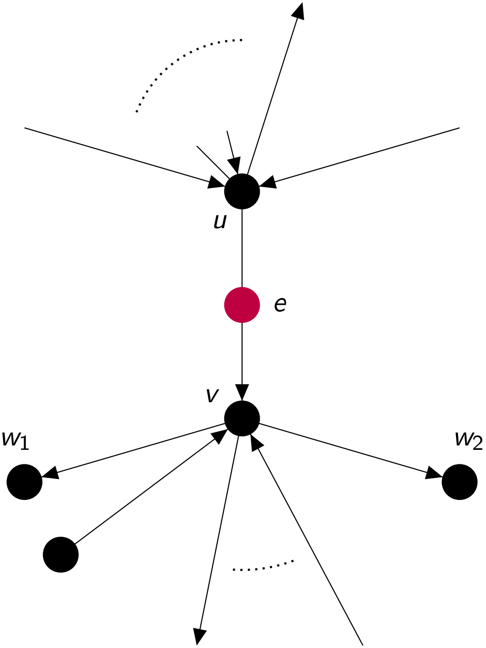

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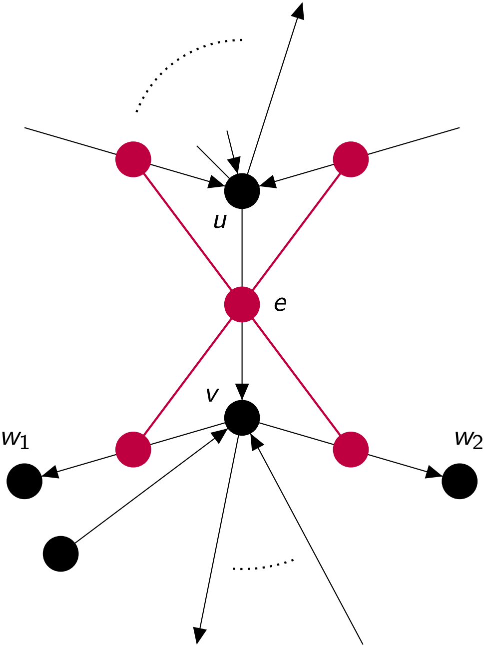

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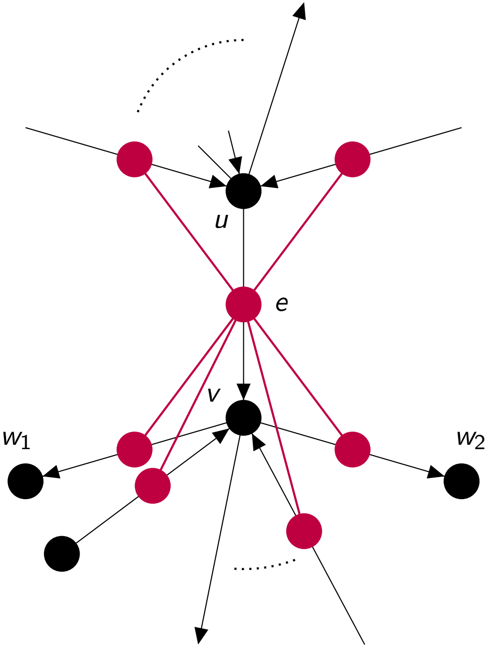

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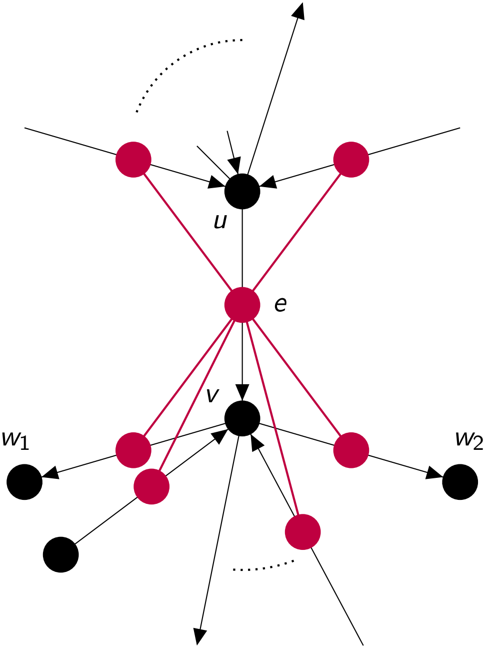


Theorem

All pairs of reductions on G commute if and only if the triloops of G form a vertex cover in $\text{tri}(G)$.

Non-commutativity

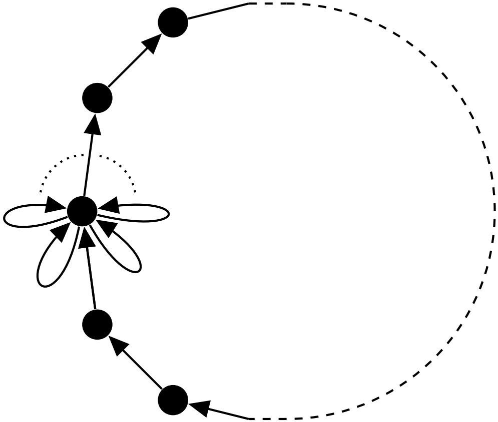
Theorem

All **sequences** *of reductions on G commute if and only if each component of G has the form . . .*

Non-commutativity

Theorem

All **sequences** *of reductions on G commute if and only if each component of G has the form ...*



Non-commutativity

Problem

Characterise alternating dimaps such that all pairs of reductions commute *up to isomorphism*:

$$\forall \mu, \nu, e, f : \quad G[\mu]f[\nu]e \cong G[\nu]e[\mu]f$$

Excluded minors for bounded genus

k-posy:

An alternating dimap with ...

- one vertex,
- $2k + 1$ edges,
- two faces.

$$V - E + F = 1 - (2k + 1) + 2 = 2 - 2k$$

Genus of k-posy $= k$

Theorem

A nonempty alternating dimap G has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus k.

Excluded minors for bounded genus

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cf. Courcelle & Dussaux (2002): ordinary maps, surface minors, bouquets.

Excluded minors for bounded genus

0-posy:

Excluded minors for bounded genus

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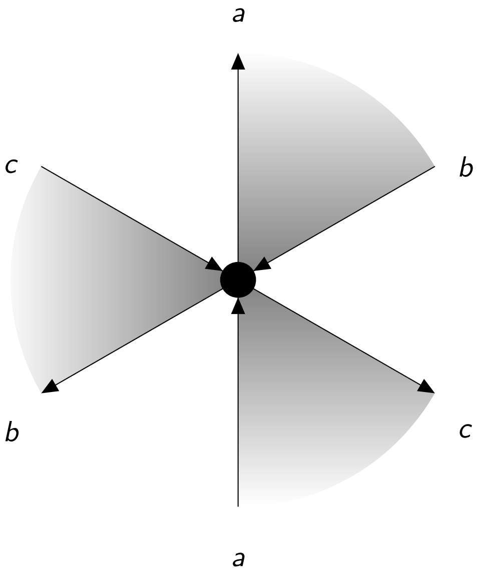
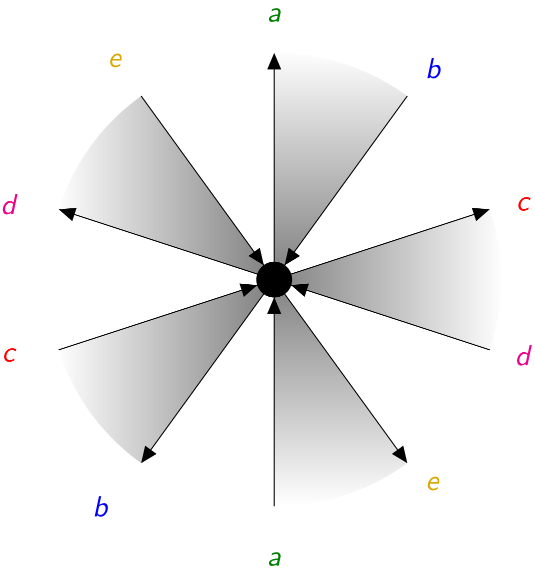
Excluded minors for bounded genus

1-posy:



Excluded minors for bounded genus

2-posy: first:



Excluded minors for bounded genus

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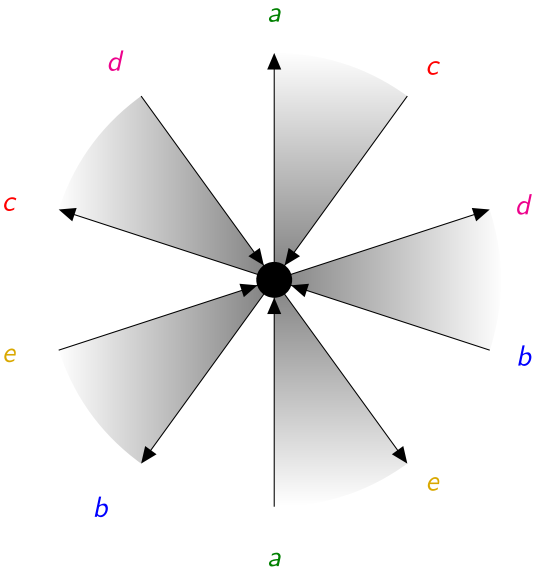
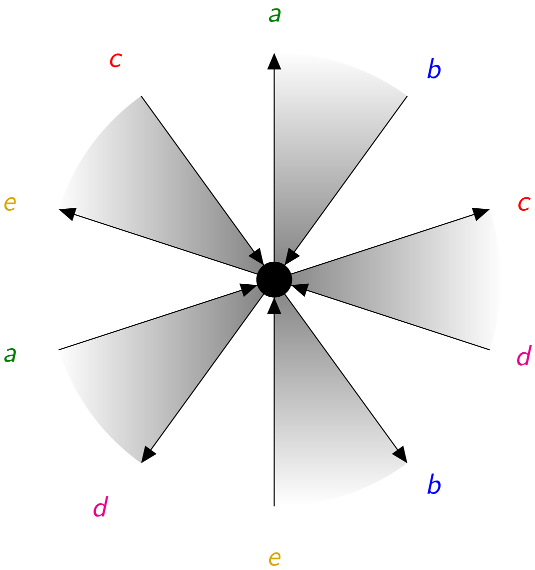


Excluded minors for bounded genus

2-posy: third:



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$(\Longrightarrow)$ Easy.

$(\Longleftarrow)$ Show:

$$\gamma(G) \geq k \implies \exists \, \text{minor} \cong \text{disjoint union of posies, total genus } k.$$

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Induction on $|E(G)|$.

Excluded minors for bounded genus

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$(\Longrightarrow)$ Easy.

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Induction on $|E(G)|$.

Inductive basis:
$|E(G)| = 1 \implies G$ is an ultraloop $\implies$ 0-posy minor.

Excluded minors for bounded genus

Showing ...

$\gamma(G) \geq k \implies \exists\,\text{minor} \cong$ disjoint union of posies, total genus k.

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Excluded minors for bounded genus

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Inductive step: Suppose true for alt. dimaps of $< m$ edges.
Let G be an alternating dimap with $|E(G)| = m$.

Excluded minors for bounded genus

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$G[1]e$, $G[\omega]e$, $G[\omega^2]e$ each have $m-1$ edges.

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$\therefore$ by inductive hypothesis, these each have, as a minor, a disjoint union of posies of total genus ...
$\gamma(G[1]e)$, $\gamma(G[\omega]e)$, $\gamma(G[\omega^2]e)$, respectively.

Excluded minors for bounded genus

Showing . . .

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If any of these $= \gamma(G)$: **done**.

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It remains to consider:
$\gamma(G[1]e) = \gamma(G[\omega]e) = \gamma(G[\omega^2]e) = \gamma(G) - 1$.

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↑
proper
1-semiloop

Excluded minors for bounded genus

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Let G be an alternating dimap with $|E(G)| = m$.
$G[1]e$, $G[\omega]e$, $G[\omega^2]e$ each have $m - 1$ edges.
$\therefore$ by inductive hypothesis, these each have, as a minor, a disjoint union of posies of total genus ...
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If any of these $= \gamma(G)$: **done**.
It remains to consider:

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↑
proper
ω-semiloop

Excluded minors for bounded genus

Showing ...

$\gamma(G) \geq k \implies \exists$ minor $\cong$ disjoint union of posies, total genus k.

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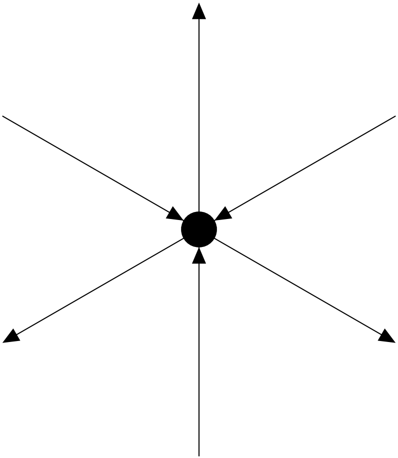
Excluded minors for bounded genus



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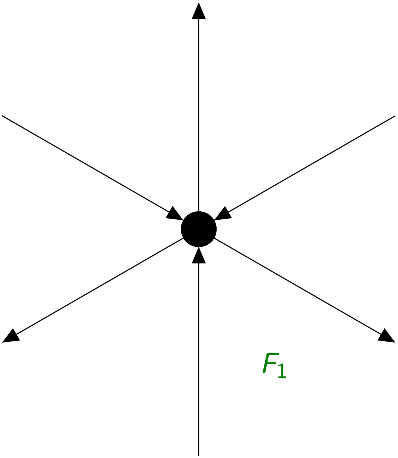

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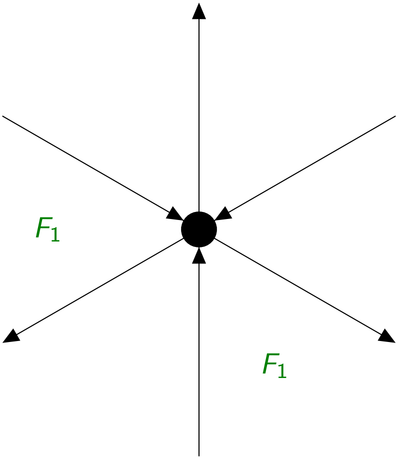

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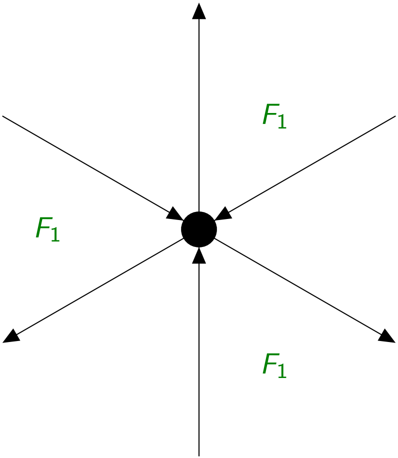

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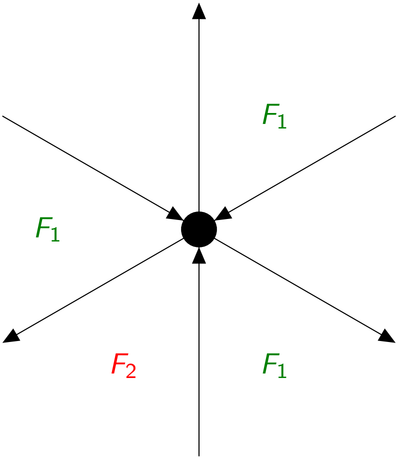

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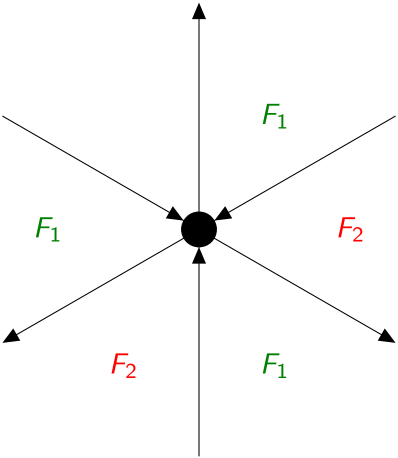

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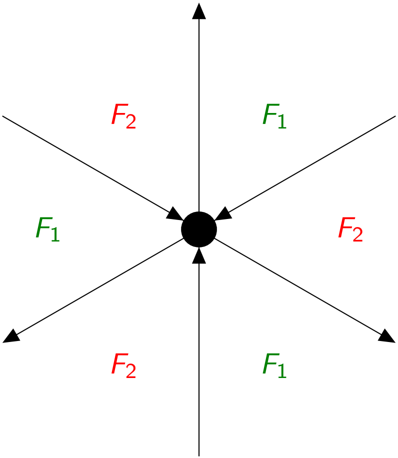

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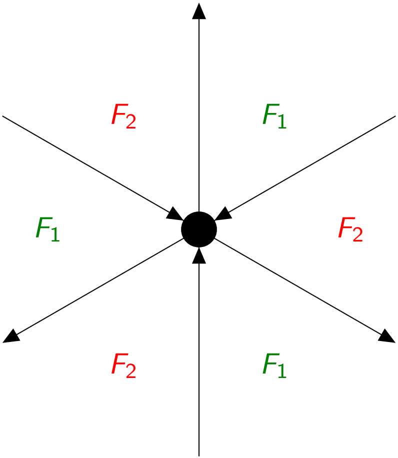

Tutte polynomial of a graph (or matroid)

$$T(G; x, y) = \sum_{X \subseteq E} (x-1)^{\rho(E)-\rho(X)} (y-1)^{\rho^*(E)-\rho^*(E \setminus X)}$$

where

$$\begin{aligned}
\rho(Y) &= \text{rank of } Y \\
&= (\#\text{vertices that meet } Y) - (\#\text{ components of } Y), \\
\rho^*(Y) &= \text{rank of } Y \text{ in the dual, } G^* \\
&= |X| + \rho(E \setminus X) - \rho(E).
\end{aligned}$$

Tutte polynomial of a graph (or matroid)

$$T(G;x,y) = \sum_{X \subseteq E} (x-1)^{\rho(E)-\rho(X)} (y-1)^{\rho^*(E)-\rho^*(E \setminus X)}$$

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By appropriate substitutions, it yields:

numbers of colourings, acyclic orientations, spanning trees, spanning subgraphs, forests, ...

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chromatic polynomial, flow polynomial, reliability polynomial, Ising and Potts model partition functions, weight enumerator of a linear code, Jones polynomial of an alternating link, ...

Tutte polynomial of a graph (or matroid)

Deletion-contraction relation:

$T(G; x, y) =$

$$\begin{cases} 1, & \text{if } G \text{ is empty,} \\ x\, T(G \setminus e; x, y), & \text{if } e \text{ is a coloop (i.e., bridge),} \\ y\, T(G/e; x, y), & \text{if } e \text{ is a loop,} \\ T(G \setminus e; x, y) + T(G/e; x, y), & \text{if } e \text{ is neither a coloop nor a loop.} \end{cases}$$

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Recipe Theorem (in various forms: Tutte, 1948; Brylawski, 1972; Oxley & Welsh, 1979):

If F is an isomorphism invariant and satisfies ...

$F(G) =$

$$\begin{cases} x\,F(G \setminus e), & \text{if } e \text{ is a coloop (i.e., bridge),} \\ y\,F(G/e), & \text{if } e \text{ is a loop,} \\ a\,F(G \setminus e) + b\,F(G/e), & \text{if } e \text{ is neither a coloop nor a loop.} \end{cases}$$

... then it can be obtained from the Tutte polynomial using appropriate substitutions and factors.

Tutte invariant for alternating dimaps

– an isomorphism invariant F such that:
$F(G) =$

$$\begin{cases} 1, & \text{if } G \text{ is empty,} \\ w\,F(G-e), & \text{if } e \text{ is an ultraloop,} \\ x\,F(G[1]e), & \text{if } e \text{ is a proper 1-loop,} \\ y\,F(G[\omega]e), & \text{if } e \text{ is a proper } \omega\text{-loop,} \\ z\,F(G[\omega^2]e), & \text{if } e \text{ is a proper } \omega^2\text{-loop,} \\ a\,F(G[1]e) + b\,F(G[\omega]e) + c\,F(G[\omega^2]e), & \text{if } e \text{ is not a triloop.} \end{cases}$$

Tutte invariant for alternating dimaps

Theorem

The only Tutte invariants of alternating dimaps are:

(a) $F(G) = 0$ *for nonempty* G,

(b) $F(G) = 3^{|E(G)|} a^{|V(G)|} b^{\text{c-faces}(G)} c^{\text{a-faces}(G)}$,

(c) $F(G) = a^{|V(G)|} b^{\text{c-faces}(G)} (-c)^{\text{a-faces}(G)}$,

(d) $F(G) = a^{|V(G)|} (-b)^{\text{c-faces}(G)} c^{\text{a-faces}(G)}$,

(e) $F(G) = (-a)^{|V(G)|} b^{\text{c-faces}(G)} c^{\text{a-faces}(G)}$.

Extended Tutte invariant for alternating dimaps

– an isomorphism invariant F such that:

$$F(G) =$$

$$\begin{cases} 1, & \text{if } G \text{ is empty,} \\ w\,F(G-e), & \text{if } e \text{ is an ultraloop,} \\ x\,F(G[1]e), & \text{if } e \text{ is a proper 1-loop,} \\ y\,F(G[\omega]e), & \text{if } e \text{ is a proper } \omega\text{-loop,} \\ z\,F(G[\omega^2]e), & \text{if } e \text{ is a proper } \omega^2\text{-loop,} \\ a\,F(G[1]e) + b\,F(G[\omega]e) + c\,F(G[\omega^2]e), & \text{if } e \text{ is a proper 1-semiloop,} \\ d\,F(G[1]e) + e\,F(G[\omega]e) + f\,F(G[\omega^2]e), & \text{if } e \text{ is a proper } \omega\text{-semiloop,} \\ g\,F(G[1]e) + h\,F(G[\omega]e) + i\,F(G[\omega^2]e), & \text{if } e \text{ is a proper } \omega^2\text{-semiloop,} \\ j\,F(G[1]e) + k\,F(G[\omega]e) + l\,F(G[\omega^2]e), & \text{if } e \text{ is not a triloop.} \end{cases}$$

Extended Tutte invariant for alternating dimaps

For any alternating dimap G, define $T_c(G;x,y)$ and $T_a(G;x,y)$ as follows.

$$T_c(G;x,y) = \begin{cases} 1, & \text{if } G \text{ is empty,} \\ T_c(G[*]e;x,y), & \text{if } e \text{ is an } \omega^2\text{-loop;} \\ x\,T_c(G[\omega^2]e;x,y), & \text{if } e \text{ is an } \omega\text{-semiloop;} \\ y\,T_c(G[1]e;x,y), & \text{if } e \text{ is a proper 1-semiloop or an } \omega\text{-loop;} \\ T_c(G[1]e;x,y) + T_c(G[\omega^2]e;x,y), & \text{if } e \text{ is not a semiloop.} \end{cases}$$

$$T_a(G;x,y) = \begin{cases} 1, & \text{if } G \text{ is empty,} \\ T_a(G[*]e;x,y), & \text{if } e \text{ is an } \omega\text{-loop;} \\ x\,T_a(G[\omega]e;x,y), & \text{if } e \text{ is an } \omega^2\text{-semiloop;} \\ y\,T_a(G[1]e;x,y), & \text{if } e \text{ is a proper 1-semiloop or an } \omega^2\text{-loop;} \\ T_a(G[1]e;x,y) + T_a(G[\omega]e;x,y), & \text{if } e \text{ is not a semiloop.} \end{cases}$$

Extended Tutte invariant for alternating dimaps

Theorem

For any plane graph G,

$$T(G; x, y) = T_c(\mathrm{alt}_c(G); x, y)$$

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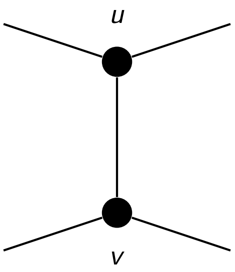

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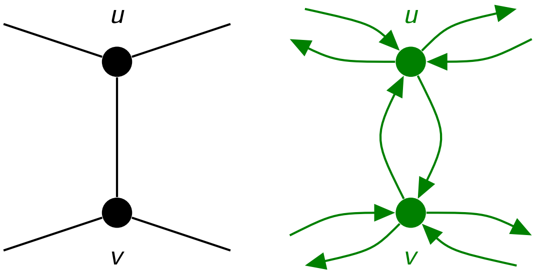

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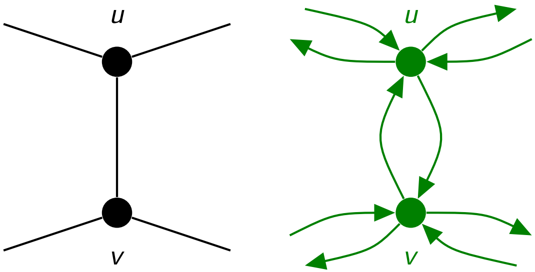

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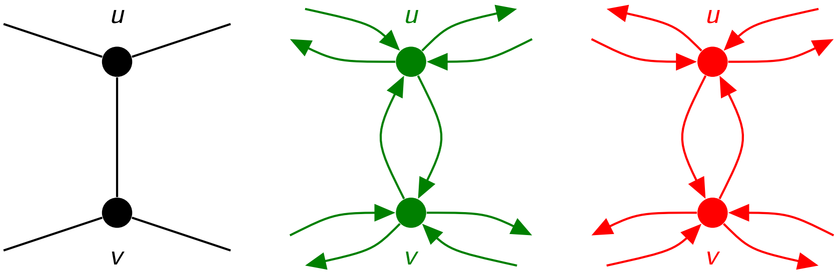

Extended Tutte invariant for alternating dimaps

$T_i(G;x) =$

$$\begin{cases} 1, & \text{if } G \text{ is empty,} \\ T_i(G[*]e;x), & \text{if } e \text{ is a 1-loop (including an ultraloop);} \\ x\,T_i(G[\omega^2]e;x), & \text{if } e \text{ is a proper } \omega\text{-semiloop or an } \omega^2\text{-loop;} \\ x\,T_i(G[\omega]e;x), & \text{if } e \text{ is a proper } \omega^2\text{-semiloop or an } \omega\text{-loop;} \\ T_i(G[\omega]e;x) + T_i(G[\omega^2]e;x), & \text{if } e \text{ is not a semiloop.} \end{cases}$$

Extended Tutte invariant for alternating dimaps

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For any plane graph G,

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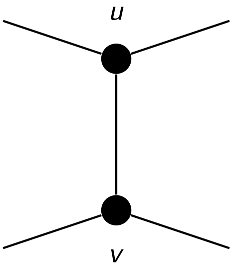

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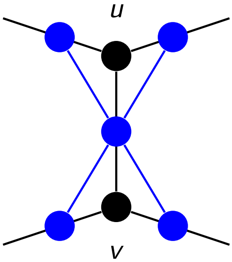

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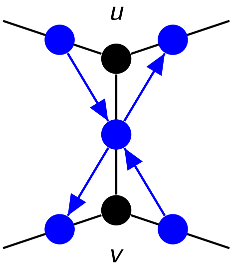

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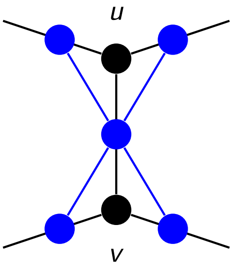

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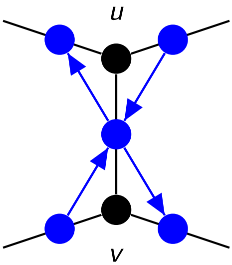

References

- R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, The dissection of rectangles into squares, *Duke Math. J.* **7** (1940) 312–340.
- W. T. Tutte, The dissection of equilateral triangles into equilateral triangles, *Proc. Cambridge Philos. Soc.* **44** (1948) 463–482.
- W. T. Tutte, Duality and trinity, in: *Infinite and Finite Sets (Colloq., Keszthely, 1973), Vol. III*, Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, 1975, pp. 1459–1472.
- R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, Leaky electricity and triangulated triangles, *Philips Res. Repts.* **30** (1975) 205–219.
- W. T. Tutte, Bicubic planar maps, *Symposium à la Mémoire de François Jaeger* (Grenoble, 1998), *Ann. Inst. Fourier (Grenoble)* **49** (1999) 1095–1102.

References

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- GF, Minors for alternating dimaps, preprint, 2013, `http://arxiv.org/abs/1311.2783`.
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For **less** information:

- GF, short public talk (10 mins) on 'William Tutte', *The Laborastory*, 2013, https://soundcloud.com/thelaborastory/william-thomas-tutte