A survey of Tutte-Whitney polynomials

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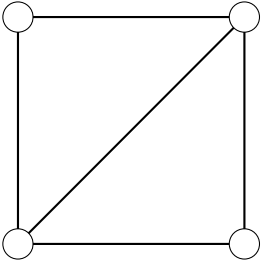
Counting colourings

- proper colourings



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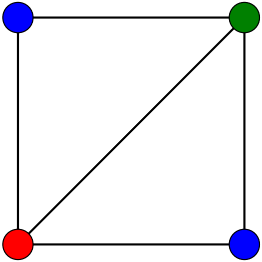

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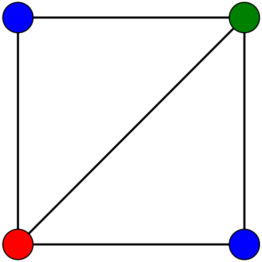

Adjacent vertices receive different colours

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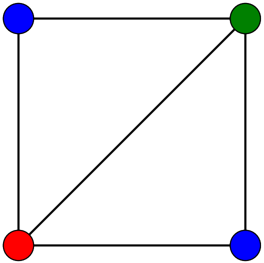


Adjacent vertices receive different colours

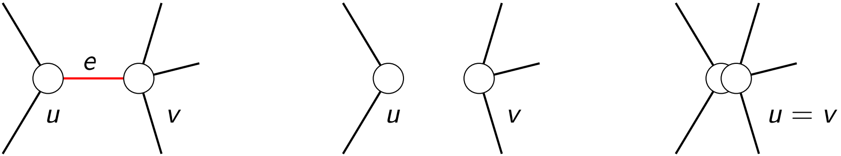
- chromatic polynomial:

$$P(G;q) = \#\ q\text{-colourings of } G$$

Deletion-contraction

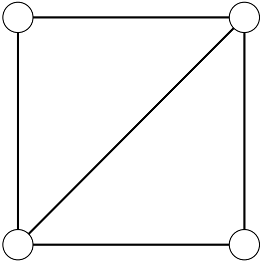
For any edge e:

$$P(G;q) \quad = \quad P(G \setminus e;q) \quad - \quad P(G/e;q)$$



Partition functions: Potts models

- general q-colourings (may be improper)



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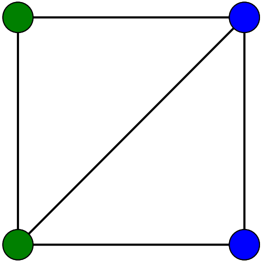

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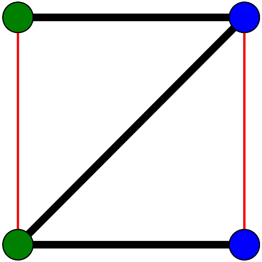

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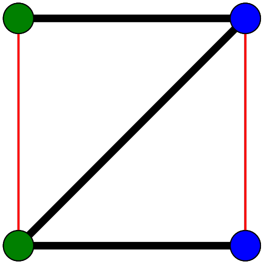

Good and bad edges

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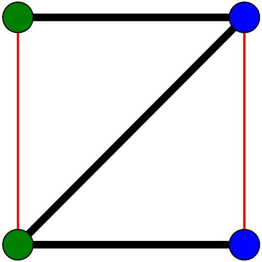

Good and bad edges

- Partition function:

$$Z(G;K,q) = \sum_{\substack{\text{all } q\text{-colourings} \\ \text{(not just proper)}}} e^{-K\cdot(\#\text{ good edges})}$$

All-terminal reliability

- Choose edges randomly: $\Pr(edge) = p$

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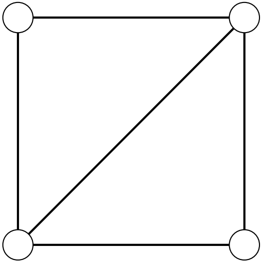

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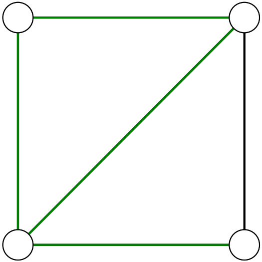

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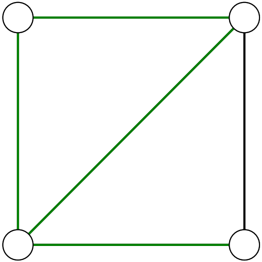

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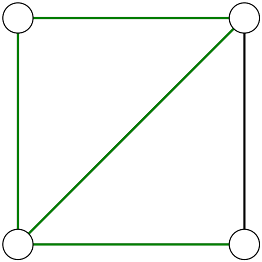

chosen edges

- Reliability:

$$\Pi(G, p) = \Pr(\text{chosen edges contain a spanning tree})$$

. . . etc

. . . etc

- flow polynomial

. . . etc

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Tutte-Whitney polynomials

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  for all $X \subseteq E$:

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- Tutte polynomial:

  $$T(G; x, y) = R(G; x-1, y-1).$$

The "Recipe Theorem"

Theorem
(Tutte 1947 $\to$ Brylawski 1972 $\to$ Oxley & Welsh 1979)
If a function f on graphs ...

- *is invariant under isomorphism,*
- *satisfies a deletion-contraction relation,*
- *is multiplicative over components (i.e., $f(G_1 \cup G_2) = f(G_1) \cdot f(G_2)$),*

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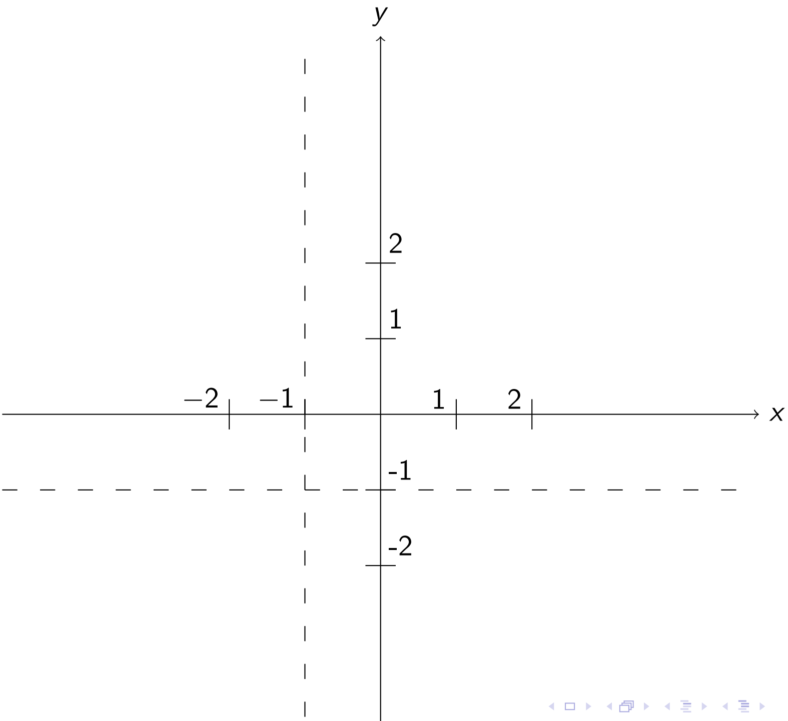
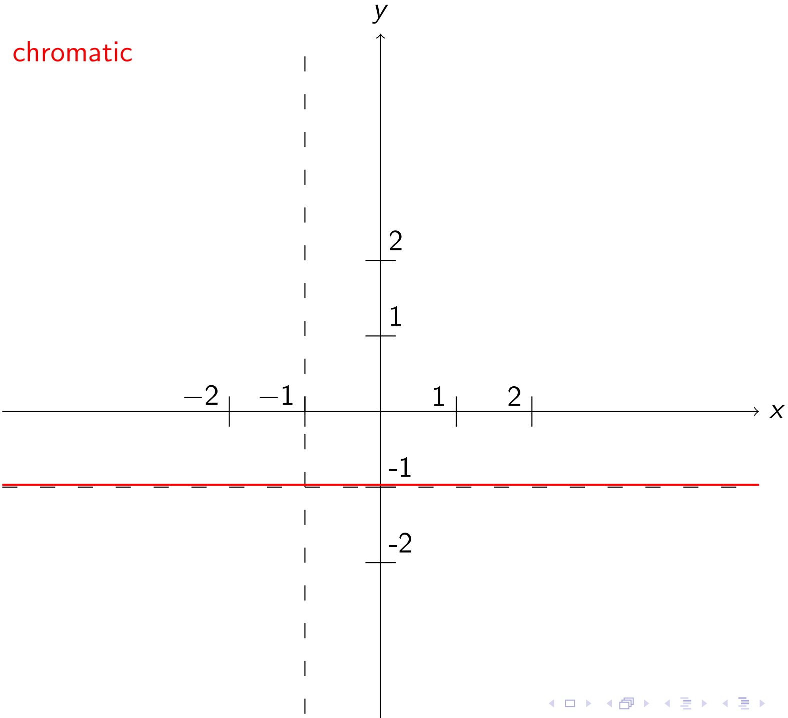
Example

$$P(G;q) = (-1)^{\rho(E)} q^{k(G)} R(G;-q,-1)$$

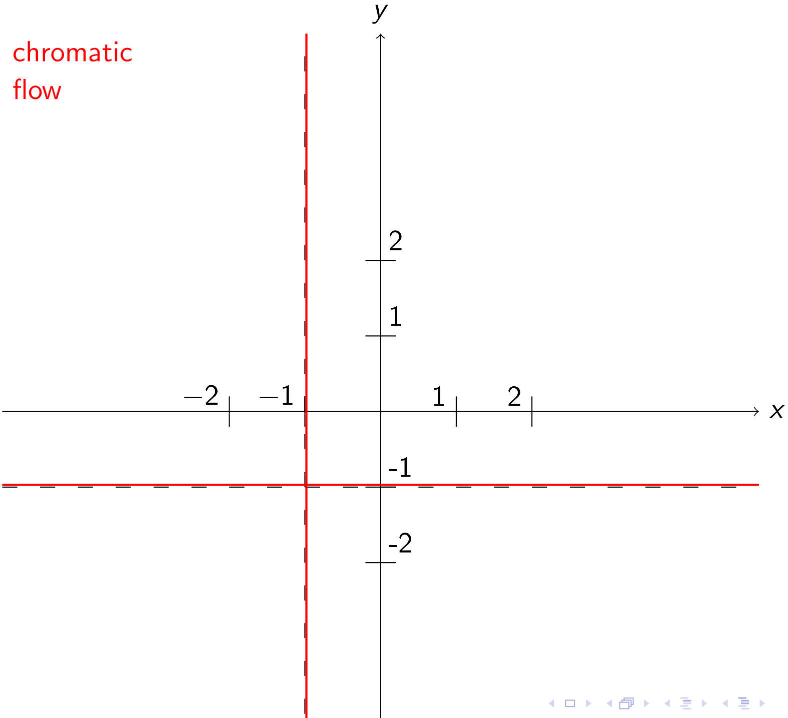

y
x
2
1
−2
−1
1
2
-1
-2

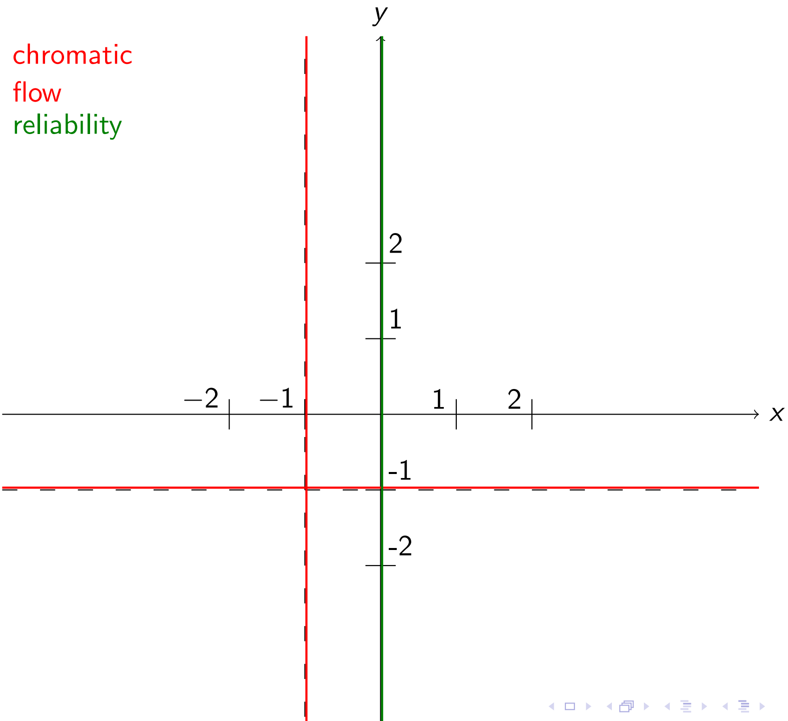
chromatic



chromatic

flow





chromatic
flow
reliability
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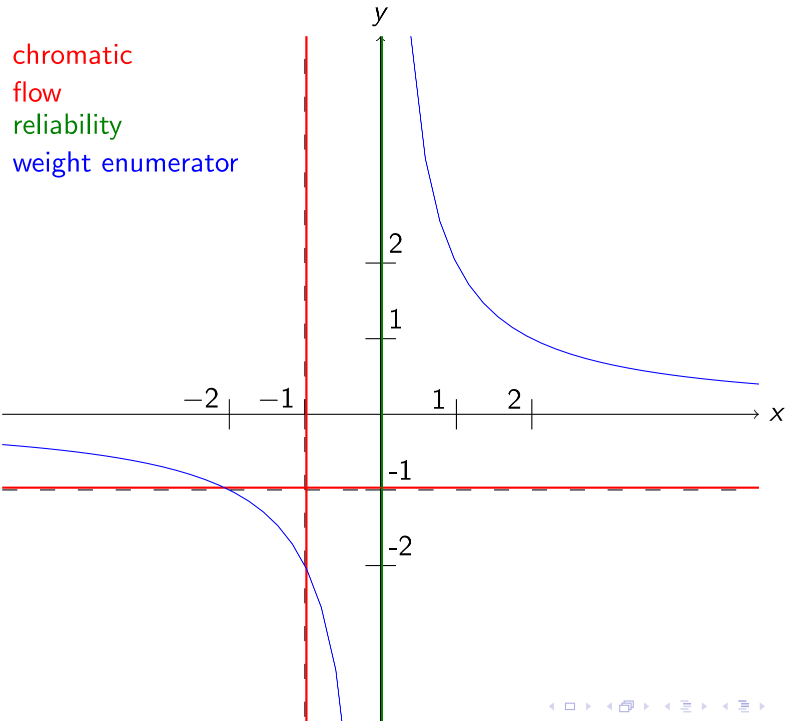
chromatic
flow
reliability
weight enumerator
y
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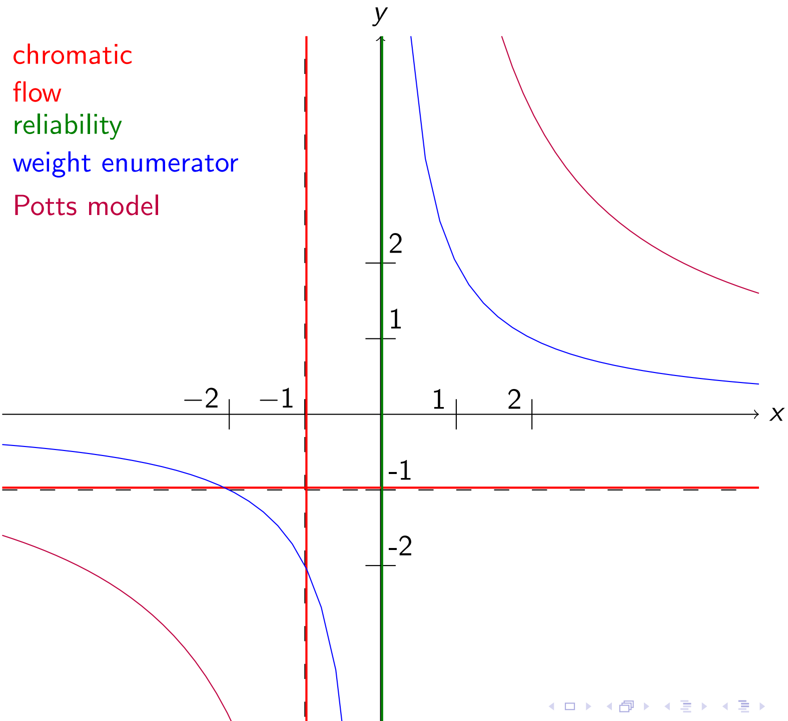

chromatic
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Potts model
y
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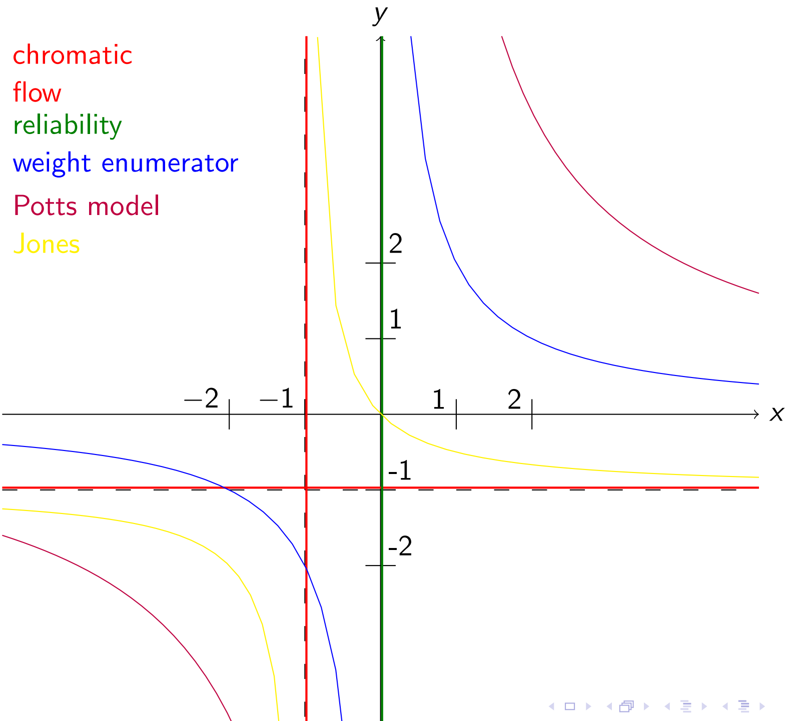
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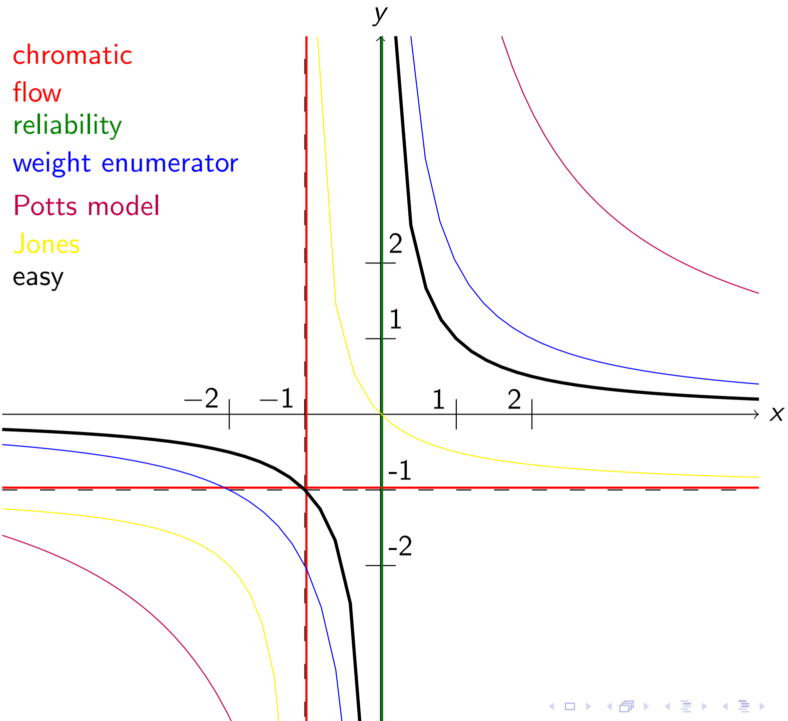
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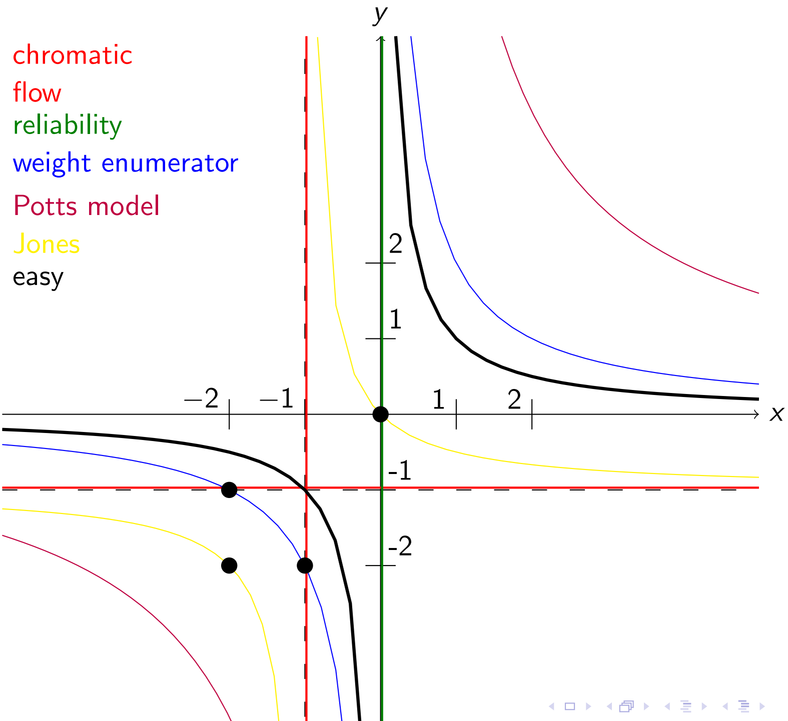
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History

History

Graphs:
Chrom.
poly

History

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Birkhoff
1912

$P(G;q)$

History

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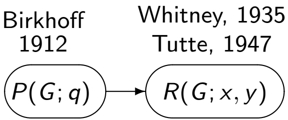

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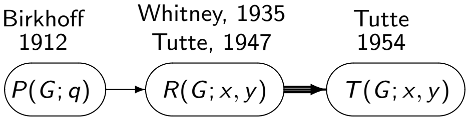

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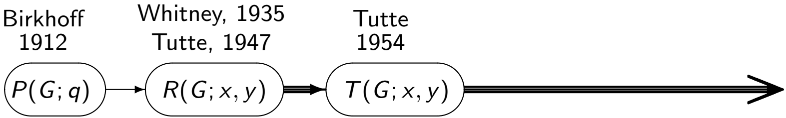

History

Stat Mech:
partition
functions

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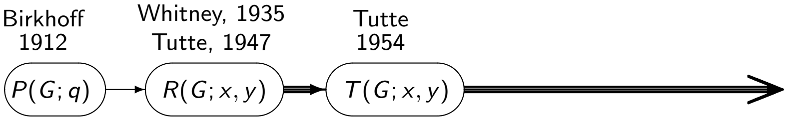

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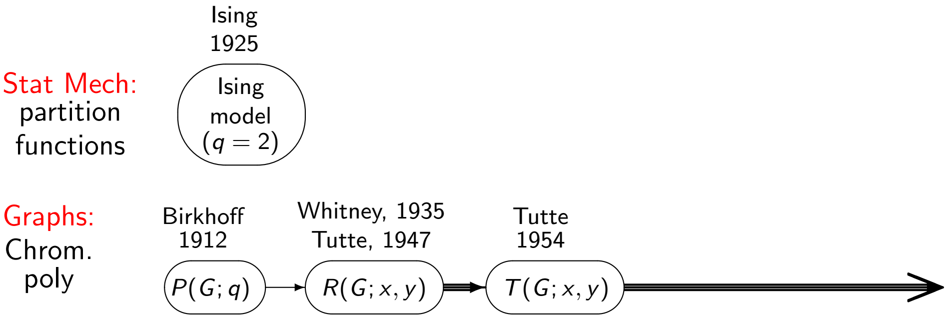
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Ising model ($q = 2$)
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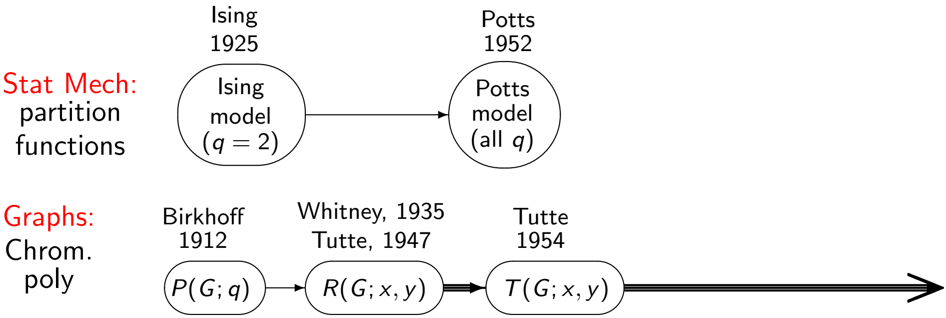
Ising
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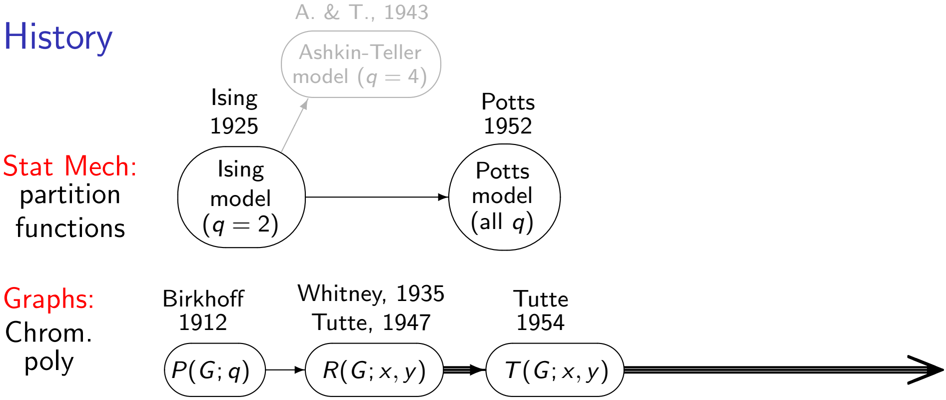
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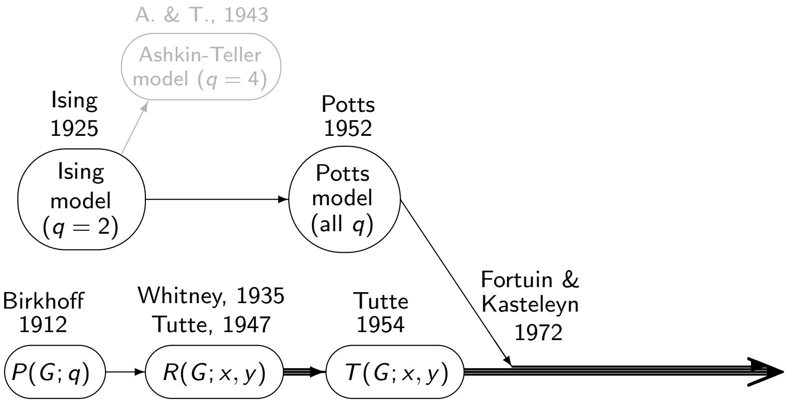

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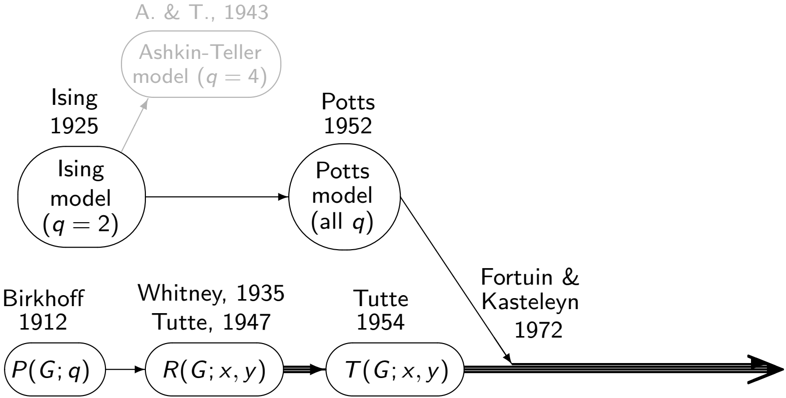
Linear codes: weight enumerator



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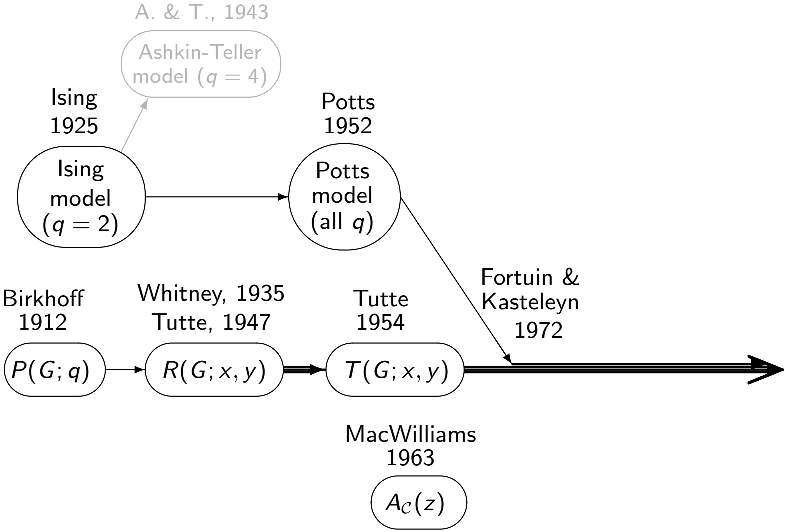

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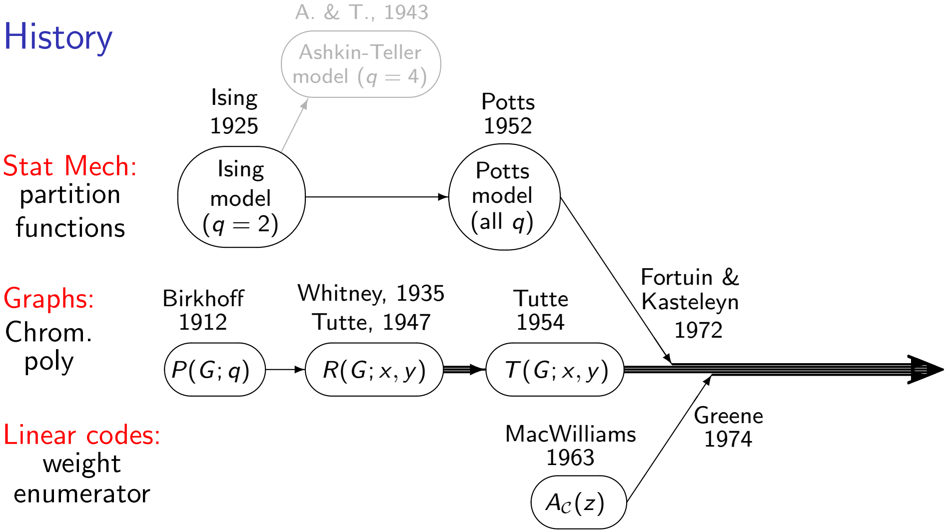

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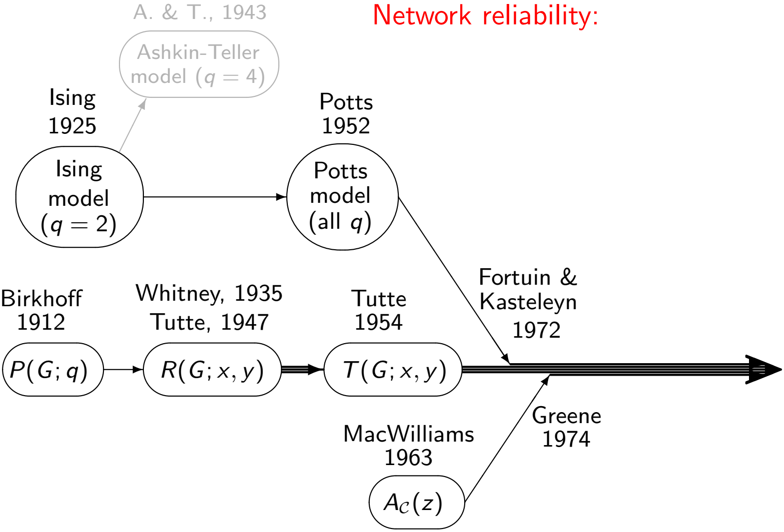

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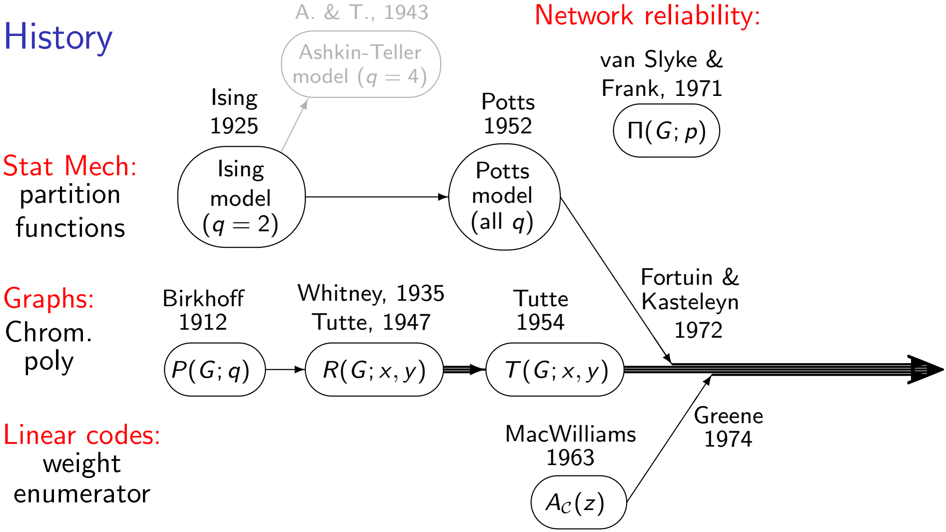
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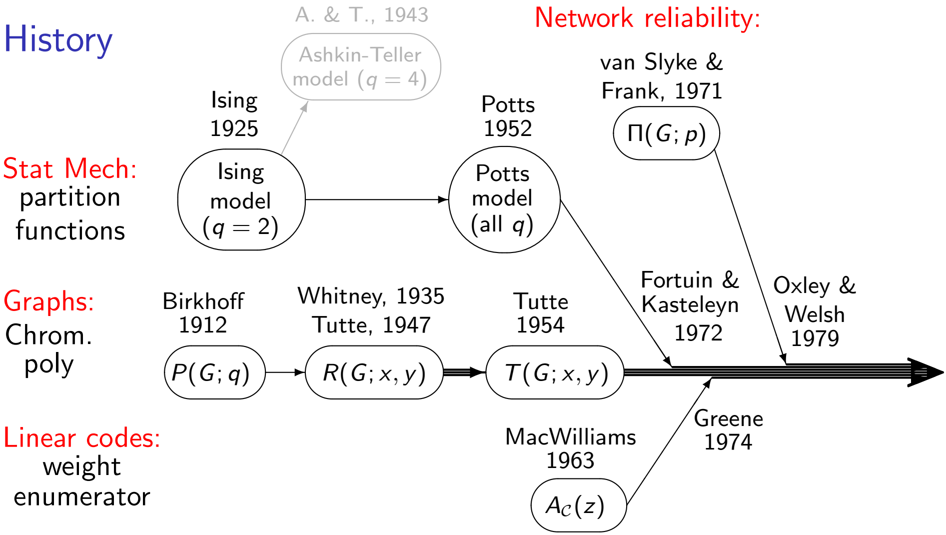
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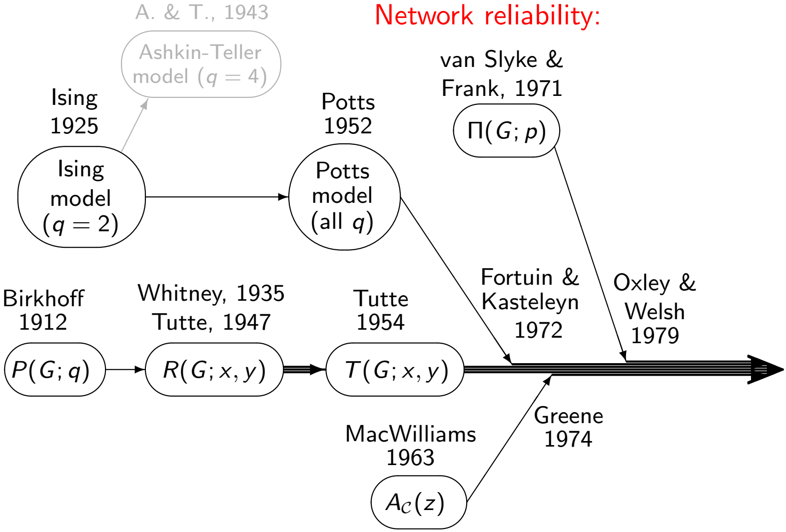


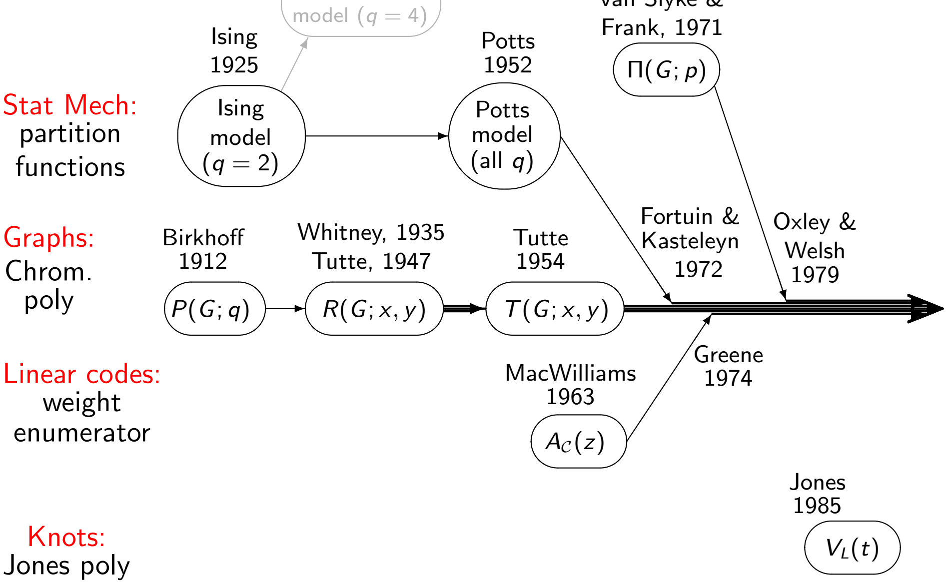
Stat Mech: partition functions

Graphs: Chrom. poly

Linear codes: weight enumerator

Knots: Jones poly

History



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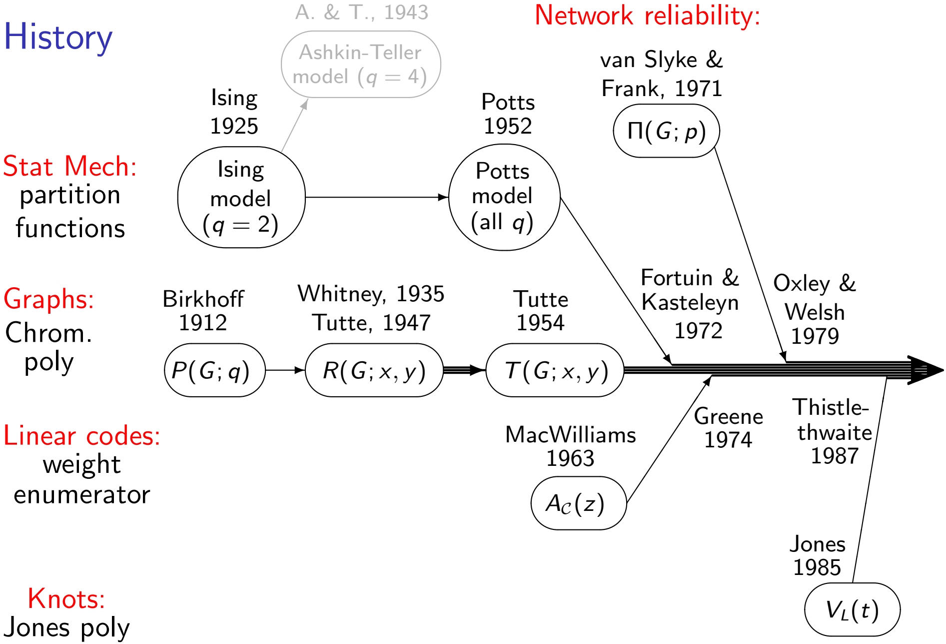
V_L(t)

History



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Complexity of computing all of $R(G; x, y)$

- Graphs: #P-hard (Linial, 1986)

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- Square grid graphs: Open (in $\#P_1$)
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- Square grid subgraphs, max deg 3: #P-hard (GF, 2006)
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Complexity of evaluating at specific points

Theorem

(Jaeger, Vertigan and Welsh, 1990)
The problem of determining $R(G; x, y)$, given a graph G, is #P-hard at all points (x, y) except those where $xy = 1$ and $(x, y) = (0, 0), (-1, -2), (-2, -1), (-2, -2)$.

Generalisations

Extensions from graphs to:

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- representable matroids (Smith), matroids (Tutte, Crapo), greedoids (Gordon & McMahon), Boolean functions or set systems (GF), hyperplane arrangements (Welsh & Whittle, Ardila), semimatroids (Ardila), signed graphs (Murasugi), rooted graphs (Wu, King & Lu), K-terminal graphs (Traldi), biased graphs (Zaslavsky), matroid perspectives (Las Vergnas), matroid pairs (Welsh & Kayibi), bimatroids (Kung), graphic polymatroids (Borzacchini), general polymatroids (Oxley & Whittle), . . .

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. . . or extend the polynomials:

- multivariate polynomials of various kinds: variables at each vertex (Noble & Welsh), or edge (Fortuin & Kasteleyn, Traldi, Kung, Sokal, Bollobás & Riordan, Zaslavsky, Ellis-Monaghan & Riordan, Britz).

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We now look at a generalisation to Boolean functions . . .

Rank $\leftrightarrow$ rowspace

Incidence matrix

$$\text{vertices} \quad \overset{\text{edges}}{\begin{pmatrix} & \vdots & \\ \cdots & 0/1 \text{ entries} & \cdots \\ & \vdots & \end{pmatrix}}$$

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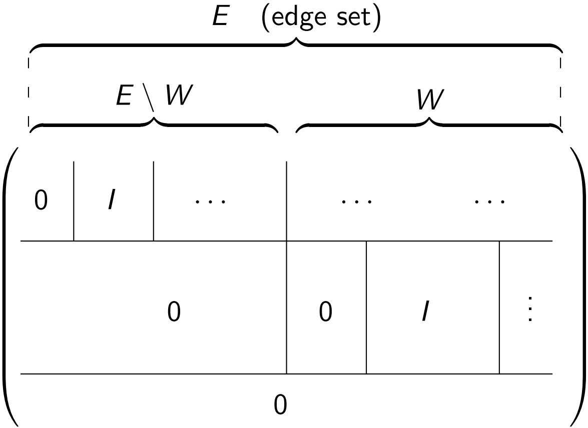

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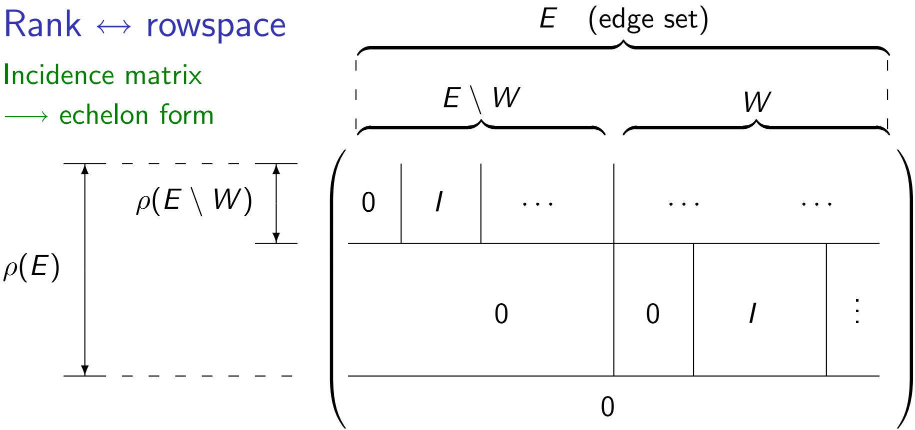

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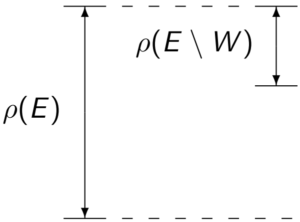

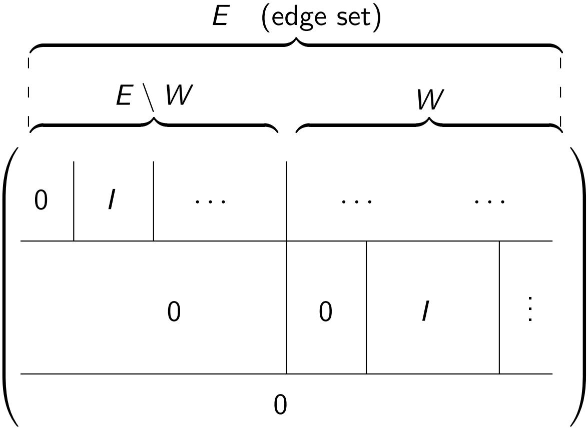

Count rowspace members that are 0 outside W:

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Basic properties:

- $(Q^\dagger Q f)(V) = \dfrac{f(V)}{f(\emptyset)}$
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$$\begin{array}{rcl} f & \xrightarrow{\quad Q \quad} & Qf \\ \text{Hadamard transform} \downarrow & & \downarrow \text{matroid-style dual} \\ \hat{f} & \xrightarrow{\quad Q \quad} & (Qf)^* = Q\hat{f} \end{array}$$

Extending the Whitney rank generating function

$$R(f;x,y) = \sum_{X \subseteq E} x^{Qf(E)-Qf(X)} y^{|X|-Qf(X)}$$

Extending the Whitney rank generating function

$$
\begin{aligned}
R(f;x,y) &= \sum_{X \subseteq E} x^{Qf(E)-Qf(X)} y^{|X|-Qf(X)} \\
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$E = \{1,2\}$

X	f
$\emptyset$	1
$\{1\}$	1
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$$R(f;x,y) = x^{\log_2 3} + 2xy^{2-\log_2 3} + y^{2-log_2 3}$$

Deletion-contraction

For $e \in E$, $X \subseteq E \setminus \{e\}$:

Deletion

$$(f \mathbin{\backslash\!\backslash} e)(X) = \frac{f(X) + f(X \cup \{e\})}{f(\emptyset) + f(\{e\})};$$

Contraction

$$(f /\!/ e)(X) = \frac{f(X)}{f(\emptyset)}.$$

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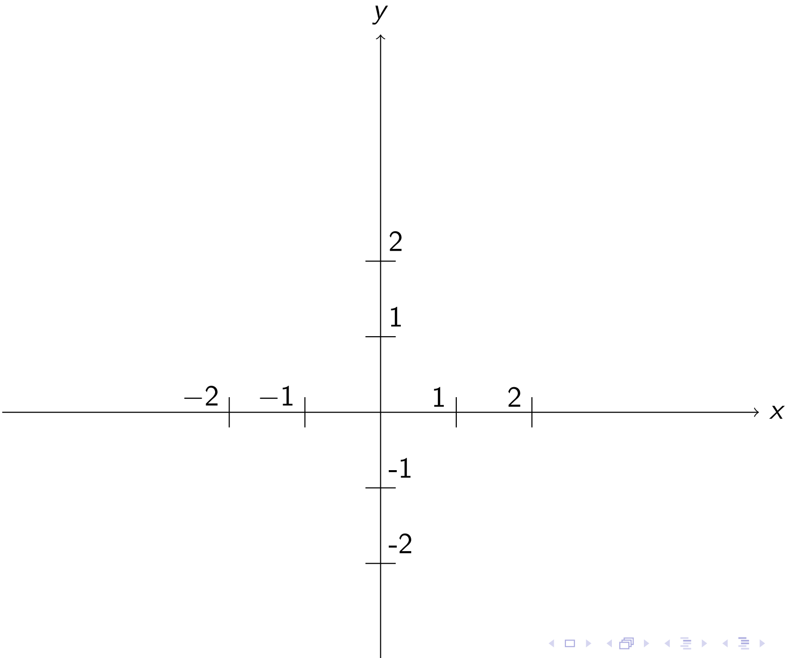
Deletion-contraction rule:

$$R(f; x, y) = x^{\log_2\left(1+\frac{f(\{e\})}{f(\emptyset)}\right)} R(f \setminus\!\!\setminus e; x, y) + y^{\log_2\left(1+\frac{\hat{f}(\{e\})}{\hat{f}(\emptyset)}\right)} R(f /\!\!/ e; x, y)$$

Generalised Tutte-Whitney plane



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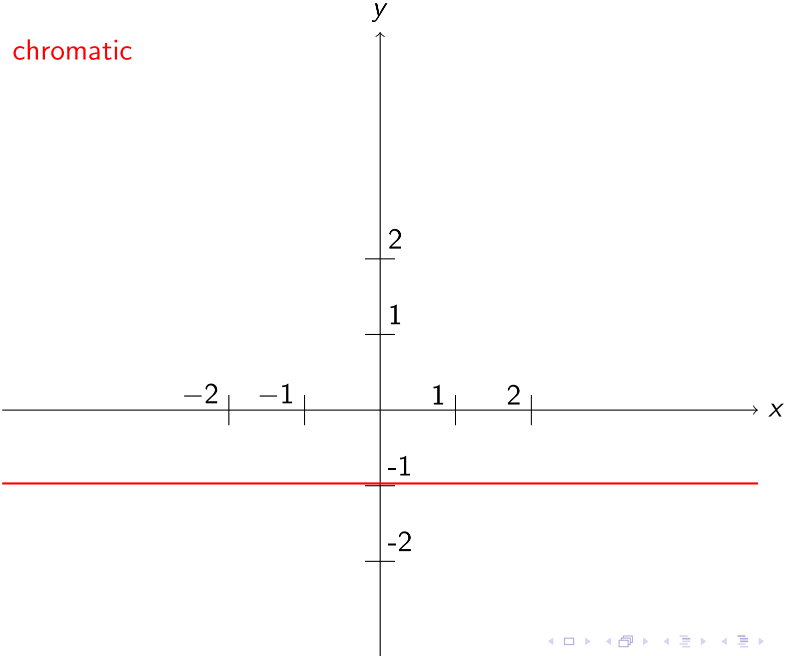

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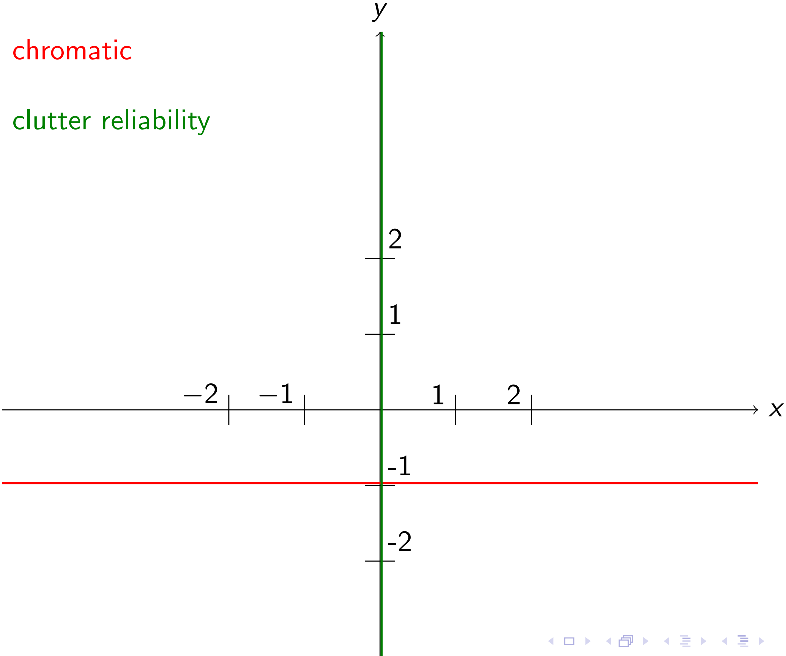

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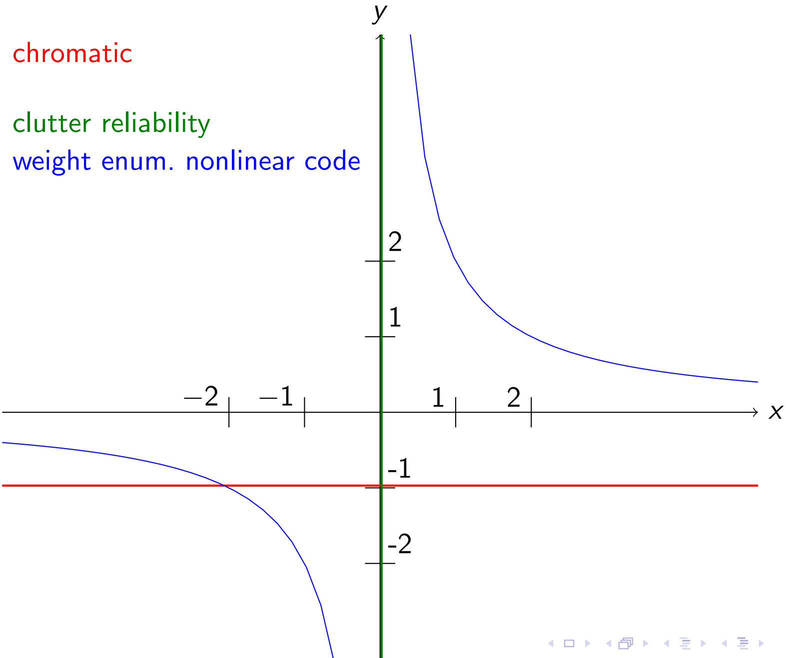

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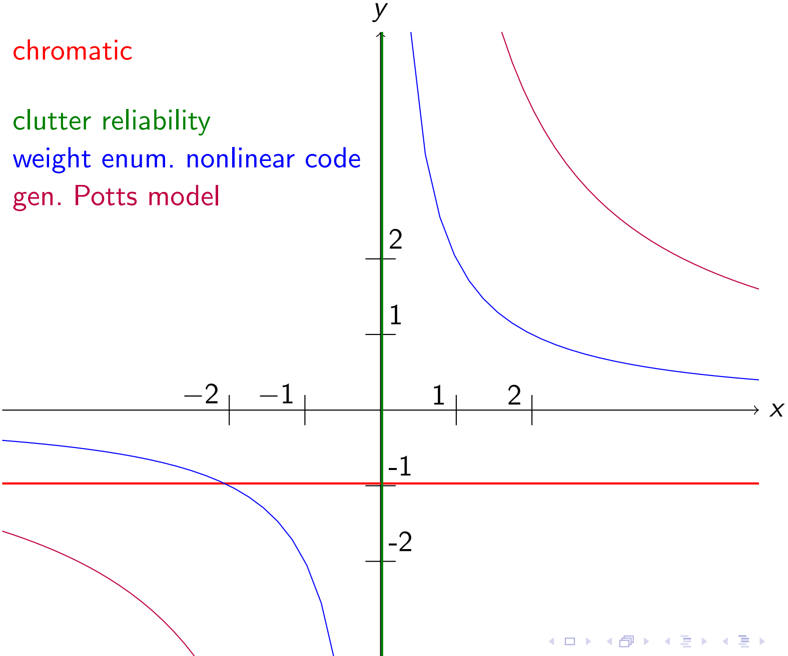

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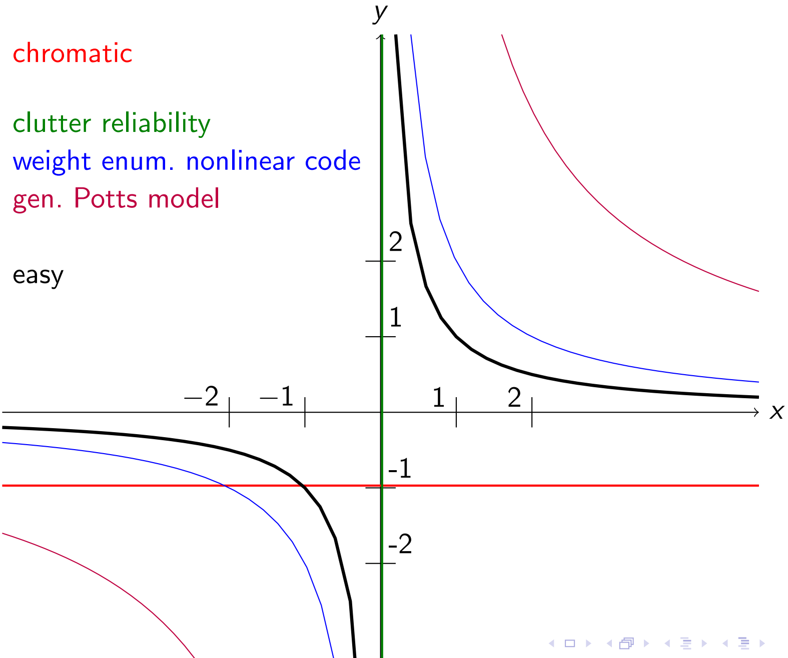


Generalised Tutte-Whitney plane

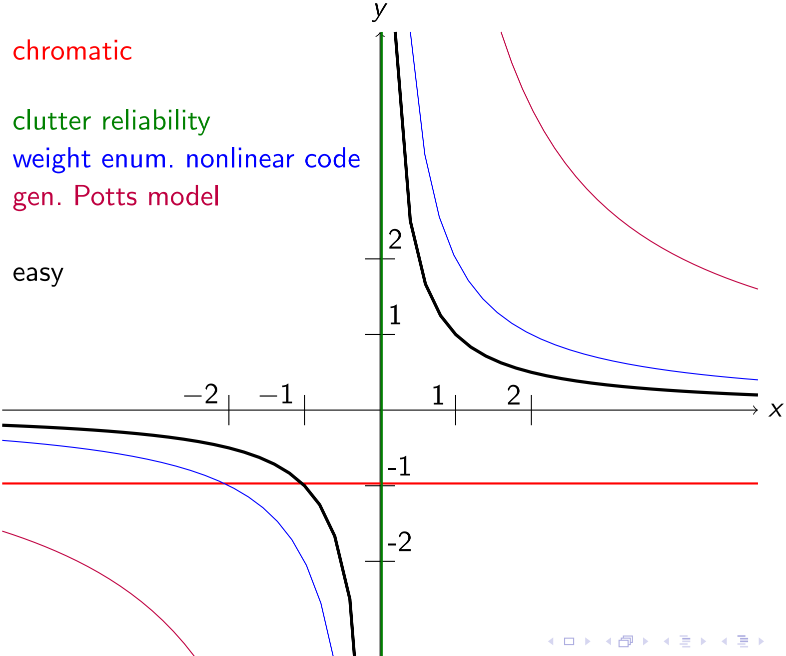
chromatic

clutter reliability

weight enum. nonlinear code

gen. Potts model

easy



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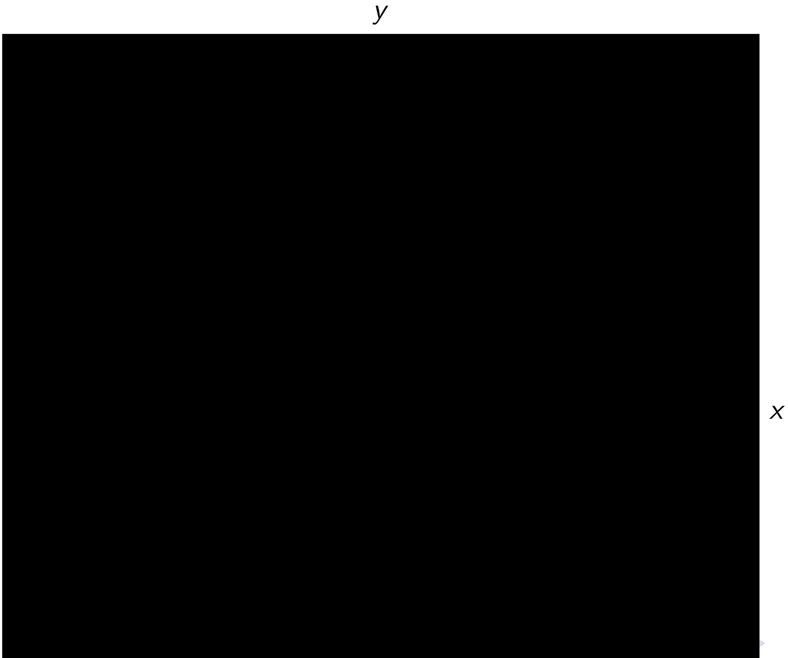

Interpolating between contraction and deletion

For $e \in E$, $X \subseteq E \setminus \{e\}$:

Contraction

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Deletion

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Contraction	λ-minor	Deletion
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Contraction ($\lambda = 0$)	λ-minor	Deletion ($\lambda = 1$)
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Fixed points:

$$\lambda = \pm\sqrt{2} - 1$$

λ-rank functions

Define $Q^{(\lambda)} f$ by:

$$(Q^{(\lambda)} f)(W) = \log_2 \left(\frac{(1+\lambda^*)^{|V|} \sum_{X \subseteq E} \lambda^{|X|} f(X)}{\sum_{X \subseteq E \setminus W} \lambda^{|W \cap \bar{V}|} (\lambda^*)^{|W \cap V|} f(X)} \right)$$

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Inversion:

$$(Q^{\dagger(\lambda)}\rho)(V) =$$

$$(-1)^{|V|}(\lambda - \lambda^*)^{-|S|} \times$$

$$\sum_{W \subseteq V} (-1)^{|W|}(1+\lambda^*)^{-|W|}(\lambda^*)^{|W \cap \bar{V}|}\lambda^{|\bar{W} \cap \bar{V}|}2^{\rho(E)-\rho(E \setminus W)}$$

A continuum of λ-Whitney functions

$$R^{(\lambda)}(f;x,y) = \sum_{X \subseteq E} x^{Q^{(\lambda)}f(E) - Q^{(\lambda)}f(X)} y^{|X| - Q^{(\lambda)}f(X)}$$

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$R(\hat{f};x,y)$	$(\sqrt{x}+\sqrt{y})^{\lvert E\rvert}$	$R^{(\lambda)}(f;x,y)$	$R(f;x,y)$
0	$\sqrt{2}-1$	λ	1

Properties of the λ-Whitney function:

$R^{(\lambda)}(f; x, y)$

- obeys a deletion-contraction-type relation (with operations $\|_{\lambda}$, $\|_{\lambda^*}$);

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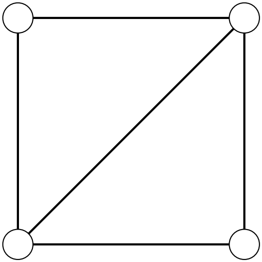
- contains the weight enumerator of a nonlinear code ... but this is also in $R(f; x, y)$, i.e., don't need λ;
- contains the partition function of the **Ashkin-Teller model** on a graph G ... which is **not** determined by $R(G; x, y)$, so **do** need λ.

Ashkin-Teller model (1943)

- 4-colourings (may be improper): colours are $(\pm 1, \pm 1)$



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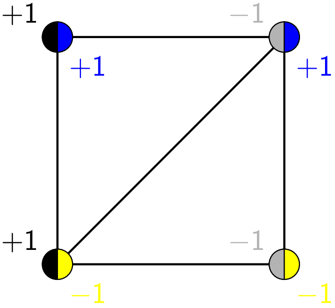

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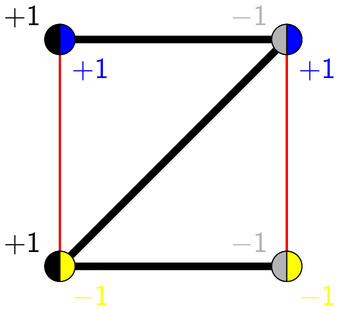

Left colours:
Good and bad edges

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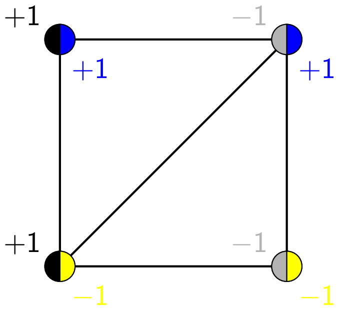

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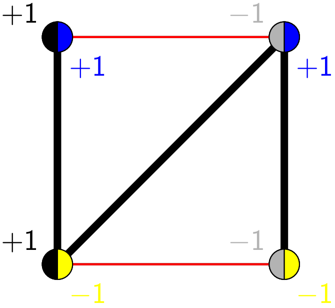

Right colours:
Good and bad edges

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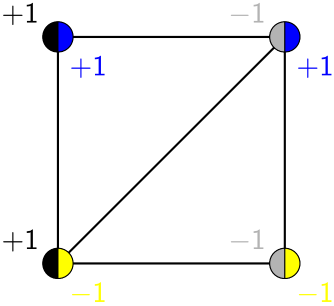

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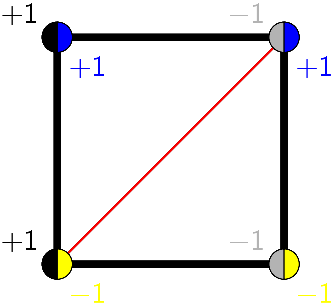

Product colours:
Good and bad edges

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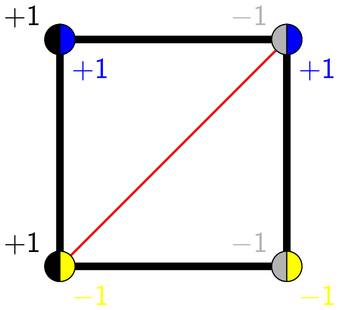

Product colours:
Good and bad edges

- Partition function (*symmetric* Ashkin-Teller):
$Z_{\mathrm{AT}}(G; K, K', q) =$

$$e^{(2K+K')|E|} \sum e^{-\left(\begin{array}{ll} & K \cdot (\#\text{ good ``left'' edges}) \\ + & K \cdot (\#\text{ good ``right'' edges}) \\ + & K' \cdot (\#\text{ good ``product'' edges}) \end{array} \right)}$$

Ashkin-Teller model (1943)

Special cases:

- $K = K'$: Potts model (up to a factor)
- $K' = 0$: product of two Ising models (each $q = 2$)

For these, $Z_{\mathrm{AT}}(G)$ is a specialisation of $R(G : x, y)$.

Ashkin-Teller model (1943)

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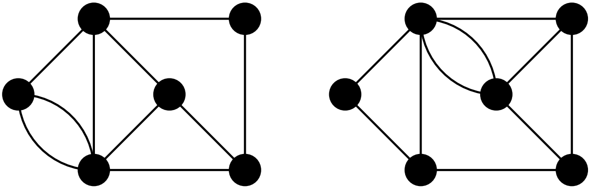
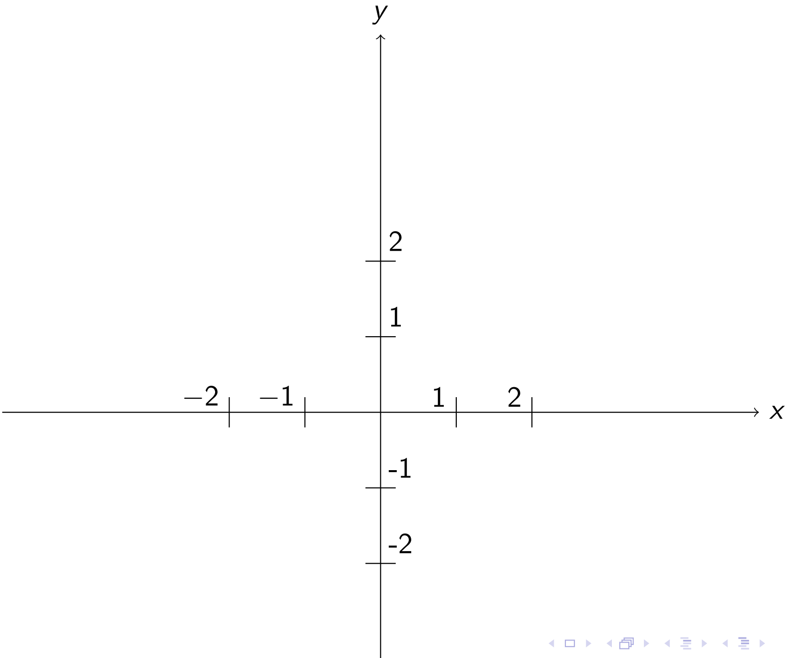
For these, $Z_{\mathrm{AT}}(G)$ is a specialisation of $R(G : x, y)$.

In general, $Z_{\mathrm{AT}}(G)$ is *not* a specialisation of $R(G : x, y)$.
Example (M. C. Gray; see Tutte (1974)):

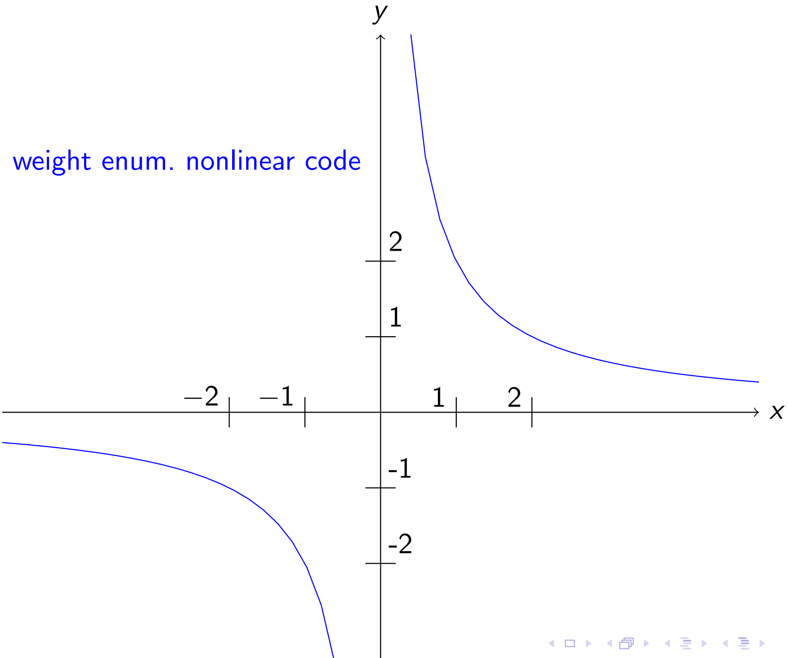


These graphs have same $R(G; x, y)$, but different $Z_{\mathrm{AT}}(G)$ (even in symmetric case).

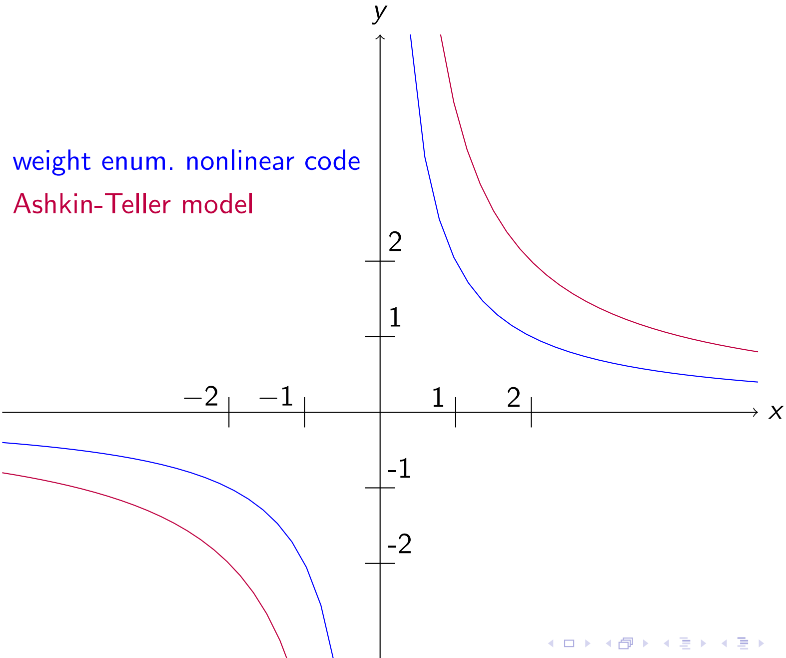
λ-Tutte-Whitney plane



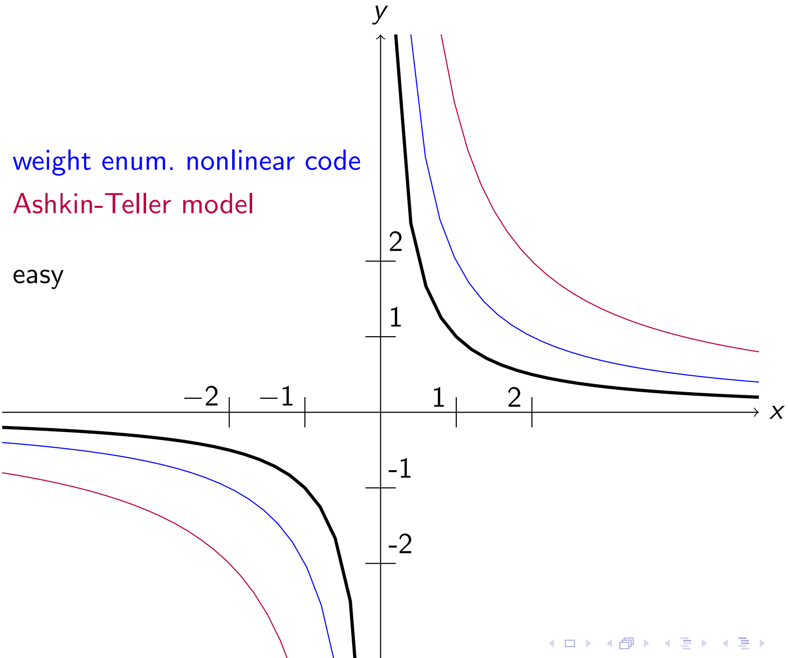
λ-Tutte-Whitney plane



λ-Tutte-Whitney plane



λ-Tutte-Whitney plane



λ-Tutte-Whitney plane