Eigencircles of 2×2 matrices

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18 July 2007

Joint work with Michael Englefield
(School of Mathematical Sciences, Monash)

Eigenvalues and eigenpairs

Eigenvalue of a 2×2 matrix: number λ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

with x, y not both 0.

To start with, $\lambda \in \mathbb{R}$.

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Eigen*pair* of a 2×2 matrix: $(\lambda, \mu) \in \mathbb{R}^2$ such that

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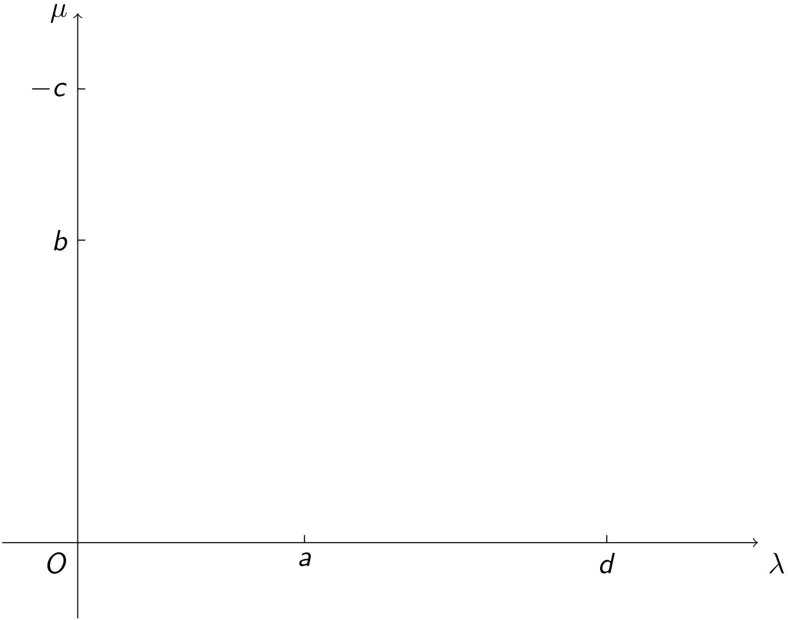
Eigenpairs form a circle, the *eigencircle*:

$$\left(\lambda - \frac{a+d}{2}\right)^2 + \left(\mu - \frac{b-c}{2}\right)^2 = \left(\frac{a+d}{2}\right)^2 + \left(\frac{b-c}{2}\right)^2 - (ad - bc)$$

$$(\lambda - f)^2 + (\mu - g)^2 = f^2 + g^2 - \det A$$

The Eigencircle



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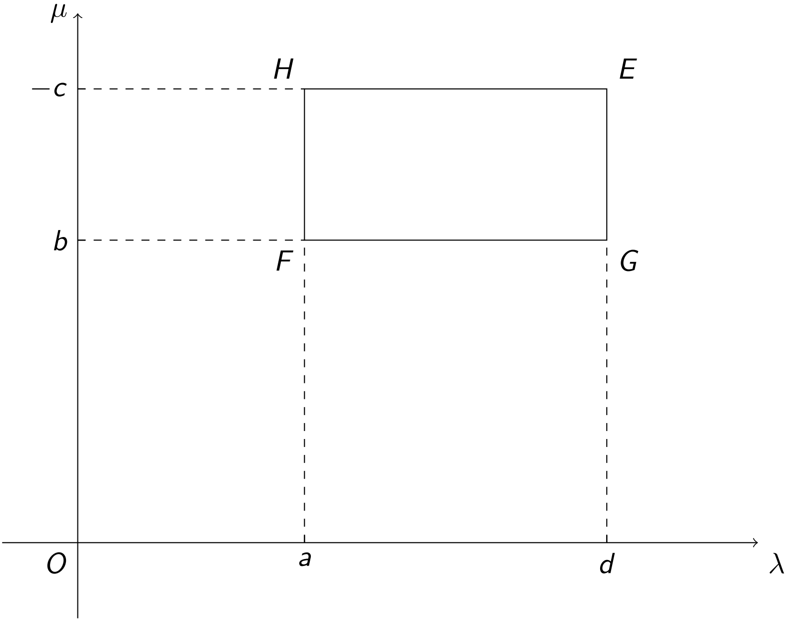

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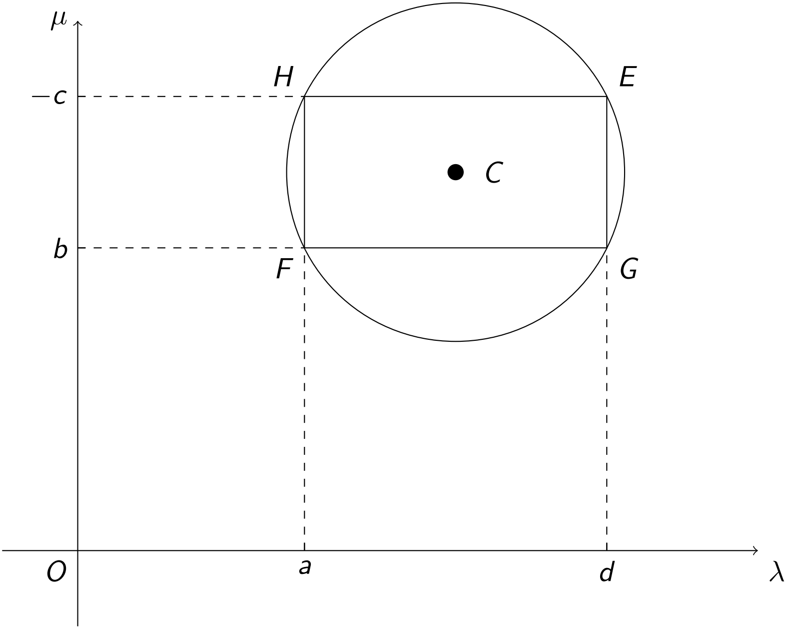

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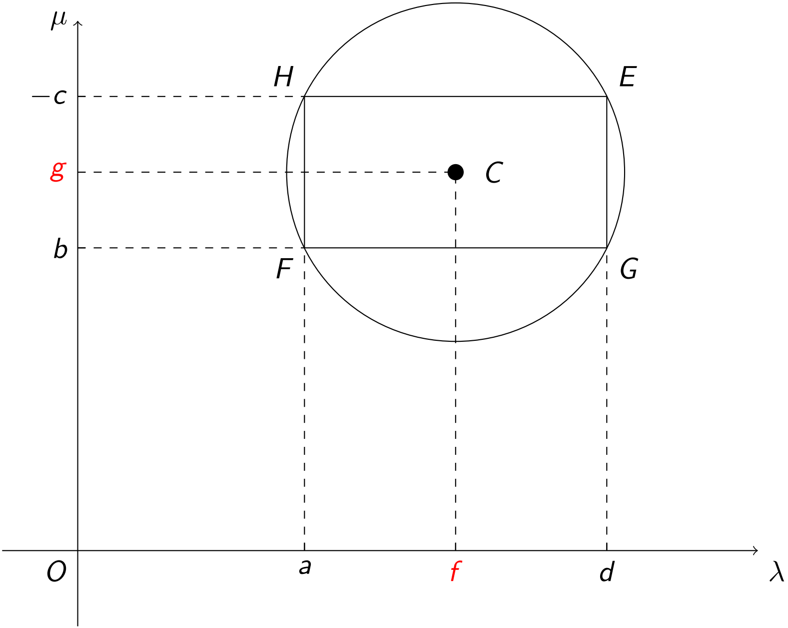

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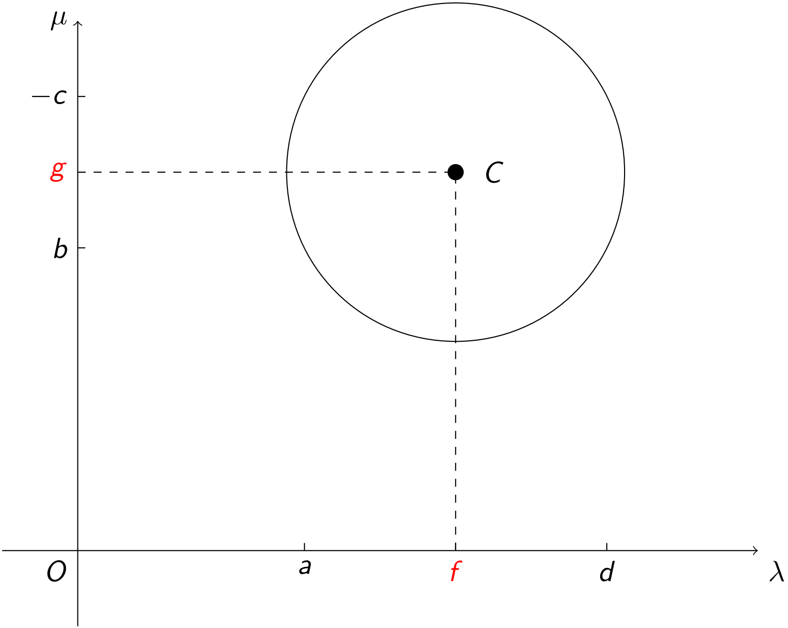

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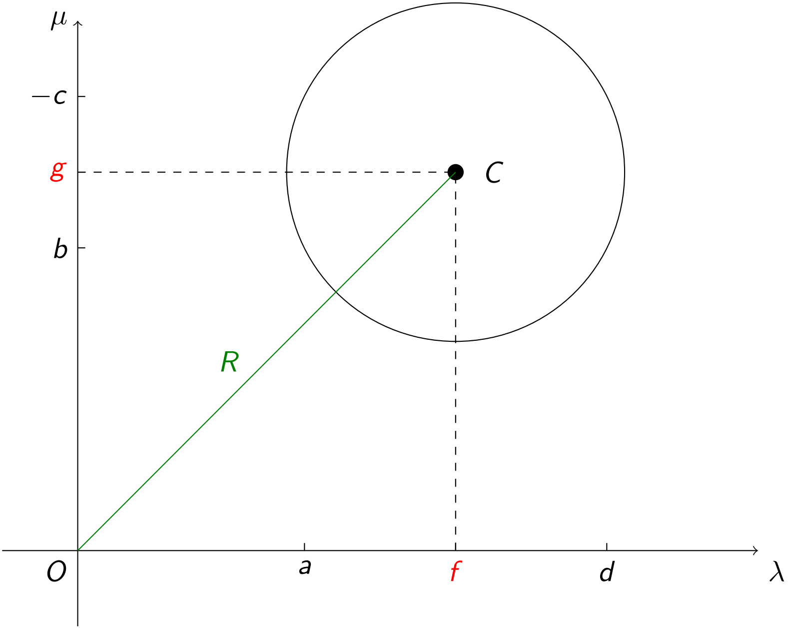

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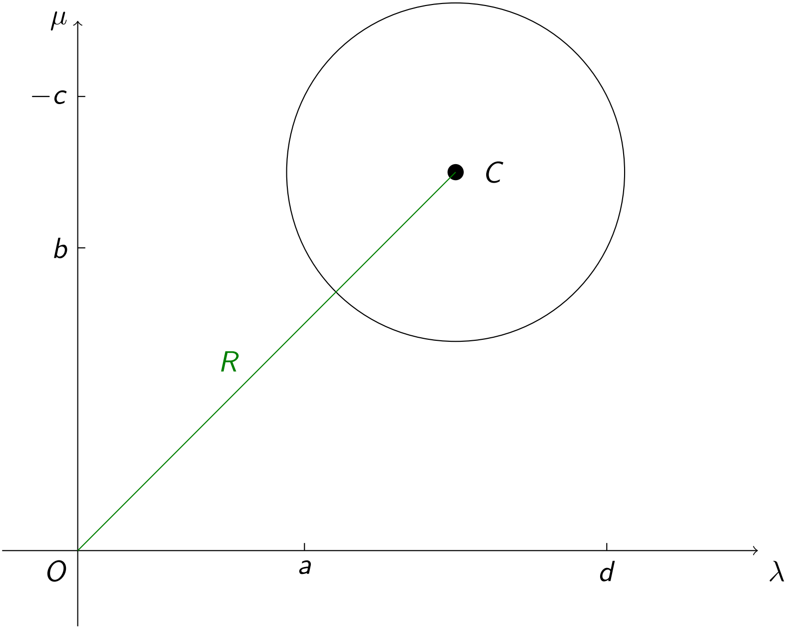

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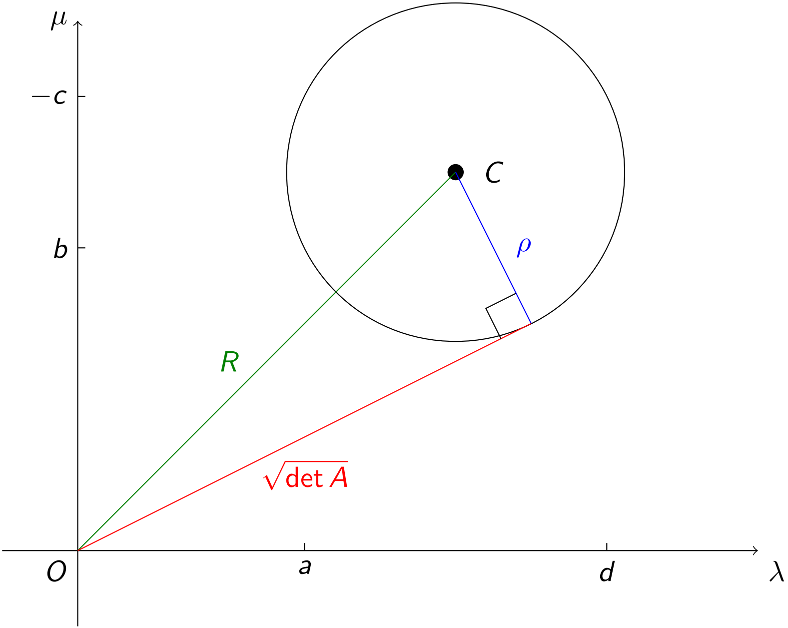

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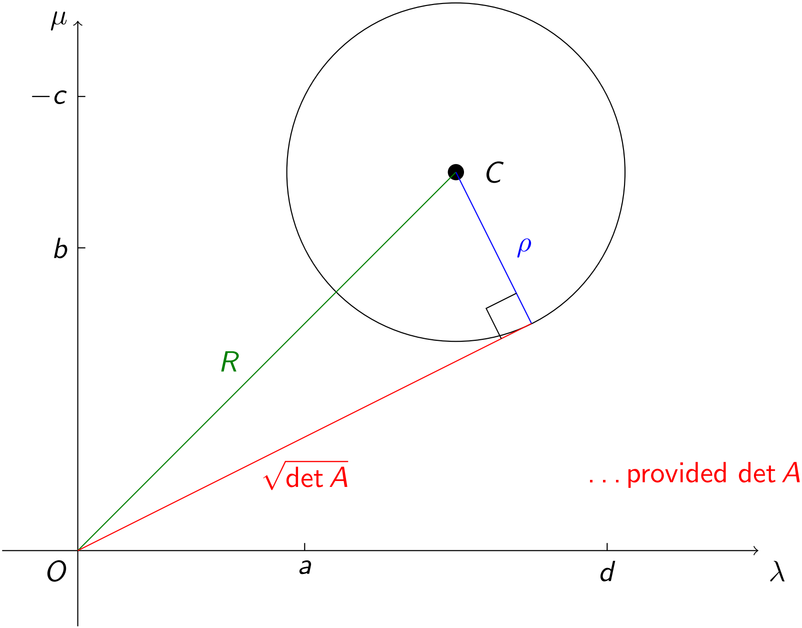

. . . provided det $A > 0$

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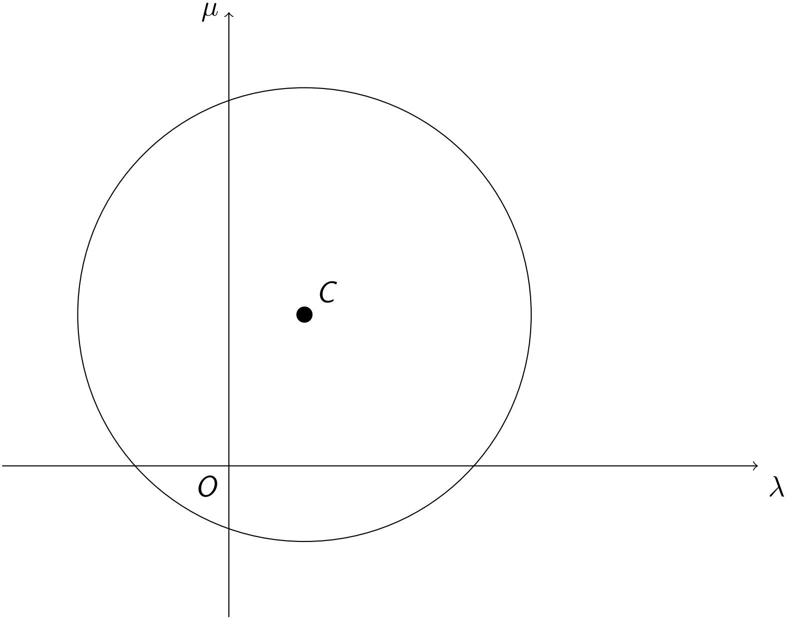

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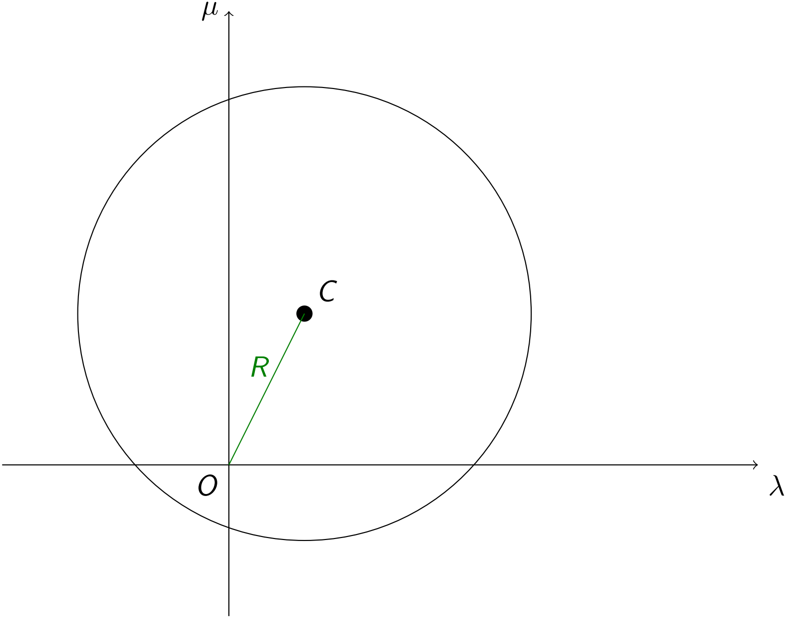

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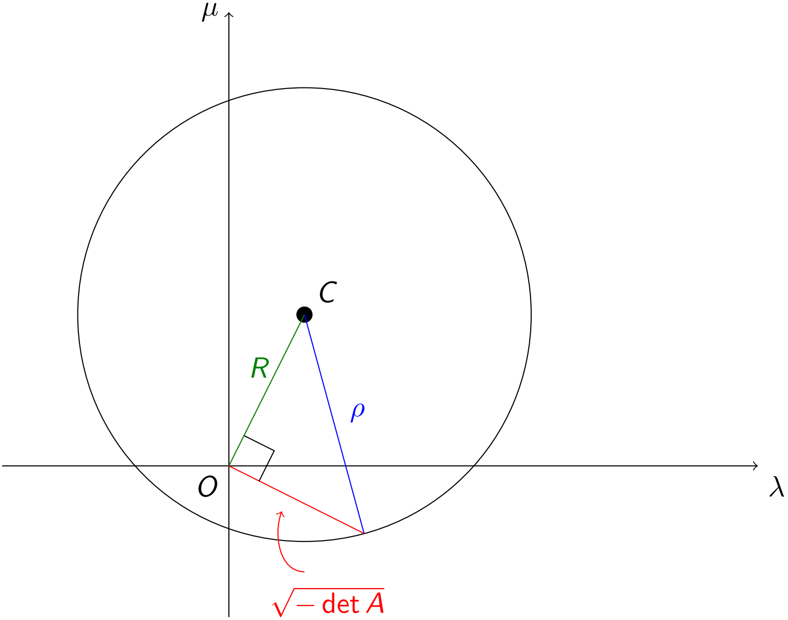

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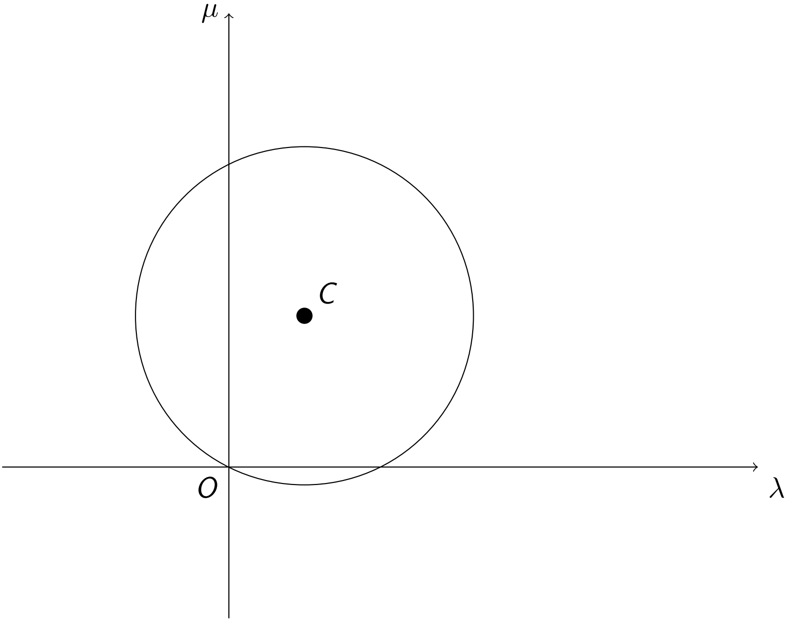

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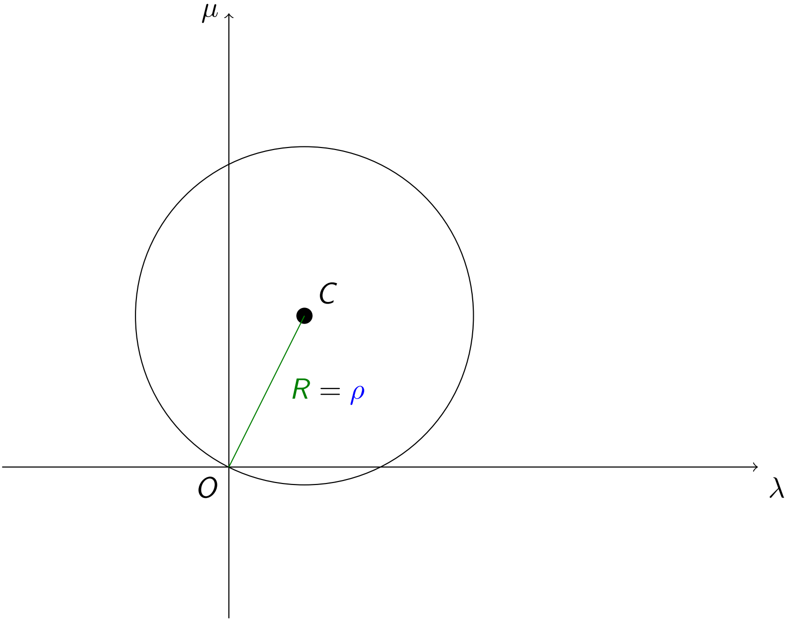

The Eigencircle

Determinant

$$\text{Origin is} \left\{ \begin{array}{c} \text{outside} \\ \text{on} \\ \text{inside} \end{array} \right\} \text{eigencircle} \qquad \Longleftrightarrow \qquad \det A \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

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Real eigenvalues

$$\text{Eigencircle meets } \lambda\text{-axis} \quad \Longleftrightarrow \quad \text{eigenvalues are real}$$

Eigenvectors

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Given (real) eigenvalue λ,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

get eigenvectors: $\begin{pmatrix} x \\ y \end{pmatrix} = \text{any multiple of} \quad \begin{pmatrix} d \\ -c \end{pmatrix} - \begin{pmatrix} \lambda \\ 0 \end{pmatrix}.$

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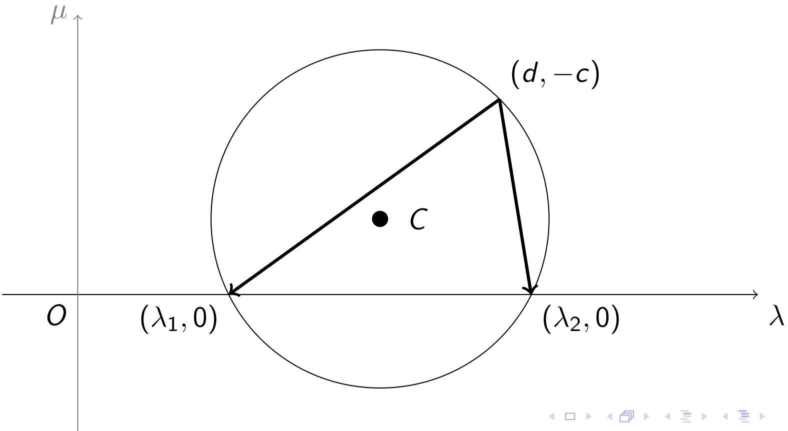

Eigenvectors

For a real symmetric 2×2 *matrix, distinct real eigenvalues have perpendicular eigenvectors.*

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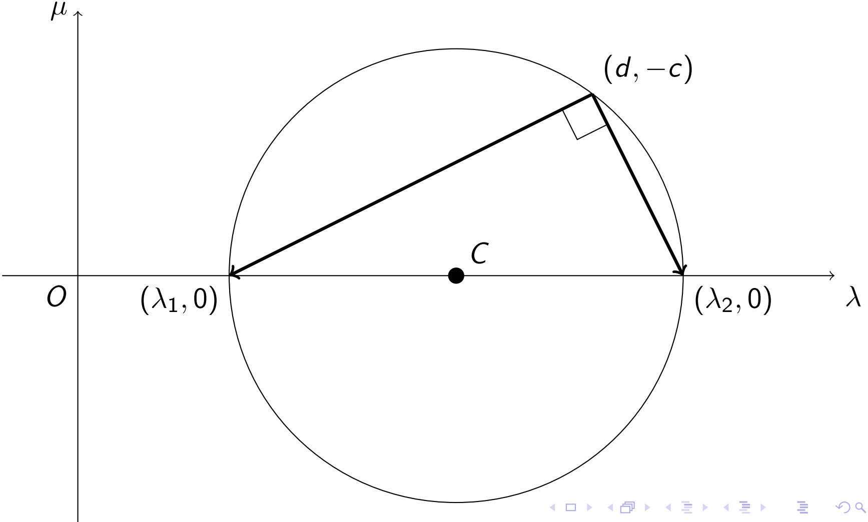
Proof without words:



(λ, μ)-eigenvectors

A (λ, μ)-*eigenvector* is a nonzero $\begin{pmatrix} x \\ y \end{pmatrix}$ corresponding to the eigenpair (λ, μ).

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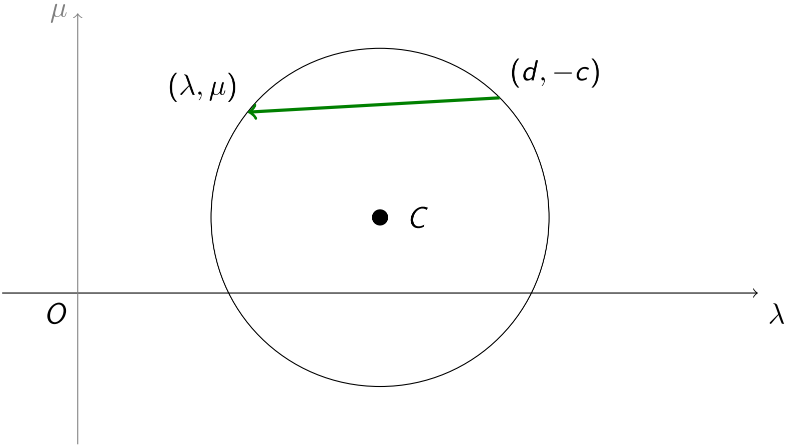

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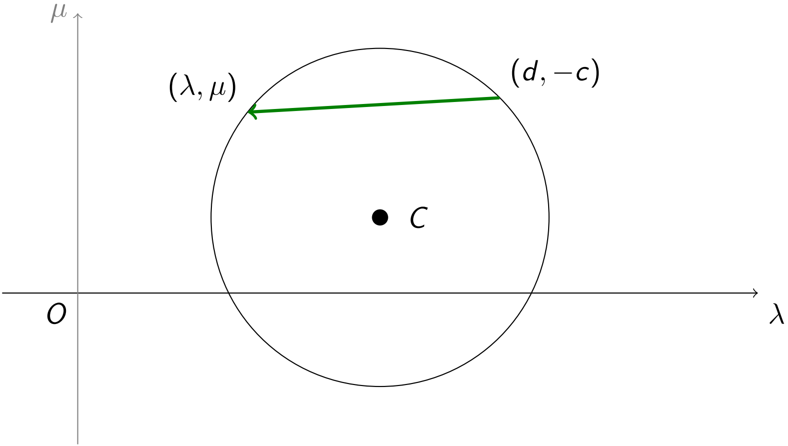

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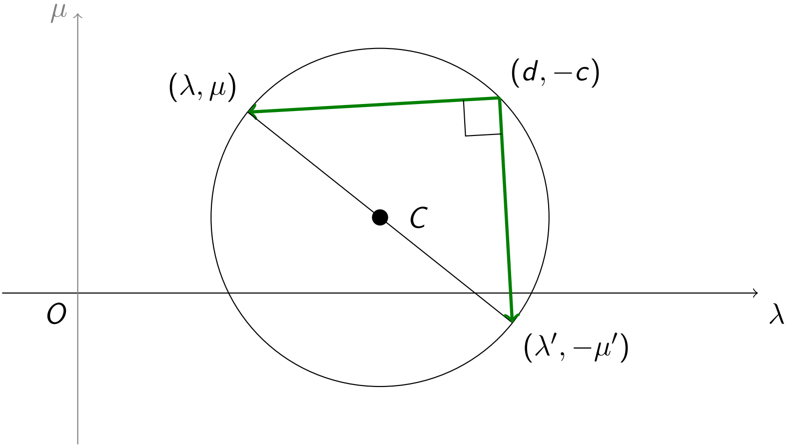

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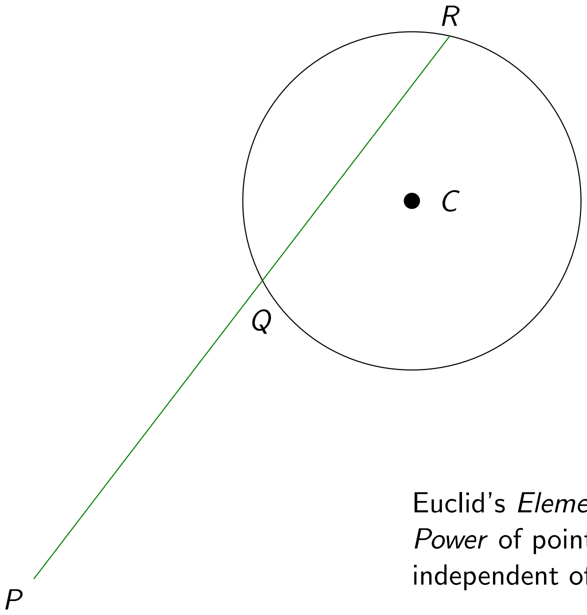
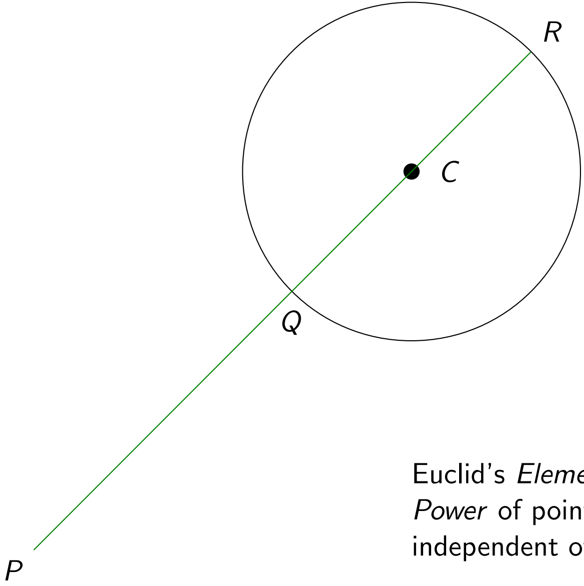
Power and determinant



Euclid's *Elements* III.35–36:
Power of point $= PQ \cdot PR$,
independent of direction of line

Power and determinant



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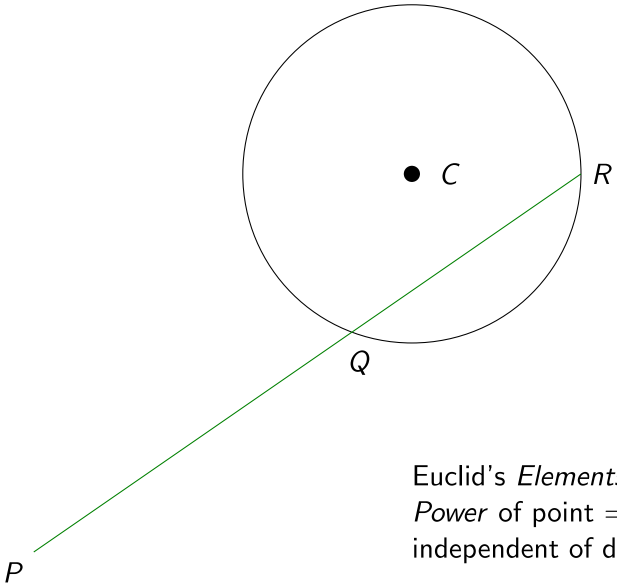

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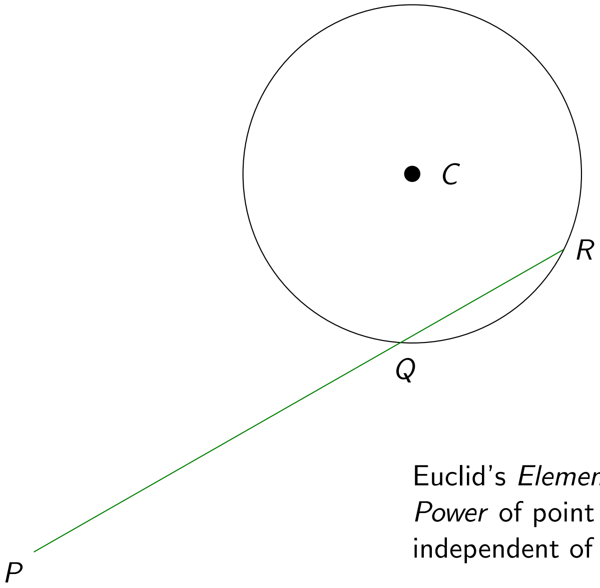

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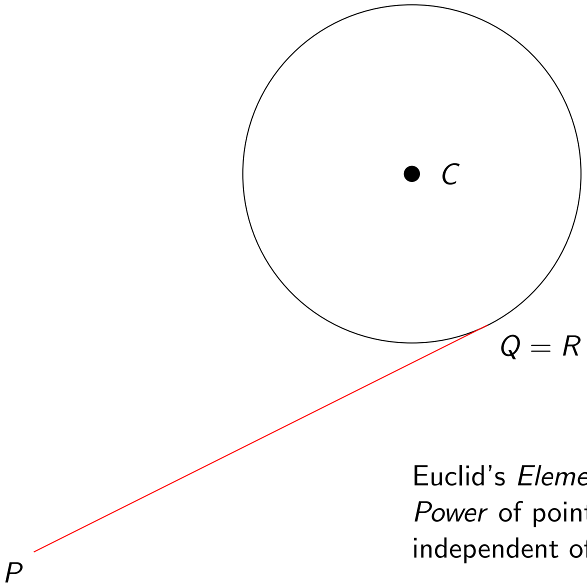

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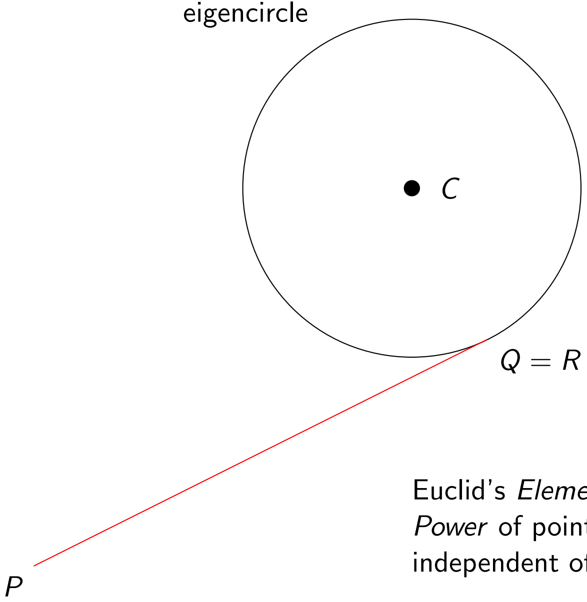

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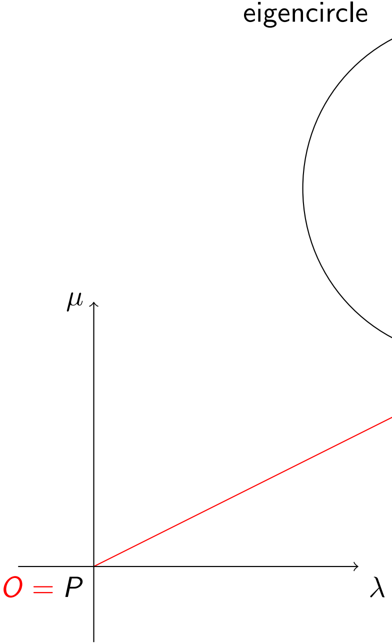

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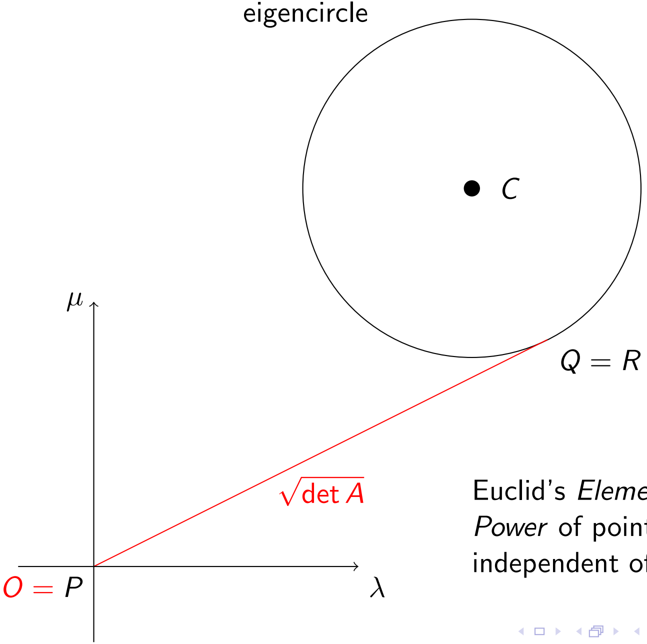

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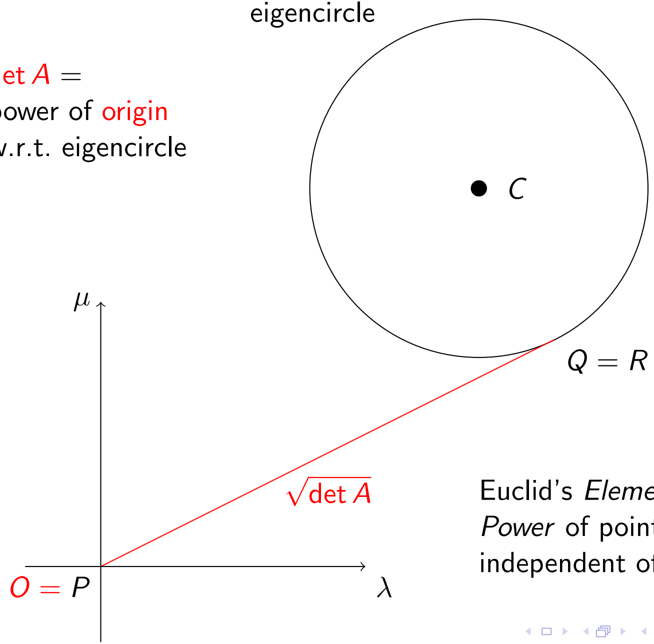

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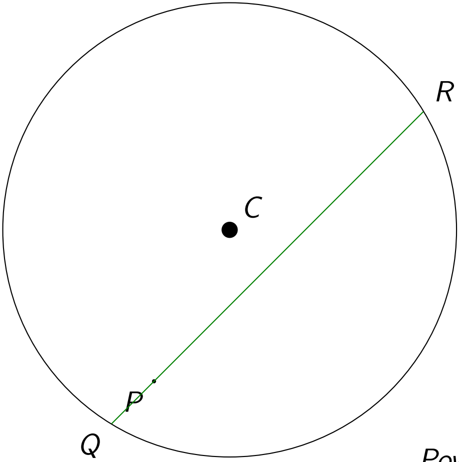
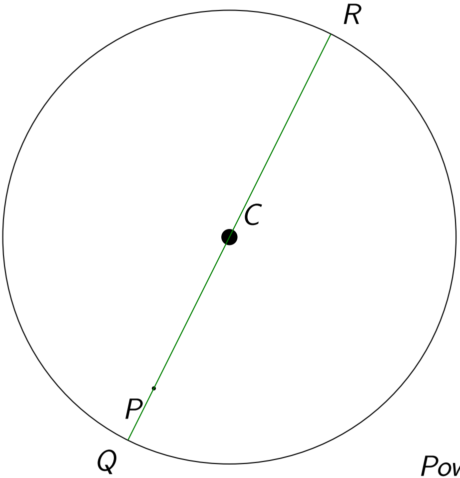


Power of point $= PQ \cdot PR$;
lengths are *signed*, so now < 0

Power and determinant



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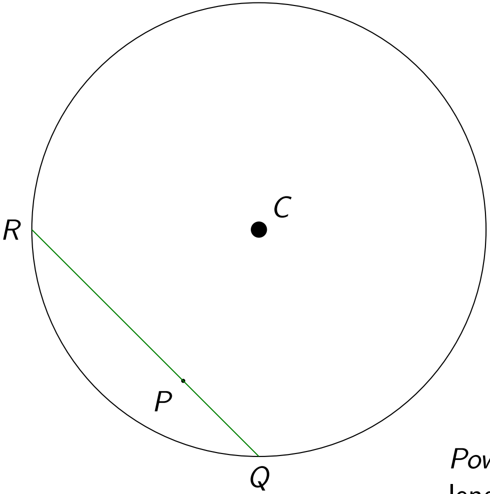

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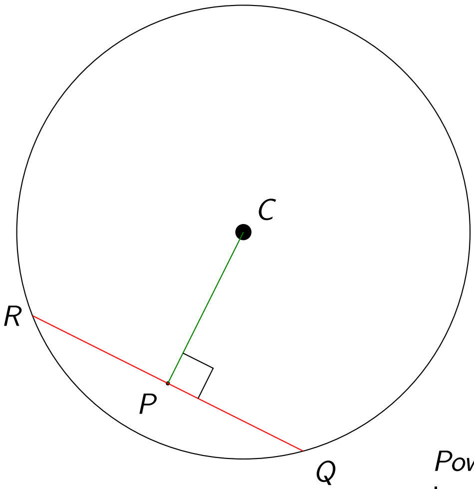

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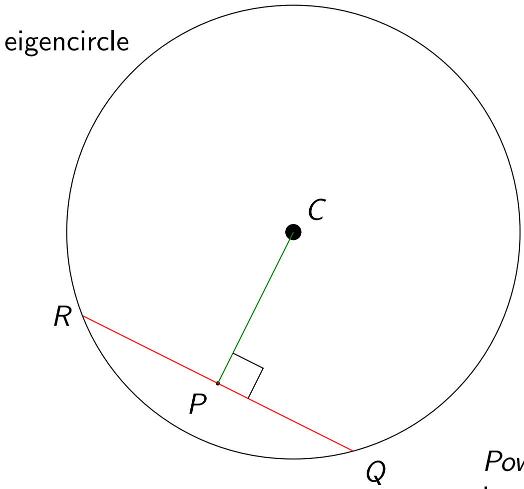

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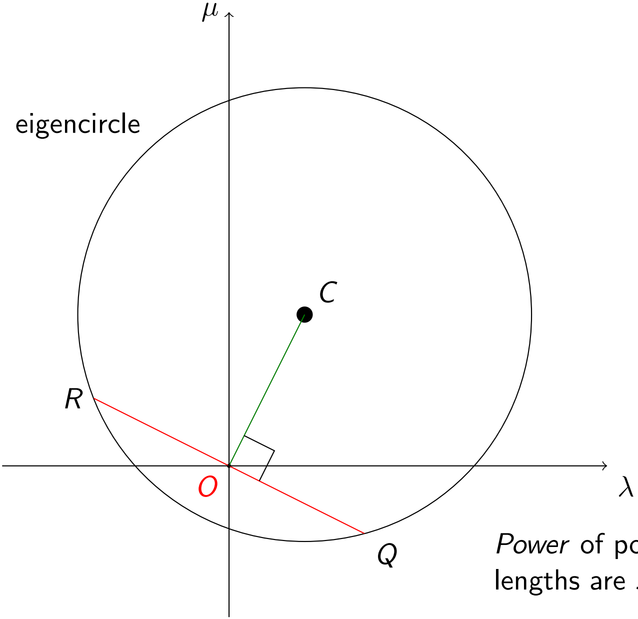

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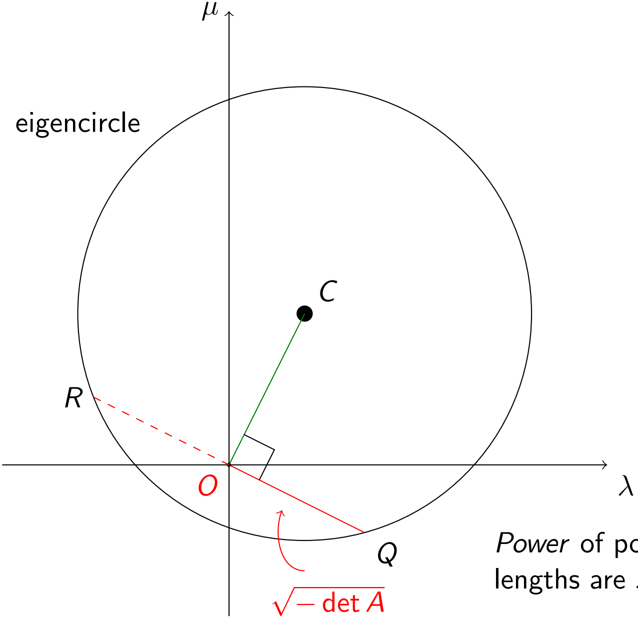

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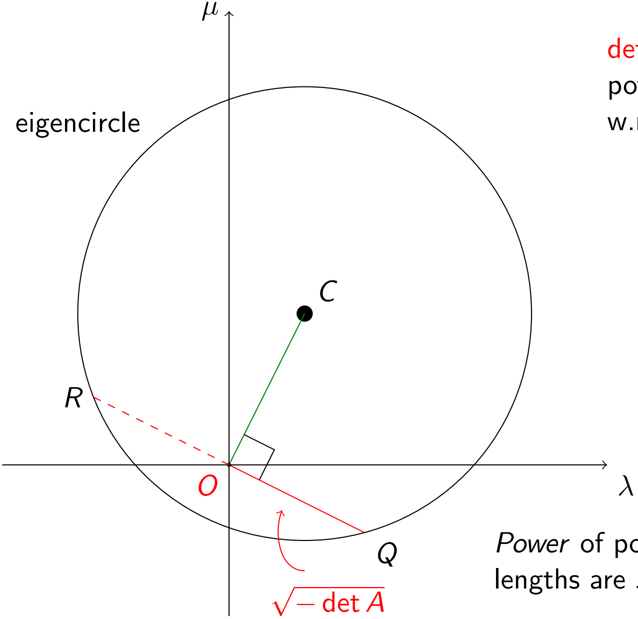

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Power and discriminant

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Characteristic equation of A: $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$

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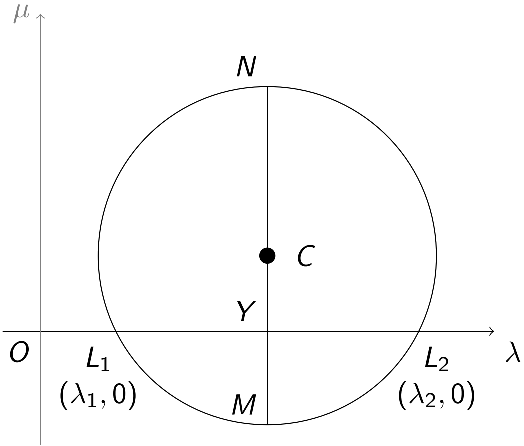

Discriminant:

$$\begin{aligned}\Delta &= (a+d)^2 - 4\det A \\ &= 4(f^2 - \det A) \\ &= 4(\rho^2 - g^2) \\ &= -4(g-\rho)(g+\rho) \\ &= -4 \cdot YM \cdot YN \\ &= -4 \cdot (\text{power of } Y)\end{aligned}$$

Power and discriminant

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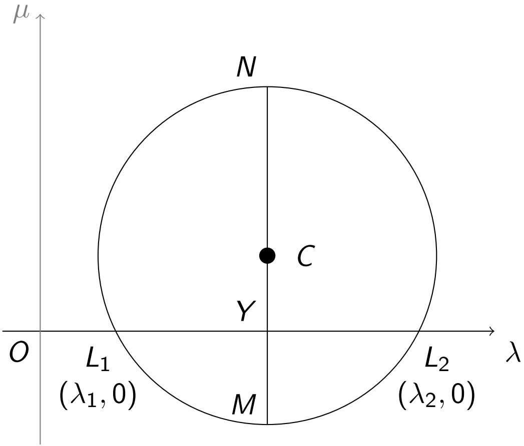

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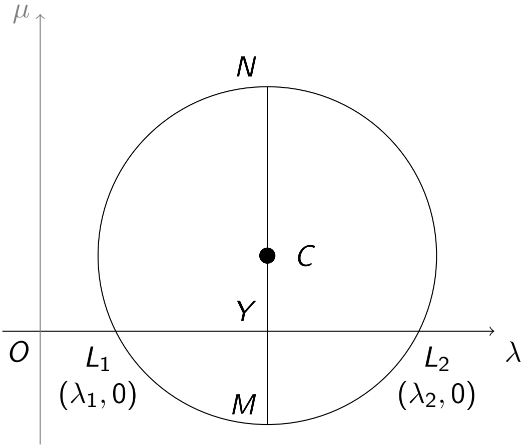

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$$\lambda_1, \lambda_2 = OY \pm YL_i$$

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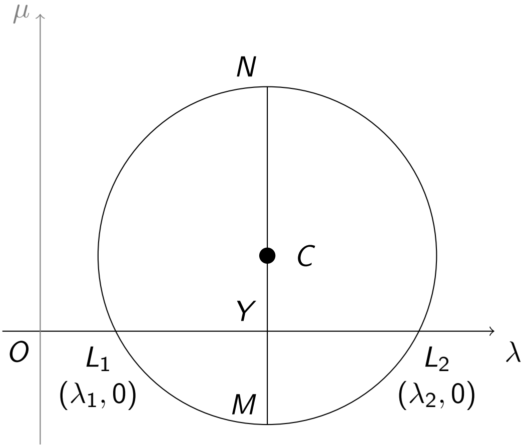

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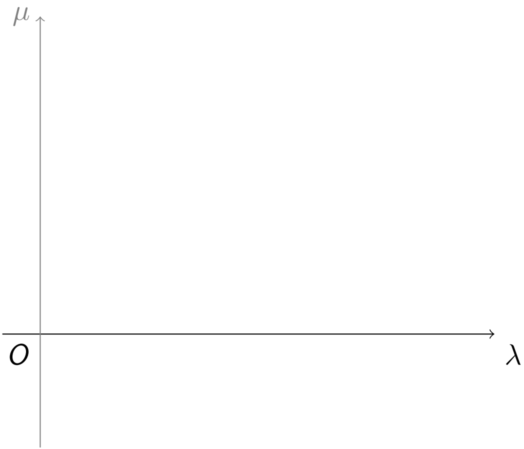
Complex eigenvalues:



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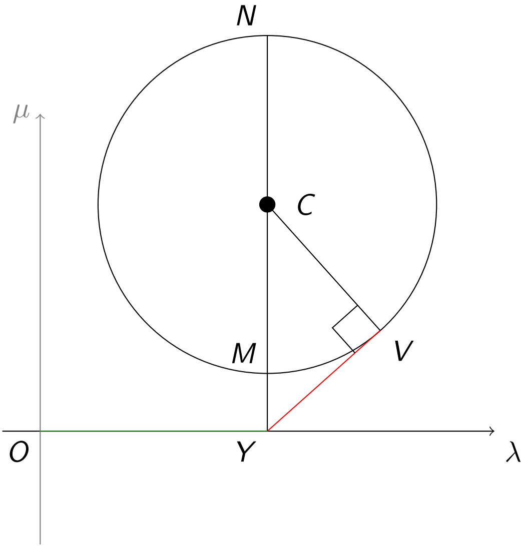
Power and discriminant

Characteristic equation of A: $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

Complex eigenvalues:



Discriminant:

$$
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&= 4(\rho^2 - g^2) \\
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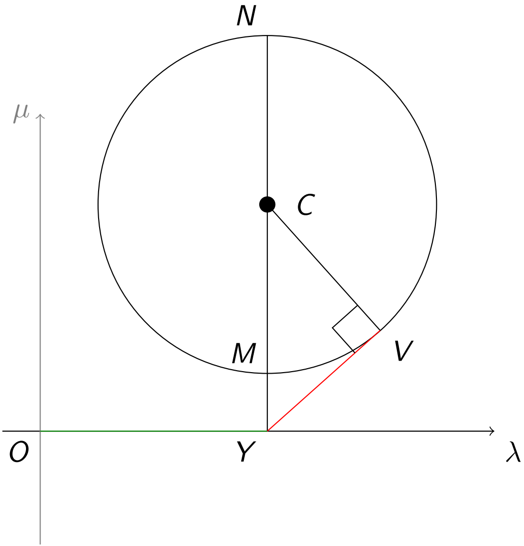

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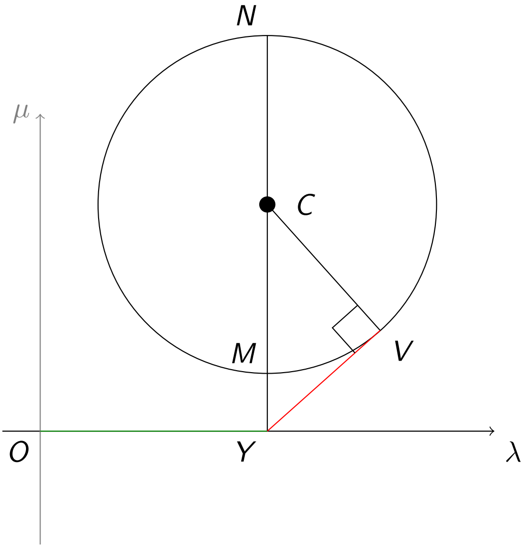

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&= -4 \cdot (\text{power of } Y) \\
&= -4 \cdot (YV)^2\ ;
\end{aligned}$$

$$\lambda_1, \lambda_2 = OY \pm i\, YV$$

Multiparameter eigenvalue problems

Multiparameter eigenvalue problems

Typical system: find k-tuple $(\lambda_1, \ldots, \lambda_k)$ such that

$$\sum_{j=1}^{k} \lambda_j A_{ij} \mathbf{x}_i = \mathbf{0},$$

where:

- each A_{ij} is an $m_i \times n_i$ matrix,
- $\mathbf{x}_i$ is a *nonzero* n_i-element vector.

Work includes:

- R D Carmichael, *Amer. J. Math.*, 1921;
- F V Atkinson, *Bull. Amer. Math. Soc.*, 1968;
- P Binding and P J Browne, *Linear Algebra Appl.*, 1989;
- B D Sleeman, *Mulitparameter Spectral Theory in Hilbert Space*, Pitman, 1978;
- H Volkmer, *Multiparameter eigenvalue problems and expansion theorems*, Springer, 1988.

Other results

Can use eigencircle to illustrate:

- geometric proof that product of eigenvalues is determinant
- expression for angle between eigenvectors
- set of all matrices with a given pair of eigenvalues
- matrices are linear combinations of rotation and reflection matrices
- complex eigenpairs: can illustrate using hyperbola in a 3rd dimension, and a hyperboloid in 4D
- quaternions: can illustrate using an *eigensphere*

Note: all this is for 2×2 matrices.

Future work

- Pick your favourite property of matrices.
  See if the 2×2 case can be illustrated using the eigencircle.
- Extend to larger matrices: 3×3, 4×4, $n \times n$.
- Develop further for quaternions (eigensphere).

Further information

- M J Englefield and G E Farr, Eigencircles of 2x2 matrices, *Mathematics Magazine* **79** (October 2006) 281–289.
- M J Englefield and G E Farr, Eigencircles and associated surfaces, preprint, 2006.