Covering Arrays with Row Limit: Bounds and Constructions

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Covering arrays

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- … is a test suite …

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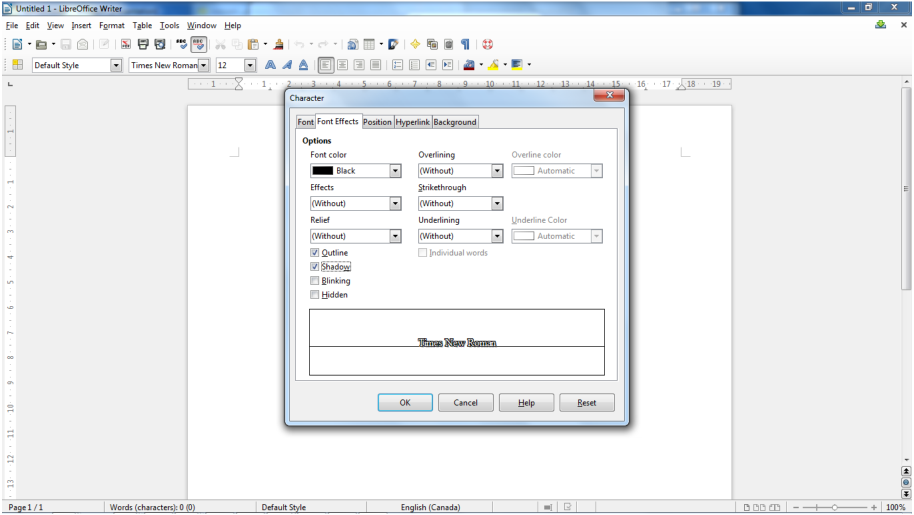
- … for verification of interactions between components.

Example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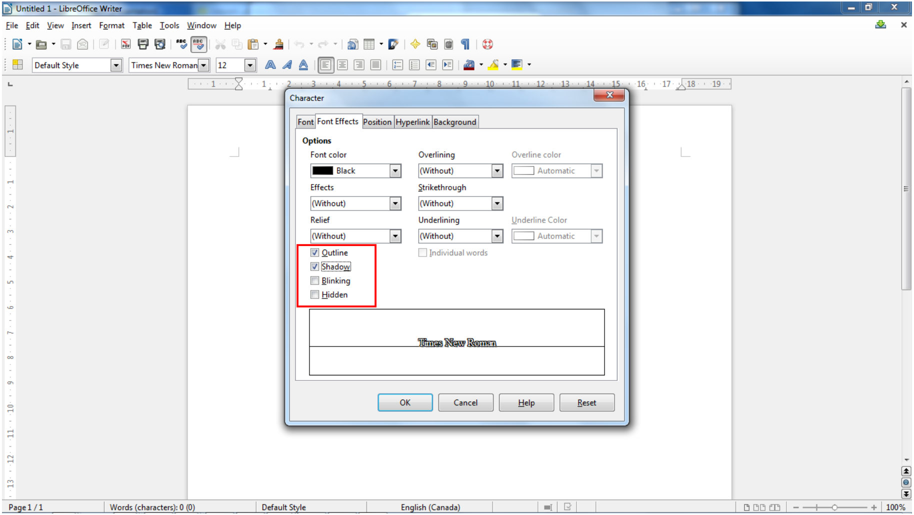
Example



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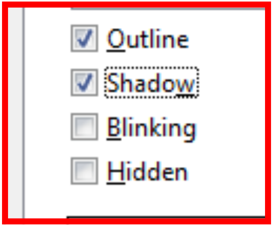

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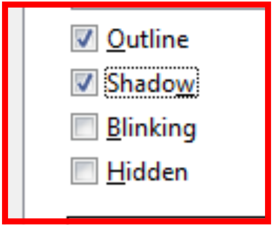

- Components: outline, shadow, blinking, hidden

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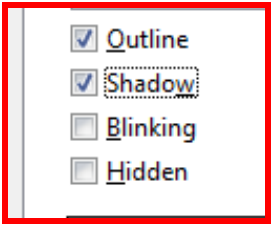

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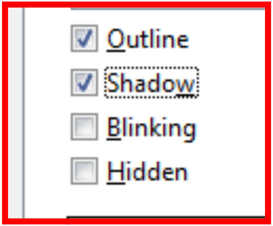

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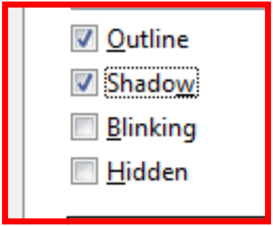

- Components: outline, shadow, blinking, hidden
- Levels for each component: present or absent (1 or 0)
- What interactions? Pairwise interactions.

Testing two at time: it would take $\binom{4}{2}2^2 = 24$ tests.

Example

Outline	Shadow	Blinking	Hidden
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

Example

Outline	Shadow	Blinking	Hidden
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

Example

Outline	Shadow	Blinking	Hidden
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

Testing can be done in only 5 iterations.

Example

Outline	Shadow	Blinking	Hidden
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

Parameters of a Covering Array

A covering array is characterized by:

- k: the number of components (columns)
- v: the number of levels for each component (alphabet size)
- t: strength $\Rightarrow$ testing interactions between t columns

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Goal:

- find the smallest number of rows, called size N of a covering array. Size of an optimal CA is usually denoted by $CAN(t, k, v)$.

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$$CA(N;2,k,2) \text{ exists for all } k \leq \binom{N-1}{\lfloor \frac{N}{2} - 1 \rfloor}$$

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- Finding an optimal covering array is NP-complete when extra constraints are imposed even when $v = 2$ (Maltais and Moura, 2011).

More on covering arrays

`http://www.pairwise.org/tools.asp` contains:

- 39 software tools for constructing *CA*s
- both commercial and open source

Covering array with row limit (*CARL*)

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$CARL(N = 12; t = 2, k = 6, v = 2 : w = 4)$

0	0	–	1	–	0
0	–	1	1	1	–
0	1	0	–	–	1
0	–	0	0	0	–
1	0	1	0	–	–
1	–	0	–	1	0
1	–	1	–	0	1
1	1	–	1	–	1
–	0	0	1	0	–
–	1	1	–	1	0
–	0	–	0	1	1
–	1	–	0	0	0

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$CARL(N = 12; t = 2, k = 6, v = 2 : w = 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
3	0	1	0	–	–	1
4	0	–	0	0	0	–
5	1	0	1	0	–	–
6	1	–	0	–	1	0
7	1	–	1	–	0	1
8	1	1	–	1	–	1
9	–	0	0	1	0	–
10	–	1	1	–	1	0
11	–	0	–	0	1	1
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- k = the number of columns (components)

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2	0	–	1	1	1	–
3	0	1	0	–	–	1
4	0	–	0	0	0	–
5	1	0	1	0	–	–
6	1	–	0	–	1	0
7	1	–	1	–	0	1
8	1	1	–	1	–	1
9	–	0	0	1	0	–
10	–	1	1	–	1	0
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3	0	1	0	–	–	1
4	0	–	0	0	0	–
5	1	0	1	0	–	–
6	1	–	0	–	1	0
7	1	–	1	–	0	1
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2	0	–	1	1	1	–
3	0	1	0	–	–	1
4	0	–	0	0	0	–
5	1	0	1	0	–	–
6	1	–	0	–	1	0
7	1	–	1	–	0	1
8	1	1	–	1	–	1
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- t = strength

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1	0	0	–	1	–	0
2	0	–	1	1	1	–
3	0	1	0	–	–	1
4	0	–	0	0	0	–
5	1	0	1	0	–	–
6	1	–	0	–	1	0
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- k = the number of columns (components)
- v = the alphabet size; the number of different values assigned to a column
- w = row limit; the number of non-empty cells in a row
- t = strength
- N = size; goal find minimum N

Covering arrays vs *CARL*s

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Theorem (Gargano et al.(1993))

$$\limsup_{k \to \infty} CAN(2, k, v) = \Theta(\log k).$$

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$$\Omega(\log k) = CAN(2,k,v) \leq CAN(t,k,v) \leq CARLN(t,k,v: w(k))$$

Schönheim lower bound

Theorem

$$CARLN_{\lambda}(t, k, v : w) \geq SB(t, k, v : w)$$

$$SB(t, k, v : w) = \left\lceil \frac{vk}{w} \left\lceil \frac{v(k-1)}{w-1} \cdots \left\lceil \frac{v(k-t+1)}{w-t+1} \right\rceil \cdots \right\rceil \right\rceil.$$

$CARL(12; 2, 6, 2 : 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
3	0	1	0	–	–	1
4	0	–	0	0	0	–
5	1	0	1	0	–	–
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Schönheim lower bound

- $w(k) = c$, $c \in \mathbb{N}$:

$$CARLN(t, k, v\colon w) = \frac{\binom{k}{t}}{\binom{w}{t}} v^t (1 + \mathrm{o}(1))$$

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$$CARLN(t, k, v\colon w) = \Theta(k^t)$$

- $\lim_{k \to \infty} \frac{k^t}{w(k)^t \log k} = 0$

$$\lim_{k \to \infty} \frac{SB(t, k, v\colon w(k))}{\log k} = 0$$

A probabilistic upper bound

$$CARL(t = 2, k = 6, v = 2 \colon w = 4)$$

1	2	3	4	5	6
0	0	0	0	—	—
0	0	0	1	—	—
0	0	1	0	—	—
0	0	1	1	—	—
⋮	⋮	⋮	⋮	⋮	⋮
0	0	0	—	0	—
0	0	0	—	1	—
0	0	1	—	0	—
0	0	1	—	1	—
⋮	⋮	⋮	⋮	⋮	⋮
—	0	1	—	0	0
—	0	1	—	0	1
—	0	1	—	1	0
—	0	1	—	1	1
⋮	⋮	⋮	⋮	⋮	⋮
—	—	1	1	0	0
—	—	1	1	0	1
—	—	1	1	1	0
—	—	1	1	1	1

1	2	3	4	5	6

A probabilistic upper bound

$$CARL(t=2, k=6, v=2: w=4)$$

1	2	3	4	5	6	
0	0	0	0	–	–	IN
0	0	0	1	–	–	
0	0	1	0	–	–	
0	0	1	1	–	–	
⋮	⋮	⋮	⋮	⋮	⋮	
0	0	0	–	0	–	
0	0	0	–	1	–	
0	0	1	–	0	–	
0	0	1	–	1	–	
⋮	⋮	⋮	⋮	⋮	⋮	
–	0	1	–	0	0	
–	0	1	–	0	1	
–	0	1	–	1	0	
–	0	1	–	1	1	
⋮	⋮	⋮	⋮	⋮	⋮	
–	–	1	1	0	0	
–	–	1	1	0	1	
–	–	1	1	1	0	
–	–	1	1	1	1	

1	2	3	4	5	6
0	0	0	0	–	–

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1	2	3	4	5	6	
0	0	0	0	–	–	
0	0	0	1	–	–	OUT
0	0	1	0	–	–	
0	0	1	1	–	–	
⋮	⋮	⋮	⋮	⋮	⋮	
0	0	0	–	0	–	
0	0	0	–	1	–	
0	0	1	–	0	–	
0	0	1	–	1	–	
⋮	⋮	⋮	⋮	⋮	⋮	
–	0	1	–	0	0	
–	0	1	–	0	1	
–	0	1	–	1	0	
–	0	1	–	1	1	
⋮	⋮	⋮	⋮	⋮	⋮	
–	–	1	1	0	0	
–	–	1	1	0	1	
–	–	1	1	1	0	
–	–	1	1	1	1	

1	2	3	4	5	6
0	0	0	0	–	–

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1	2	3	4	5	6	
0	0	0	0	–	–	
0	0	0	1	–	–	
0	0	1	0	–	–	OUT
0	0	1	1	–	–	
⋮	⋮	⋮	⋮	⋮	⋮	
0	0	0	–	0	–	
0	0	0	–	1	–	
0	0	1	–	0	–	
0	0	1	–	1	–	
⋮	⋮	⋮	⋮	⋮	⋮	
–	0	1	–	0	0	
–	0	1	–	0	1	
–	0	1	–	1	0	
–	0	1	–	1	1	
⋮	⋮	⋮	⋮	⋮	⋮	
–	–	1	1	0	0	
–	–	1	1	0	1	
–	–	1	1	1	0	
–	–	1	1	1	1	

1	2	3	4	5	6
0	0	0	0	–	–

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1	2	3	4	5	6	
0	0	0	0	–	–	
0	0	0	1	–	–	
0	0	1	0	–	–	
0	0	1	1	–	–	IN
⋮	⋮	⋮	⋮	⋮	⋮	
0	0	0	–	0	–	
0	0	0	–	1	–	
0	0	1	–	0	–	
0	0	1	–	1	–	
⋮	⋮	⋮	⋮	⋮	⋮	
–	0	1	–	0	0	
–	0	1	–	0	1	
–	0	1	–	1	0	
–	0	1	–	1	1	
⋮	⋮	⋮	⋮	⋮	⋮	
–	–	1	1	0	0	
–	–	1	1	0	1	
–	–	1	1	1	0	
–	–	1	1	1	1	

1	2	3	4	5	6
0	0	0	0	–	–
0	0	1	1	–	–

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1	2	3	4	5	6	
0	0	0	0	–	–	
0	0	0	1	–	–	
0	0	1	0	–	–	
0	0	1	1	–	–	
⋮	⋮	⋮	⋮	⋮	⋮	
0	0	0	–	0	–	
0	0	0	–	1	–	
0	0	1	–	0	–	
0	0	1	–	1	–	
⋮	⋮	⋮	⋮	⋮	⋮	
–	0	1	–	0	0	
–	0	1	–	0	1	
–	0	1	–	1	0	
–	0	1	–	1	1	
⋮	⋮	⋮	⋮	⋮	⋮	
–	–	1	1	0	0	
–	–	1	1	0	1	
–	–	1	1	1	0	
–	–	1	1	1	1	IN

1	2	3	4	5	6
0	0	0	0	–	–
0	0	1	1	–	–
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
–	–	1	1	1	1

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1	2	3	4	5	6
0	0	0	0	—	—
0	0	0	1	—	—
0	0	1	0	—	—
0	0	1	1	—	—
⋮	⋮	⋮	⋮	⋮	⋮
0	0	0	—	0	—
0	0	0	—	1	—
0	0	1	—	0	—
0	0	1	—	1	—
⋮	⋮	⋮	⋮	⋮	⋮
—	0	1	—	0	0
—	0	1	—	0	1
—	0	1	—	1	0
—	0	1	—	1	1
⋮	⋮	⋮	⋮	⋮	⋮
—	—	1	1	0	0
—	—	1	1	0	1
—	—	1	1	1	0
—	—	1	1	1	1

1	2	3	4	5	6
0	0	0	0	—	—
0	0	1	1	—	—
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
—	—	1	1	1	1

For each uncovered pair, add a row to cover it.

A probabilistic upper bound (UB_1)

Theorem (N.F., Danziger, Mendelsohn)

Let $c_1, c_2 > 1$ such that $\frac{1}{c_1} + \frac{1}{c_2} < 1$. Then,

$$CARLN(t,k,v\colon w) \leq UB_1(t,k,v\colon w)$$

$$UB_1(t,k,v\colon w) = c_1 \frac{\binom{k}{t}}{\binom{w}{t}} v^t \left(1 + \ln \frac{c_2}{c_1}\binom{w}{t}\right)$$

Asymptotic size of UB_1

Theorem (N.F., Danziger, Mendelsohn)

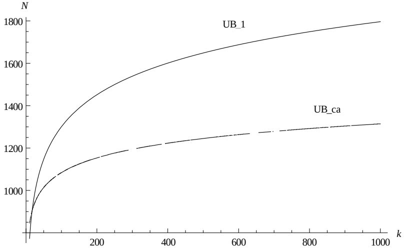
If $w(k) = \Theta(k)$, *then*

$$\limsup_{k \to \infty} CARLN(t, k, v \colon w(k)) = \Theta(\log k)$$

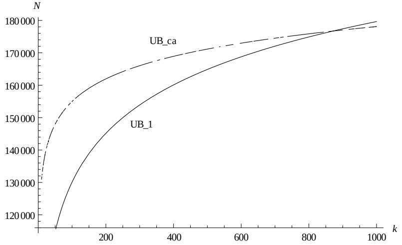
UB_1 for covering arrays

Theorem (Godbole et al., 1996)

$$CAN(t,k,v) \leq \frac{-\ln\left(ev^t t\binom{k-1}{t-1}\right)}{\ln\left(1-\frac{1}{v^t}\right)} = UB_{ca}(t,k,v).$$



$UB_1(2,k,10\colon k)$, $UB_{ca}(2,k,10)$



$UB_1(2,k,100\colon k)$, $UB_{ca}(2,k,100)$

Improvement to UB_1

Theorem (N.F., Danziger, Mendelsohn)

If $w(k) \ln w(k) = \mathrm{o}(k)$, *then*

$$CARLN(t,k,v\colon w(k)) \leq \frac{\binom{k}{t}}{\binom{w}{t}} v^t \left(1 + \ln \binom{w}{t}\right)(1+\mathrm{o}(1)).$$

Summary

$w = const$	Schönheim bound $CARLN = \Theta(k^t)$	constructive
$w(k) \ln w(k) = \mathrm{o}(k)$	improved UB_1	constructive
$\lim_{k\to\infty} \frac{k^t}{w(k)^t \log k} \neq 0$	Schöheim bound is $\Omega(\log k)$	
any $w(k)$	UB_1	not constructive
$w(k) = \Theta(k)$	$CARLN = \Theta(\log k)$	

Greedy algorithm

A generalization of *AETG* algorithm for covering arrays (Cohen et al., 1997).

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- Build an array one row at a time.
- At each step compute the average number of uncovered t-tuples contained in any admissible row.
- Add a row which covers at least the average number of uncovered t-tuples.

Greedy algorithm

A generalization of *AETG* algorithm for covering arrays (Cohen et al., 1997).

Algorithm:

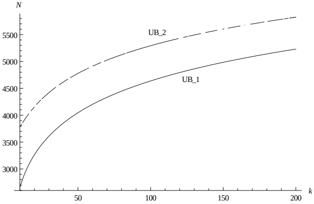
- Build an array one row at a time.
- At each step compute the average number of uncovered t-tuples contained in any admissible row.
- Add a row which covers at least the average number of uncovered t-tuples.

Theorem (N.F., Danziger, Mendelsohn)

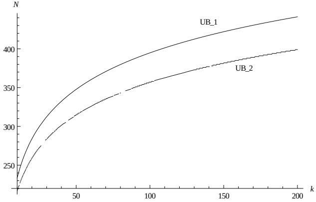
$$CARLN(t,k,v\colon w(k)) \leq UB_2(t,k,v\colon w)$$

$$UB_2(t,k,v\colon w) = \left\lceil 1 - \frac{\ln\left(\binom{k}{t}v^t - \binom{w}{t}\right)}{\ln\left(1 - \frac{\binom{w}{t}}{\binom{k}{t}v^t}\right)} \right\rceil .$$

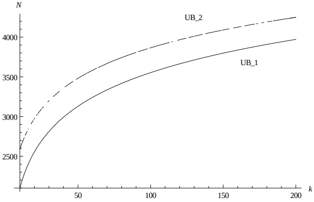
UB_1 vs UB_2



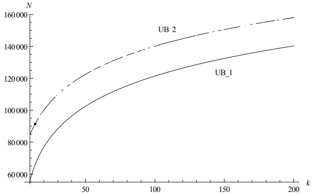
$UB_1(2, k, 10 \colon k/2)$, $UB_2(2, k, 10 \colon k/2)$



$UB_1(2, k, 5 \colon 9k/10)$, $UB_2(2, k, 5 \colon 9k/10)$



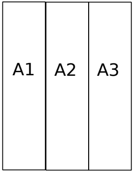
$UB_1(2, k, 15 \colon 9k/10)$, $UB_2(2, k, 15 \colon 9k/10)$



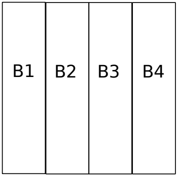
$UB_1(5, k, 5 \colon 9k/10)$, $UB_2(5, k, 5 \colon 9k/10)$

Product construction for strength $t = 2$

A:
$CARL(N_1; 2, k_1, v : w_1)$



B:
$CARL(N_2; 2, k_2, v : w_2)$



Product construction for strength $t = 2$

A:

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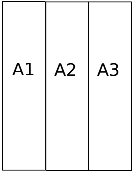
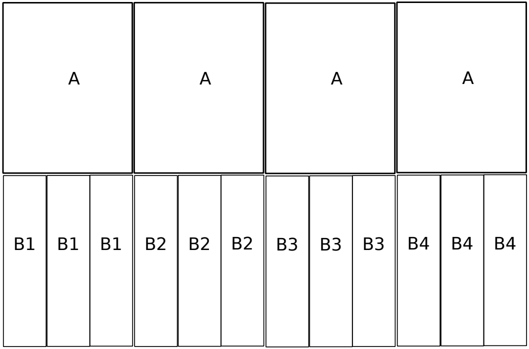


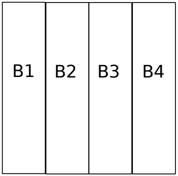
$A \times B$:

$CARL(N_1 + N_2; 2, k_1 k_2, v : \max\{k_1 w_2, k_2 w_1\})$



B:

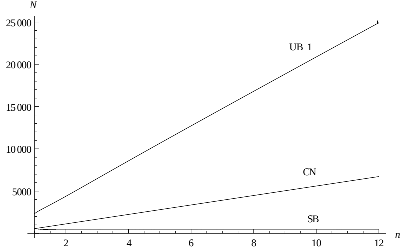
$CARL(N_2; 2, k_2, v : w_2)$



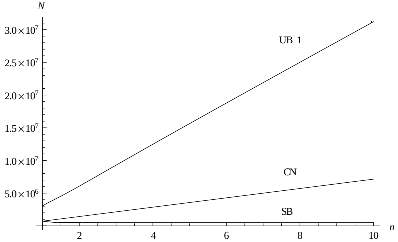
Product construction for strength $t = 2$

- Preserves the ration $\frac{w(k)}{k}$
- Retains logarithmic growth

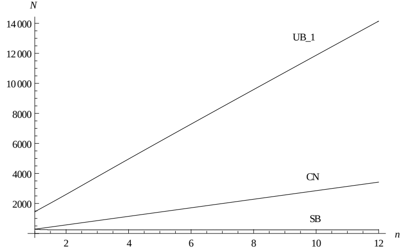
Constructed *CARLs*



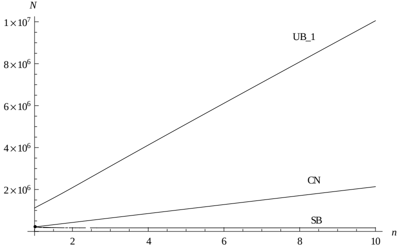
$UB_1(2, 12^n, 5: 3 \cdot 12^{n-1})$,
$CN(n) = nSB(2, 12, 5: 3)$,
$SB(2, 12^n, 5: 3 \cdot 12^{n-1})$



$UB_1(2, 21^n, 101: 3 \cdot 21^{n-1})$,
$CN(n) = nSB(2, 21, 101: 3)$,
$SB(2, 21^n, 101: 3 \cdot 21^{n-1})$

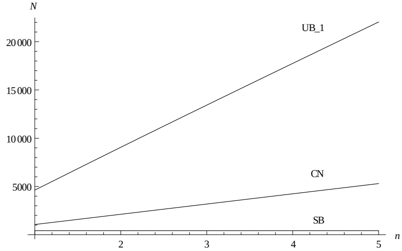


$UB_1(2, 12^n, 5: 4 \cdot 12^{n-1})$,
$CN(n) = nSB(2, 12, 5: 4)$,
$SB(2, 12^n, 5: 4 \cdot 12^{n-1})$

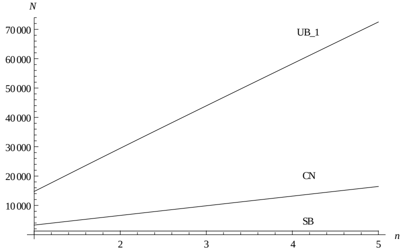


$UB_1(2, 17^n, 97: 4 \cdot 17^{n-1})$,
$CN(n) = nSB(2, 17, 97: 4)$,
$SB(2, 17^n, 97: 4 \cdot 17^{n-1})$

Constructed *CARLs*



$UB_1(2, 12^n16^n, 5\colon 3 \cdot 12^{n-1}16^n)$,
$CN(n) = nSB(2, 12, 5\colon 3) + nSB(2, 16, 6\colon 4)$,
$SB(2, 12^n16^n, 5\colon 3 \cdot 12^{n-1}16^n)$



$UB_1(2, 15^n20^n, 7\colon 3 \cdot 15^{n-1}20^n)$,
$CN(n) = nSB(2, 15, 7\colon 3) + nSB(2, 20, 7\colon 4)$
$SB(2, 15^n20^n, 7\colon 3 \cdot 15^{n-1}20^n)$

Constructions

- Wilson's Construction

$$V = \{1, 2, 3, 4, 5, 6\}$$

$$\begin{aligned}\mathcal{B} \;=\; & \{\{1,2,3\},\ \{1,4,5\},\\ & \{1,3,6\},\ \{2,4,6\},\\ & \{2,5,6\},\ \{3,4,5\}\}\end{aligned}$$

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CARL(2,3,v: w)

C1	C2	C3

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CARL(2,3,v: w)

C1	C2	C3

1	2	3	4	5	6
C1	C2	C3			
C1			C2	C3	
C1		C2			C3
	C1		C2		C3
	C1			C2	C3
		C1	C2	C3	

Wilson's Construction

$$V = \{1, 2, 3, 4, 5, 6\}$$

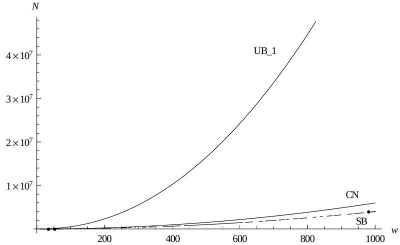
$$\mathcal{B} = \{\{1, 2, 3\},\ \{1, {\color{red}4, 5}\},\ \{1, 3, 6\},\ \{2, 4, 6\},\ \{2, 5, 6\},\ \{3, {\color{red}4, 5}\}\}$$

CARL(2,3,v: w)

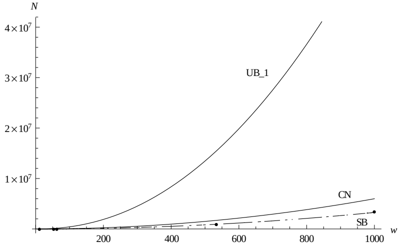
C1	C2	C3

1	2	3	4	5	6
C1	C2	C3			
C1			C2	C3	
C1		C2			C3
	C1		C2		C3
	C1			C2	C3
		C1	C2	C3	

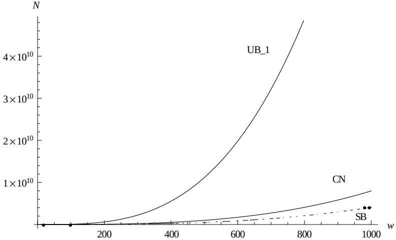
Wilson's construction using orthogonal arrays



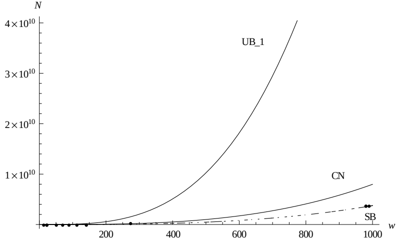
$UB_1(2, 2w, w-1\colon w)$,
$CN(w) = 6(w-1)^2$,
$SB(2, 2w, w-1\colon w)$



$UB_1(2, 1.81\,w, w-1\colon w)$
$CN(w) = 6(w-1)^2$,
$SB(2, 1.81\,w, w-1\colon w)$

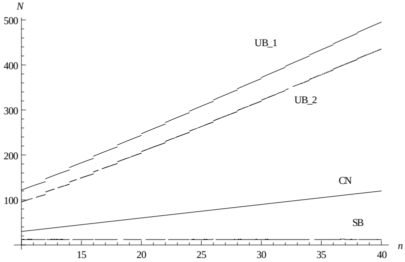


$UB_1(3, 8w/5, w-1\colon w)$,
$CN(w) = 8(w-1)^3$,
$SB(3, 8w/5, w-1\colon w)$

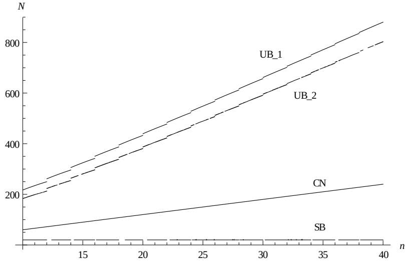


$UB_1(3, 17.1w/11, w-1\colon w)$
$CN(w) = 8(w-1)^3$,
$SB(3, 17.1w/11, w-1\colon w)$

Wilson's construction using covering arrays with $t = 2$ and $v = 2$



$UB_1(2, 3w(n)/2, 2\colon w(n))$, $UB_2(2, 3w(n)/2, 2\colon w(n))$, $CN(n) = 3n$, $SB(2, 3w(n)/2, 2\colon w(n))$



$UB_1(2, 2w(n), 2\colon w(n))$, $UB_2(2, 2w(n), 2\colon w(n))$, $CN(n) = 6n$, $SB(2, 2w(n), 2\colon w(n))$

$$w(n) = \binom{n-1}{\lfloor n/2 \rfloor - 1}$$

Summary

$w = const$	Schönheim bound $CARLN = \Theta(k^t)$	constructive
$w(k) \ln w(k) = o(k)$	improved UB_1	constructive
$\lim_{k\to\infty} \frac{k^t}{w(k)^t \log k} \neq 0$	Schöheim bound is $\Omega(\log k)$	
any $w(k)$	UB_1 UB_2	not constructive constructive
$w(k) = \Theta(k)$	$CARLN = \Theta(\log k)$	

*CARL*s with smaller size are more likely to be obtained through direct constructions.

▸ The End?

*CARL*s with constant row limit and $t = 2$

*CARL*s with constant row limit and $t = 2$

- Schönheim lower bound:

$$CARLN_\lambda(t, k, v : w) \geq \left\lceil \frac{vk}{w} \left\lceil \frac{v(k-1)}{w-1} \right\rceil \right\rceil.$$

- $w(k) = c$, $c \in \mathbb{N}$:

$$CARLN(t, k, v : w) = \frac{\binom{k}{t}}{\binom{w}{t}} v^t (1 + o(1))$$

- $$CARLN(t, k, v : w) = \Theta(k^t)$$

Back to definition

$CARL(N = 12; t = 2, k = 6, v = 2 : w = 4)$

0	0	–	1	–	0
0	–	1	1	1	–
0	1	0	–	–	1
0	–	0	0	0	–
1	0	1	0	–	–
1	–	0	–	1	0
1	–	1	–	0	1
1	1	–	1	–	1
–	0	0	1	0	–
–	1	1	–	1	0
–	0	–	0	1	1
–	1	–	0	0	0

Group divisible covering designs...

- ... are *CARL*s with $t = 2$ and constant w

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Group divisible covering designs...

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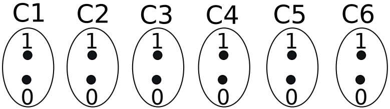
- $k - GDCD$ of type g^u, $(V, \mathcal{G}, \mathcal{B})$:

$|V| = 12$, $\mathcal{G} = \{C1, C2, \cdots, C6\}$



$CARL(12; 2, 6, 2 : 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
⋮	⋮	⋮	⋮	⋮	⋮	⋮

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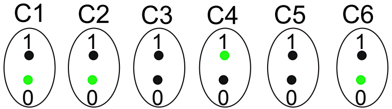

$\{(1,0), (2,0), (4,1), (6,0)\}$

$CARL(12; 2, 6, 2: 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
⋮	⋮	⋮	⋮	⋮	⋮	⋮

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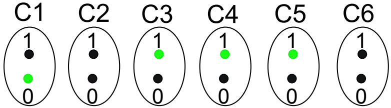


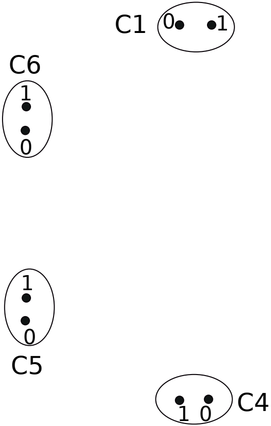
$\{(1,0),(2,0),(4,1),(6,0)\}$
$\{(1,0),(3,1),(4,1),(5,1)\}$

$CARL(12; 2, 6, 2 : 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Edge covering

- graph covering of a $K_{\underbrace{g,g,\ldots,g}_{u}}$ by K_k:



$CARL(12; 2, 6, 2: 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
⋮	⋮	⋮	⋮	⋮	⋮	⋮

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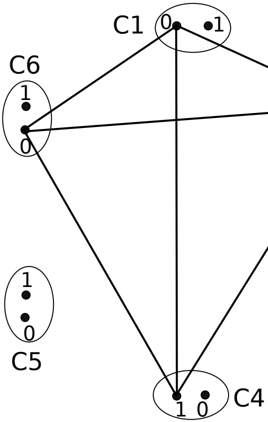

$CARL(12; 2, 6, 2 : 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
⋮	⋮	⋮	⋮	⋮	⋮	⋮

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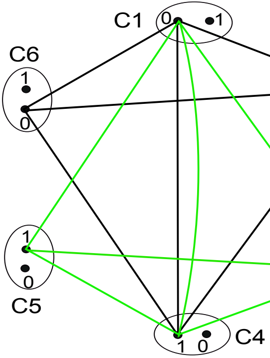
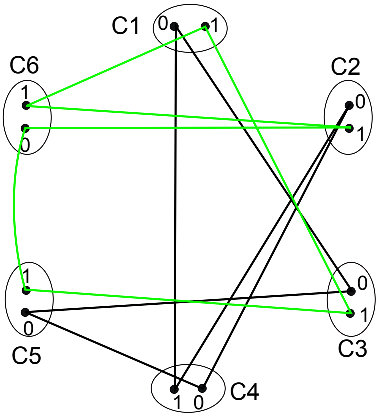


$CARL(12; 2, 6, 2 : 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Excess graph when $t = 2$



$2C_6$

$CARL(12; 2, 6, 2 : 4)$

	1	2	3	4	5	6
1	0	0	–	1	–	0
2	0	–	1	1	1	–
3	0	1	0	–	–	1
4	0	–	0	0	0	–
5	1	0	1	0	–	–
6	1	–	0	–	1	0
7	1	–	1	–	0	1
8	1	1	–	1	–	1
9	–	0	0	1	0	–
10	–	1	1	–	1	0
11	–	0	–	0	1	1
12	–	1	–	0	0	0

Group divisible covering designs

- $C(k, g^u) = CARLN(2, u, g : k)$

Group divisible covering designs

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- $k = 3$ done (Heinrich and Yin (1999))
- $k = 4$:

Group divisible covering designs

- $C(k, g^u) = CARLN(2, u, g\colon k)$
- $k = 3$ done (Heinrich and Yin (1999))
- $k = 4$:
  - $C(4, g^u) \geq \left\lceil \frac{gu}{4} \left\lceil \frac{g(u-1)}{3} \right\rceil \right\rceil$

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- $k = 3$ done (Heinrich and Yin (1999))
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  - Construction methods: Wilson's construction, double group divisible designs, and some others

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  - Construction methods: Wilson's construction, double group divisible designs, and some others
  - Two types of objects: essential and auxiliary

$4-GDCDs$

Theorem

There exists a small positive integer δ, such that for any positive integer g and $u \geq 4$,

$$C(4, g^u) \leq \left\lceil \frac{gu}{4} \left\lceil \frac{g(u-1)}{3} \right\rceil \right\rceil + \delta,$$

except possibly when (1) $g = 17$ and $u \equiv 0 \pmod 3$, or (2) $g \geq 8$, $g \equiv 2, 5 \pmod 6$, and $u \equiv 23 \pmod{24}$ or $u \in \{29, 35, 41\}$.

Optimal $4-GDCDs$

Theorem

$C(4, g^u) = \left\lceil \frac{gu}{4} \left\lceil \frac{g(u-1)}{3} \right\rceil \right\rceil$ *when* $u \geq 4$ *and one of the following holds:*

- $u \equiv 0 \pmod{12}$ *except possibly when* $g = 17$, *or when* $u = 36$ *and* $g \equiv 5 \pmod{6}$, *or when* $(g, u) = (11, 24)$;
- $u \equiv 1, 4 \pmod{12}$, *except when* $(g, u) \in \{(2, 4), (6, 4)\}$;
- $u \equiv 2 \pmod{12}$, *except possibly when* $g = 13$, *or* $g \equiv 7 \pmod{12}$, *or* $g = 17$ *and* $u \equiv 2 \pmod{24}$, *or* $(g, u) \in \{(15, 14), (21, 14), (11, 38), (17, 38)\}$;
- $u \equiv 3 \pmod{12}$, *except possibly when* $g \in \{13, 17\}$, *or* $g \equiv 7, 10 \pmod{12}$, *or* $u \in \{27, 39, 51\}$ *and* $g \equiv 5 \pmod{6}$, *or* $u = 27$ *and* $g \equiv 4 \pmod{12}$;

Theorem continued...(1)

Theorem

- $u \equiv 5 \pmod{12}$, *except possibly when* $g = \{13, 26, 44\}$, *or* $u \in \{29, 41\}$ *and* $g \equiv 2, 8 \pmod{24}$, *or* $g \equiv 5 \pmod{6}$, *or* $g \equiv 14 \pmod{24}$, *or* $u \equiv 17 \pmod{24}$ *and* $g \equiv 20 \pmod{24}$, *or* $u \equiv 17 \pmod{24}$ *and* $g \equiv 2 \pmod{24}$;
- $u \equiv 6 \pmod{12}$, *except when* $(g, u) = (3, 6)$, *and possibly when* $u = 6$ *and* $g \equiv 3 \pmod{6}$, $g \geq 9$, *or* $g \equiv 5 \pmod{6}$, *or* $(g, u) \in \{(15, 18), (21, 18)\}$;
- $u \equiv 7 \pmod{12}$, *except when* $(g, u) \in \{(1, 7), (1, 19)\}$, *and possibly when* $g \in \{5, 7\}$;
- $u \equiv 8 \pmod{12}$, *except possibly when* $(g, u) \in \{(11, 32), (11, 44), (17, 32), (17, 44)\}$;
- $u \equiv 9 \pmod{12}$, *except when* $(g, u) = (1, 9)$, *and possibly when* $g = 13$, *or* $g \equiv 5, 7, 10, 11 \pmod{12}$, *or* $u = 9$ *and* $g \equiv 1 \pmod{12}$, $g \geq 13$, *or* $u = 21$ *and* $g \equiv 4 \pmod{12}$;

Theorem continued...(2)

Theorem

- $u \equiv 10 \pmod{12}$, *except when* $(g, u) = (1, 10)$, *and possibly when* $g \in \{5, 7\}$;
- $u \equiv 11 \pmod{12}$, *except possibly when* $g = 26$, *or* $u = 35$ *and* $g \equiv 2, 8 \pmod{24}$, *or* $g \equiv 5 \pmod{6}$, *or* $u \equiv 11 \pmod{24}$ *and* $g \equiv 14, 20 \pmod{24}$, *or* $u \equiv 23 \pmod{24}$ *and* $g \equiv 2 \pmod{6}$, $g \geq 8$, *or* $(g, u) = (2, 23)$.

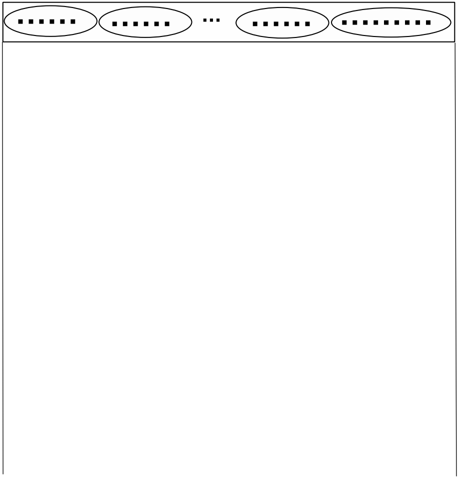
Close to optimal $4-GDCD$s

Theorem

- *When* $g \equiv 3 \pmod 6$, *there exists a close to optimal* $4-GDCD$ *of type* g^6 *with* $\left\lceil \frac{3g}{2} \left\lceil \frac{5g}{3} \right\rceil \right\rceil + 2$ *blocks, which is optimal when* $g = 3$.
- *When* $g \equiv 1 \pmod{12}$, $g \neq 13$, *there exists a close to optimal* $4-GDCD$ *of type* g^9 *having* $\left\lceil \frac{9g}{4} \left\lceil \frac{8g}{3} \right\rceil \right\rceil + 1$ *blocks, which is optimal when* $g = 1$.

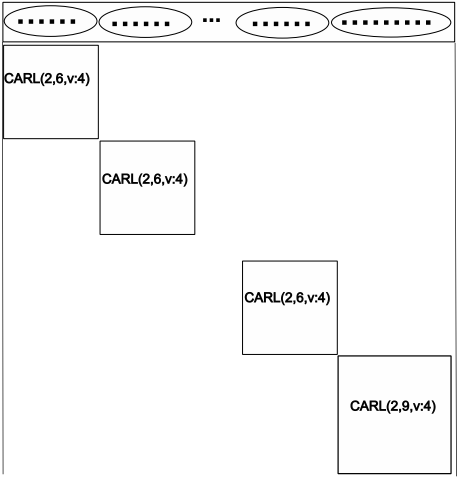
Wilson construction

Example: If $k \equiv 3 \pmod 6$, $k \geq 33$, there exists a $4-GDD$ of type $6^{\frac{k-9}{6}} 9^1$, $(V, \mathcal{G}, \mathcal{B})$.



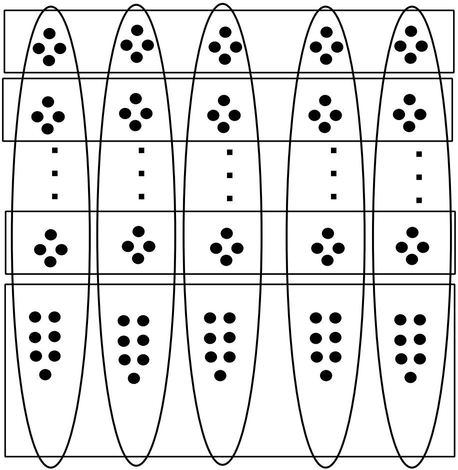
Wilson construction for *CARL*s with $w = 4$

Example: If $k \equiv 3 \pmod 6$, $k \geq 33$, there exists a $4 - GDD$ of type $6^{\frac{k-9}{6}} 9^1$, $(V, \mathcal{G}, \mathcal{B})$.



Construction of *CARL*s with smaller number of components

Example: $k = 5$ and $v \equiv 7 \pmod{12}$, $v \geq 31$:



References

Cohen, David M., Siddhartha R. Dalal, and Gardner C. Patton. 1997. *The AETG system: An approach to testing based on combinatorial design*, IEEE Transaction on Software Engineering **23**, no. 7, 437–444.

Gargano, L., J. Körner, and U. Vaccaro. 1993. *Sperner capacities*, Graphs and Combinatorics **9**, no. 1, 31–46.

Godbole, Anant P, Daphne E Skipper, and Rachel A Sunley. 1996. *t-covering arrays: upper bounds and poisson approximations*, Combinatorics, Probability and Computing **5**, no. 2, 105–117.

Katona, G. O. H. 1973. *Two applications (for search theory and truth functions) of sperner type theorems*, Periodica Mathematica Hungarica. Journal of the János Bolyai Mathematical Society **3**, 19–26. Collection of articles dedicated to the memory of Alfréd Rényi, II.

Kleitman, Daniel J. and Joel Spencer. 1973. *Families of k-independent sets*, Discrete Mathematics **6**, 255–262.

Maltais, Elizabeth and Lucia Moura. 2011. *Hardness results for covering arrays avoiding forbidden edges and error-locating arrays*, Theoretical Computer Science **412**, no. 46, 6517–6530.

Thank you!