Solution geometry of a random k-XORSAT near the clustering threshold

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Monash University – Discrete Maths Seminar

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Collaborator: Mike Molloy

A few words about random structures

- What are random structures?
- Why study random strcutures?

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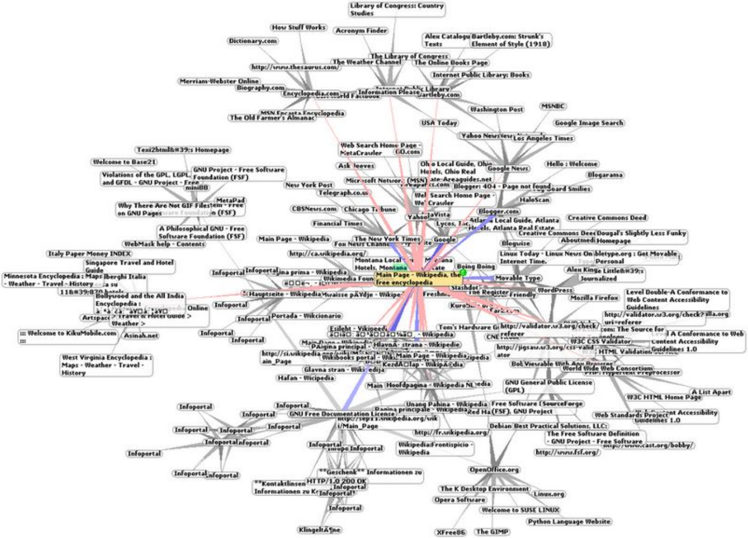
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Violations of the GPL, LGPL, and GFDL - GNU Project - Free
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New York Post
Microsoft Network (MSN)
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http://si.wikipedia.org/wiki/Main_Page
Wikibooks portal
Bob Viewable With Any Browser
World Wide Web Consortium
PHP: Hypertext Preprocessor
Glavna stran - Wikipedija
Hafan - Wicipedia
Hoofdpagina - Wikipedia NL
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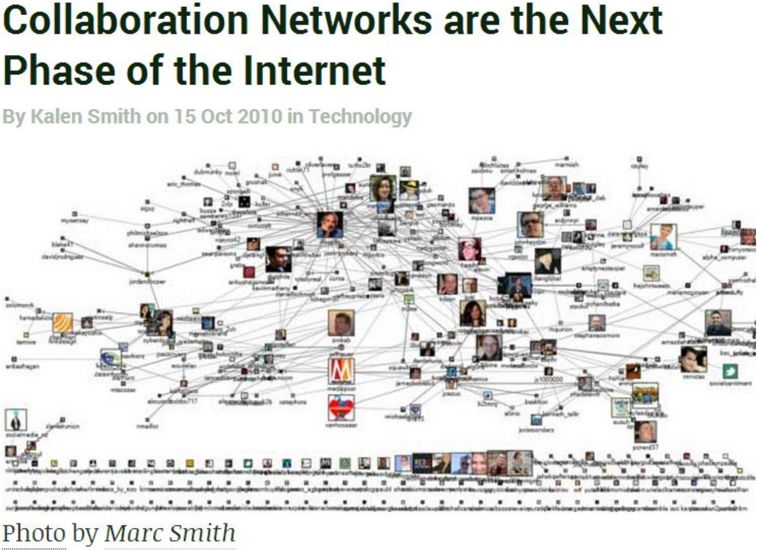
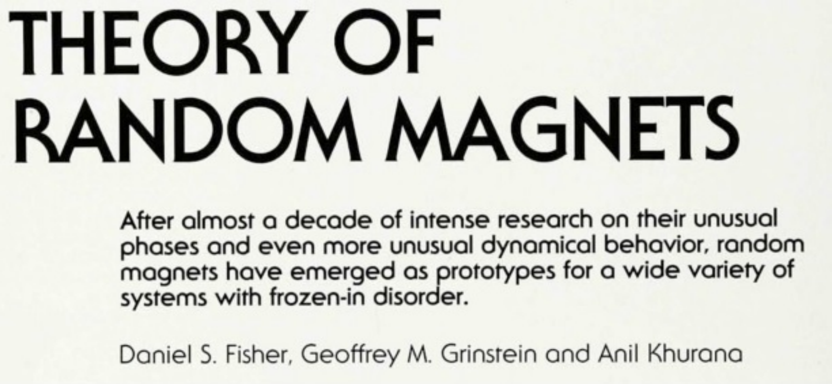


Collaboration Networks are the Next Phase of the Internet

By Kalen Smith on 15 Oct 2010 in Technology

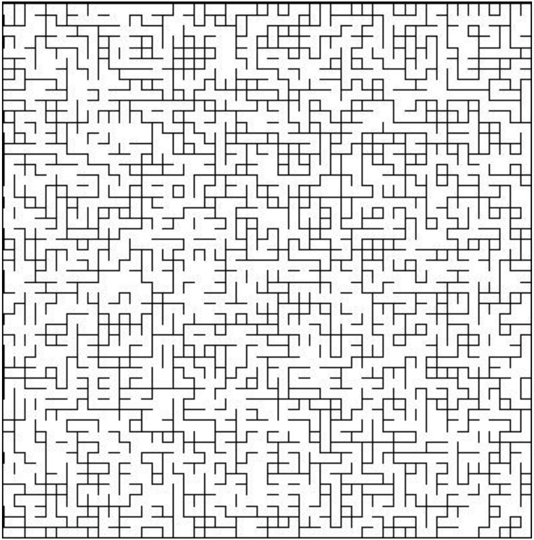
Photo by *Marc Smith*



THEORY OF RANDOM MAGNETS

After almost a decade of intense research on their unusual phases and even more unusual dynamical behavior, random magnets have emerged as prototypes for a wide variety of systems with frozen-in disorder.

Daniel S. Fisher, Geoffrey M. Grinstein and Anil Khurana



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Phase transition; threshold

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Random CSPs

Every clause in a k-SAT formula F (with m clauses on n literals) has the form

$$x_{i_1} \vee x_{i_2} \vee \bar{x}_{i_3} \vee \cdots \vee x_{i_k}.$$

F is satisfiable if there is a solution $(x_1, \ldots, x_n)$ such that every clause of F is satisfied.

A random instance F with m clauses and n literals is chosen uniformly at random from the set of $(2n)^{km}$ formulas.

More CSPs:

k-NAESAT, k-XORSAT, k-COL of graphs, 2-COL of k-uniform hypergraphs.

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Theorem (Friedgut 1999)

k-SAT has a sharp threshold at $m/n = c_k(n)$.

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Main challenge: large deviations

- Z: the number of k-SAT solutions;
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What causes large deviations?

- *k*-SAT solutions are not symmetric;
- "Degree sequence";
- Marginal distribution (The marginal distribution is computed by physicists (cavity method));
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For large k...

	SAT UB	SAT LB	Algorithmic Barrier
k-SAT	$2^k \ln 2 - (1+\ln 2)/2$	$2^k \ln 2 - (1+\ln 2)/2$	$2^k \ln k/k$
k-COL	$2k \ln k - \ln k$	$2k \ln k - \ln k - 2\ln 2$	$(1/2)k \ln k$
2-COL k-hypergraph	$2^{k-1} \ln 2 - \ln 2/2$	$2^{k-1} \ln 2 - (1+\ln 2)/2$	$2^{k-1} \ln k/k$

Connectivity of solutions

Let **x** and **y** be two solutions of F.

The Hamming distance between **x** and **y** is the number of i such that $x_i \neq y_i$.

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- F: a SAT formula.
- $f = f(n)$: any function of n between 1 and n.

Let F_f be the graphs such that

- $V(F_f)$: the set of solutions of F.
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If σ and τ are in the same component of F_f, we say they are f-connected.

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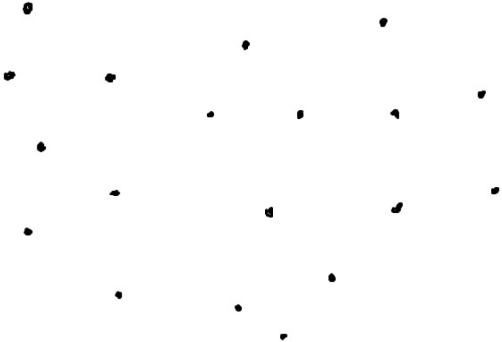

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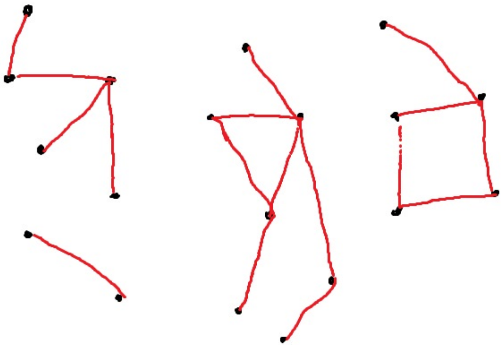

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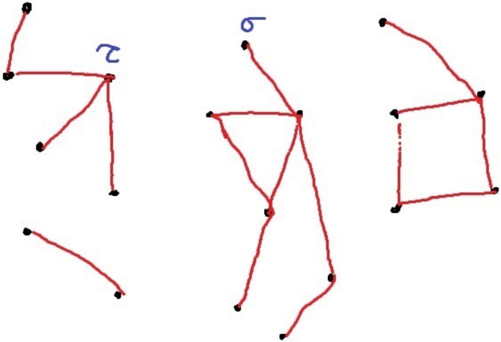

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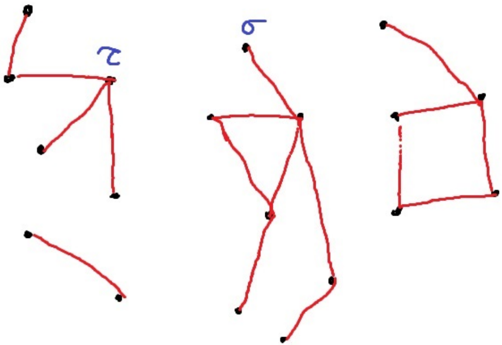

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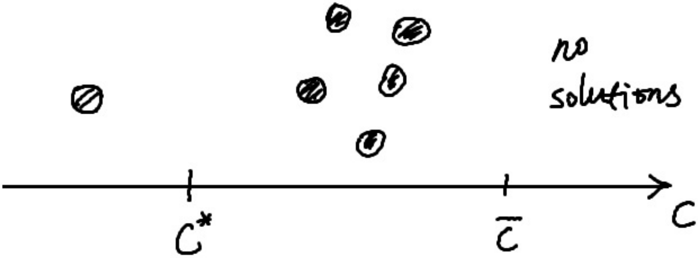

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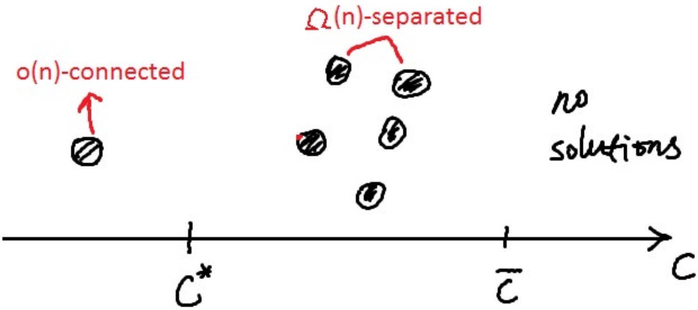

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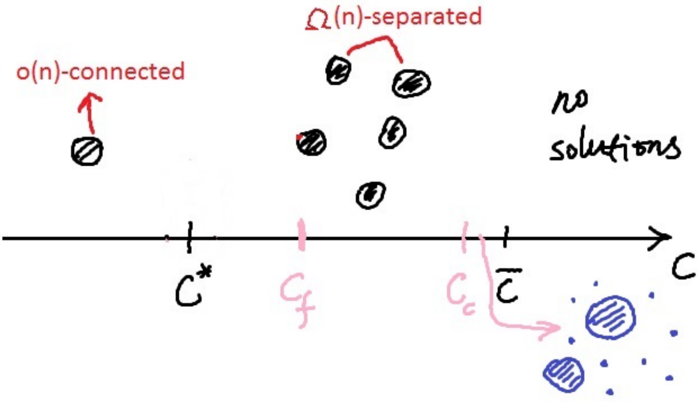

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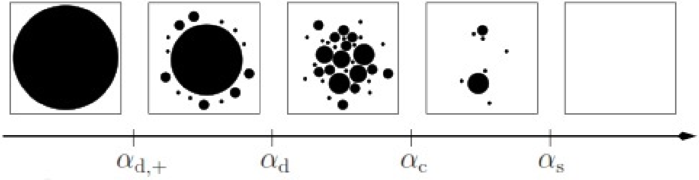
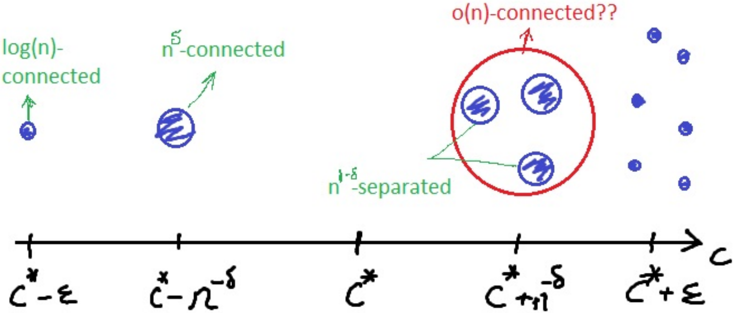


[Krzakala, Montanari, Ricci-Tersenghi, Semerjian, Zdeborova 2007]

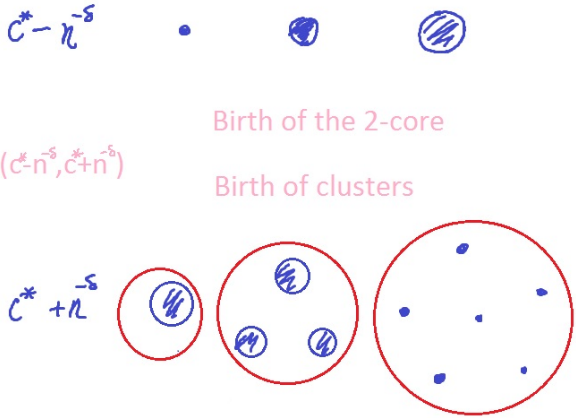
What mathematicians say...

- k-XORSAT clustering threshold determined (AM12,IKKM12)
- k-SAT has well-separated clusters (DMMZ08,MMZ05)
- k-SAT (NAESAT,COL) clusters appear after (asymptotically in k) the hypothesized clustering threshold (AC08)
- k-SAT clusters contains frozen variables (AR09)
- freezing threshold for k-COL, k-NAESAT (MR13)
- Condensation occurs in 2-COL in hypergraph (CZ12)

Birth of *k*-XORSAT clusters



Birth of k-XORSAT clusters: $c^* - \epsilon < c < c^* + \epsilon$



Birth of the 2-core

$(c^*-n^{-\delta}, c^*+n^{-\delta})$

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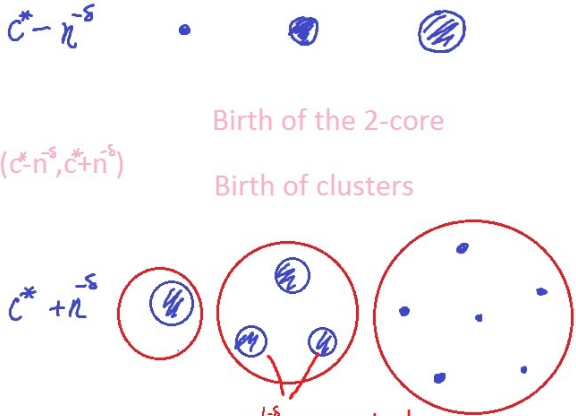

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$n^{1-\delta}$-separated

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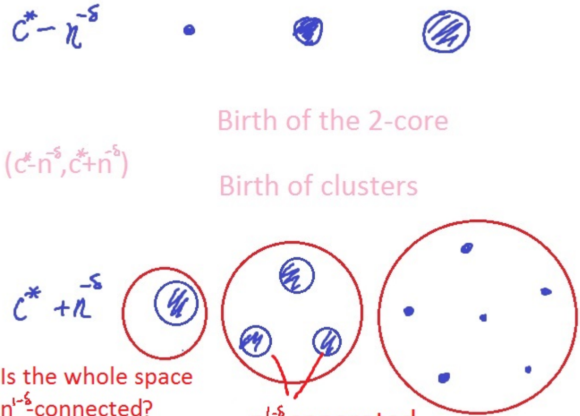

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Birth of clusters

Is the whole space $n^{1-\delta}$-connected?

$n^{1-\delta}$-separated

Linear system vs XORSAT formula

A clause in an k-XORSAT formula F (with m clauses) has the form

$$x_1 \oplus x_2 \oplus \bar{x}_3 \oplus \cdots \oplus x_k,$$

where $x_1 \oplus x_2$ is true $(= 1)$ if and only if exactly one of the variables is true $(= 1)$.
F is linear:

$$x_1 \oplus x_2 \oplus \bar{x}_3 \oplus \cdots \oplus x_k = x_1 + x_2 + (1 + x_3) + \cdots + x_k (mod\ 2).$$

An assignment satisfying F is a solution to a linear system of the following form

$$A\mathbf{x} = \mathbf{b} (\text{mod } 2),$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector of boolean variables;
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$$x_1 + x_4 + x_5 = 1$$

$$x_2 + x_4 + x_7 = 0$$

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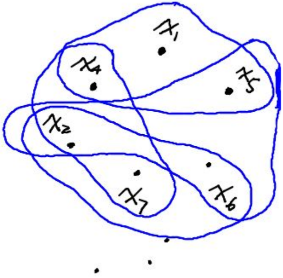

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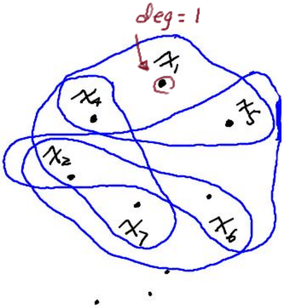

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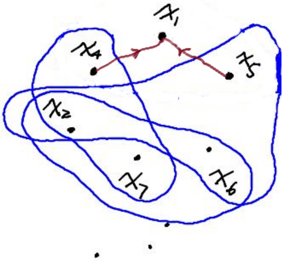

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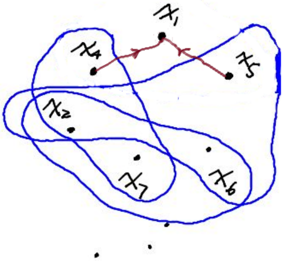

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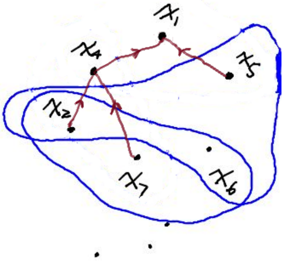

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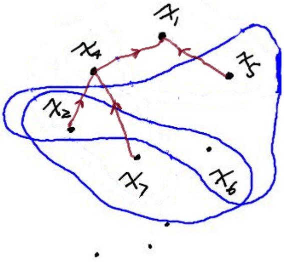

2-core of H

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Theorem (AM12,IKKM12)

A.a.s. the following statements are true.

(a) *If $m < (c^* - \epsilon)n$, then all solutions are in a single $O(\log n)$-connected cluster.*

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c^* corresponds to the emergence threshold of the 2-core.

2-core threshold

Theorem (Kim '06)

Let H be a random k-uniform hypergraph with n vertices and cn edges. There is a constant $c^* > 0$ (depending on k and can be specified) such that

(a) *if $c < c^* - n^{-1/2+\epsilon}$, then w.h.p. H has an empty 2-core;*

(b) *if $c > c^* + n^{-1/2+\epsilon}$, then w.h.p. H has a 2-core with size $\Omega(n)$.*

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Theorem (**G.** and Molloy '13)

Suppose $m = (c^* + n^{-\delta})n$, $\delta > 0$ *small. Then, a.a.s.*

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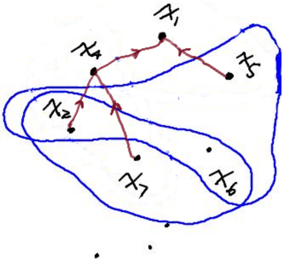

2-core of H

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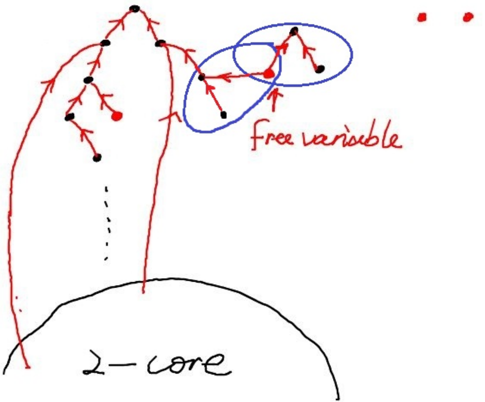

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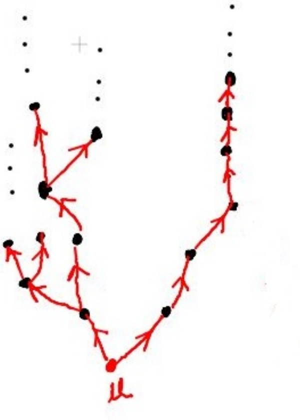

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$depth(v)$: the least number of vertices needs to be removed before v in any stripping sequence. (These are the set of vertices reachable from v in the digraph.)

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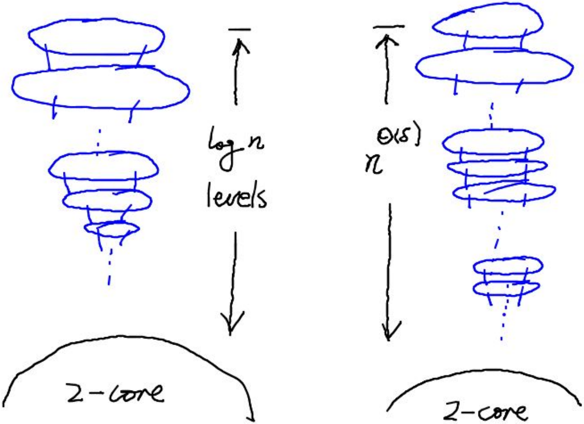
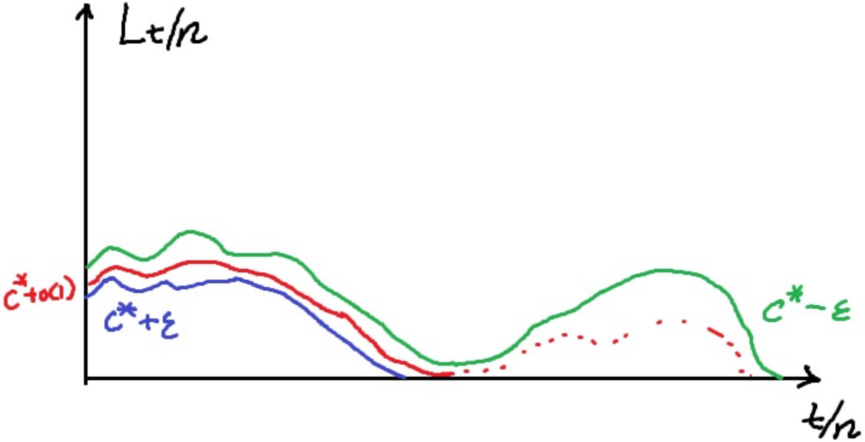


Why stripping becomes slow?



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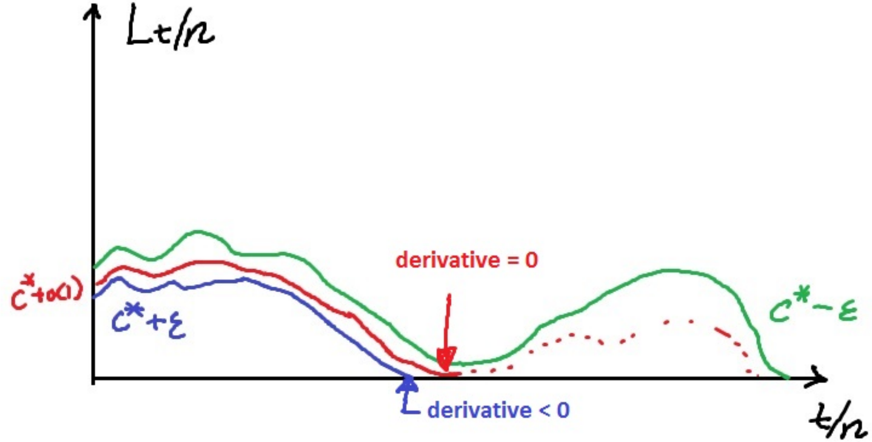

Stripping number

Theorem (G. and Molloy '13)

Assume $c = c^* + n^{-\delta}$, $0 < \delta < 1/2$, *the stripping number is between* $n^{\delta/2}$ *and* $n^{\delta/2} \log n$.

Idea behind the proof Approximate L_t by a "lazy" random walk – it takes long for L_t to reach 0. ☐

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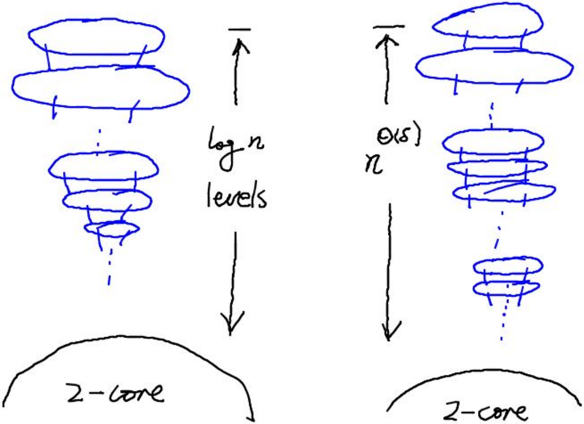
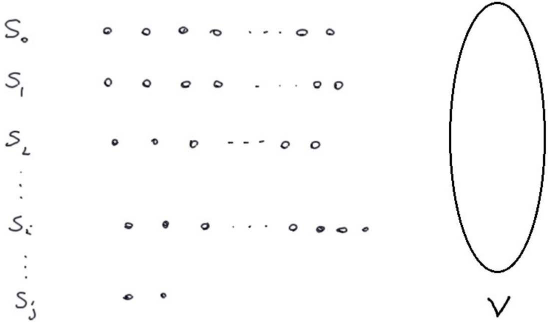


Upper bound of maximum depth

Expose $S_0, S_1, \cdots S_j, V$



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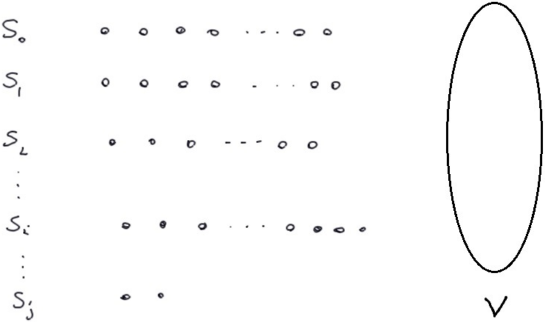

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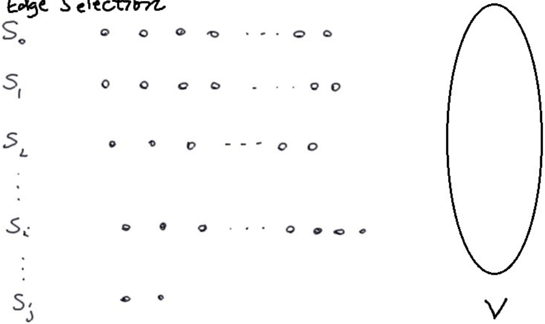

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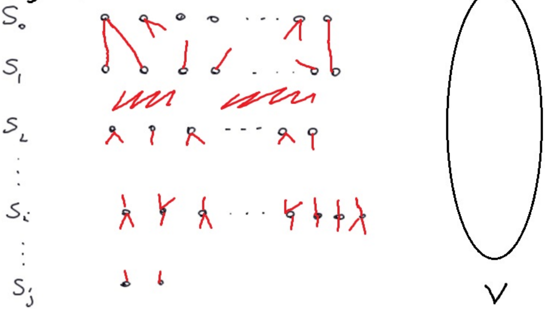

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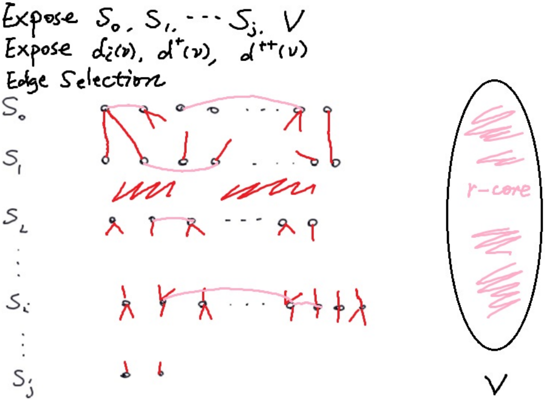

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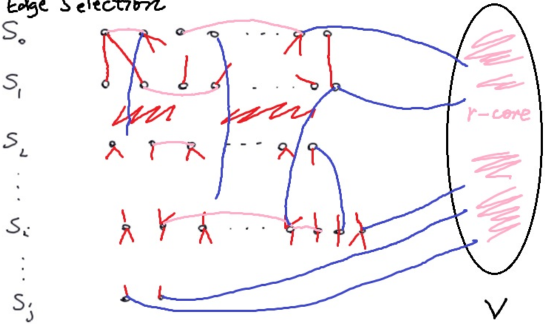

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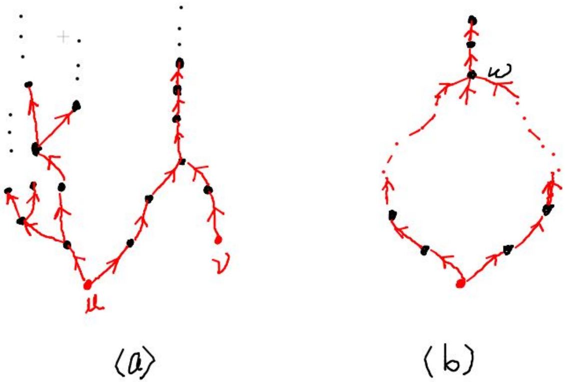

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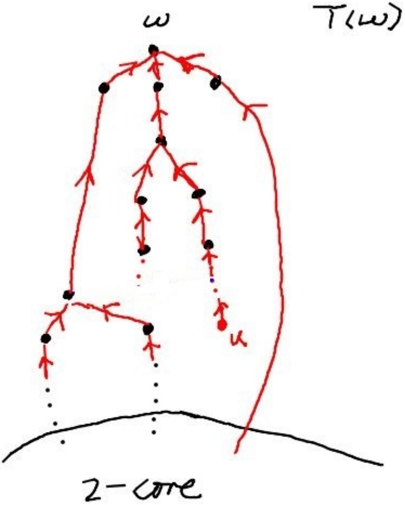

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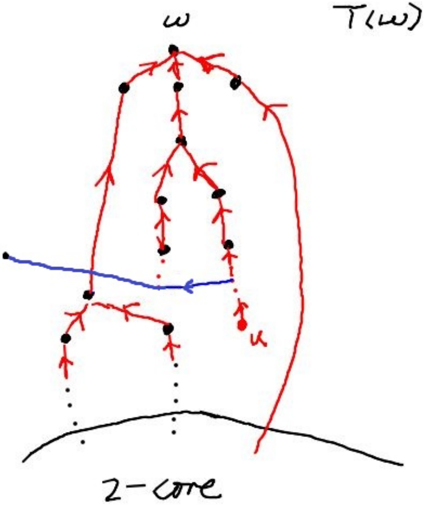

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Assume $c = c^* + n^{-\delta}$, $r \geq 3$, $\delta > 0$ small. Then a.a.s. each pair of clusters is $n^{1-O(\delta)}$-separated.

Note that this theorem does not exclude the possibility that clusters are $\Omega(n)$-separated.

Proof

Take x and y disagree on the 2-core;

$\Rightarrow$ x and y disagree on a set S of variables in the 2-core;

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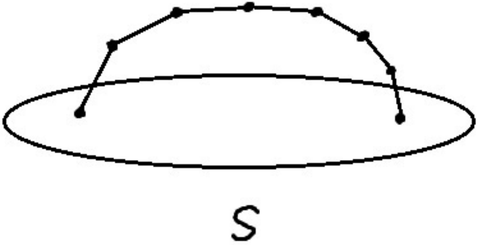

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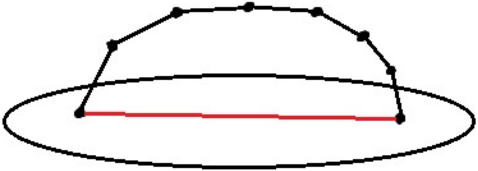

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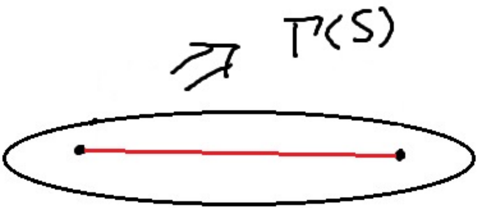

Between clusters

Theorem (G. and Molloy 2013)

Assume $c = c^* + n^{-\delta}$, $r \geq 3$, $\delta > 0$ *small. Then a.a.s. each pair of clusters is* $n^{1-O(\delta)}$*-separated.*

Note that this theorem does not exclude the possibility that clusters are $\Omega(n)$-separated.

Proof

Take x and y disagree on the 2-core;
$\Rightarrow$ x and y disagree on a set S of variables in the 2-core;
$\Rightarrow$ $|e \cap S|$ is even for every e in the 2-core;
$\Rightarrow$ $\Gamma(S)$;
$\Rightarrow$ average degree of $\Gamma(S)$ is at least 3;

Proof

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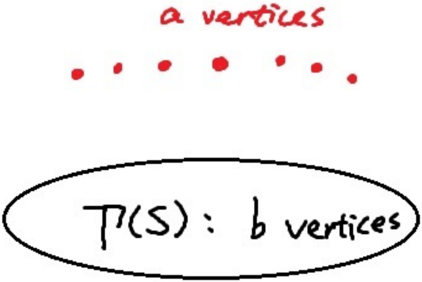
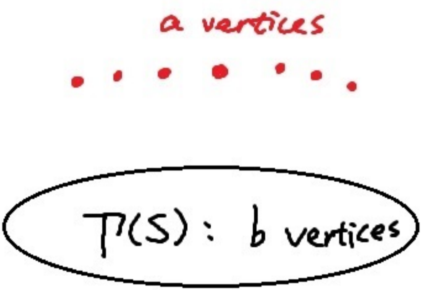


case 1: a=O(b), both a, b small.

Proof



case 1: a=O(b), both a, b small.

⇒ a small graph: average degree >2

Between clusters

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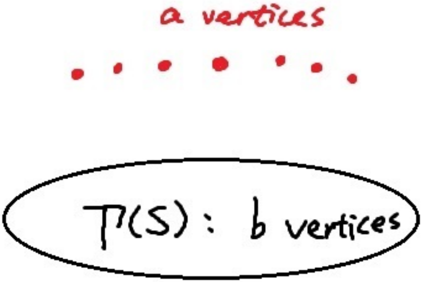
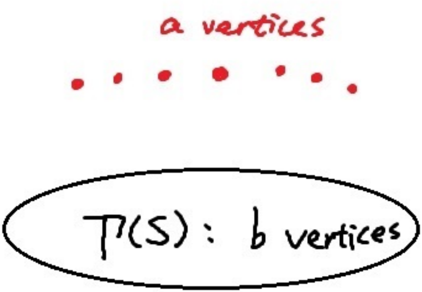


case 1: a=O(b), both a, b small.

⇒ a small graph: average degree >2

case 2: b small, a>> b.

Proof



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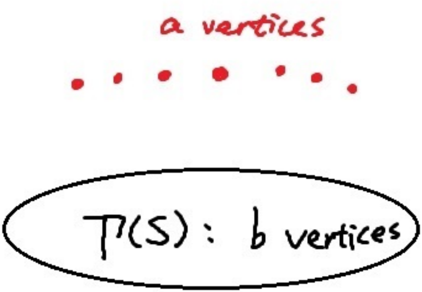


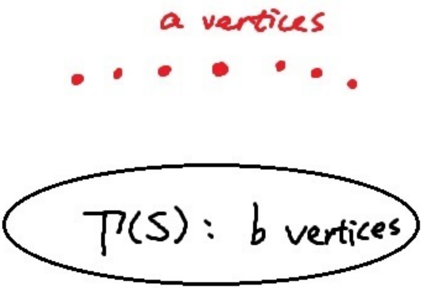
case 1: $a=O(b)$, both a, b small.

⇒ a small graph: average degree >2

case 2: b small, a>> b.

⇒ such a path cannot be long

Proof



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cases 1,2 ⇒ a+b must be large

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(a) A sparse random graph is locally tree-like; (b) Branching parameter $= 1 - \zeta$;

$\Rightarrow$ x and y are well-separated.

Future work

- For $c = c^* + n^{-\delta}$, are clusters $o(n)$-connected or $\Omega(n)$-separated?
- $c = c^* - n^{-\delta}$ (stripping number, depth, cluster connectivity)?
- $c = c^* + O(n^{-1/2})$ (stripping number, depth, cluster connectivity)?
- What happens to the first graph containing a 2-core?

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