Mixing time of the Swendsen-Wang process on the complete graph

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Collaborators

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Probability on Graphs

- Many problems in statistical mechanics are of the form:
  - Consider a sequence of finite graphs $G_n = (V_n, E_n)$ with:
    - $G_n \subset G_{n+1}$ and $|V_{n+1}| > |V_n|$
    - E.g. complete graphs K_n, or tori $\mathbb{Z}_n^d$
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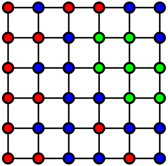

  - $\Omega = [q]^V$ for fixed $q \in \{2, 3, 4 \ldots\}$
  - $\pi(\sigma) = \dfrac{1}{Z}\, e^{-\beta H(\sigma)}$ for $\sigma \in \Omega$
    - $H(\sigma) = -\sum_{uv \in E} \delta_{\sigma_u, \sigma_v}$
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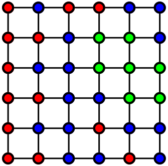

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    - $\beta = 1/$temperature
  - If $\beta \approx 0$ then $\pi(\cdot) \approx$ uniform on Ω ("Disorder")
  - If $\beta \gg 1$ preference for $u \sim v$ to have $\sigma_u = \sigma_v$ ("Order")
  - Phase transition between order and disorder at critical β_c

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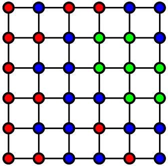

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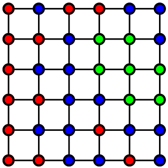

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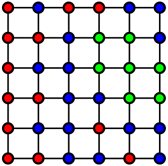

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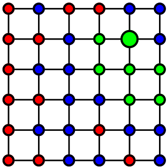

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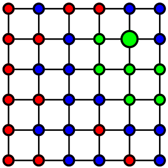

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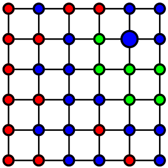

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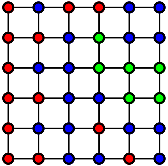

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Mixing times

- Consider an irreducible aperiodic Markov chain on a finite state space Ω with transition matrix P and stationary distribution π

$$d(t) := \max_{x \in \Omega} \|P^t(x, \cdot) - \pi(\cdot)\| \leq C\alpha^t, \qquad \textit{for } \alpha \in (0,1)$$

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  - If $t_{\mathrm{mix}} = O(\mathrm{poly}(\log|\Omega|))$ we have **rapid mixing**
  - Otherwise, we have **torpid mixing**

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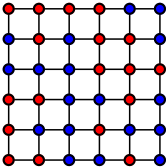
- Stationary distribution is q-state Potts model



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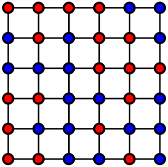

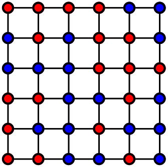

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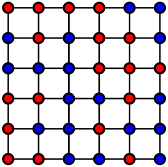

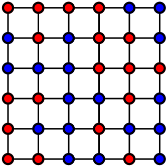

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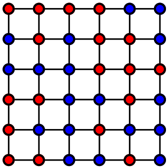

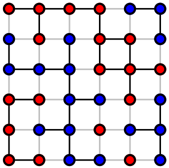

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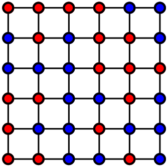

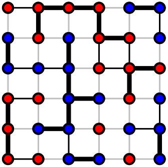

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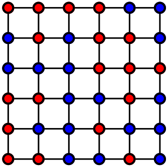

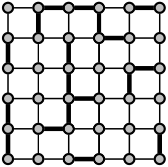

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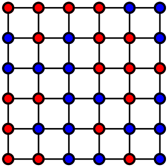

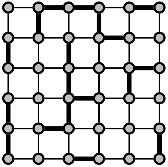

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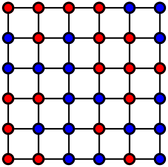
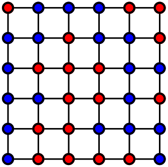

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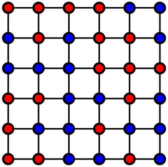

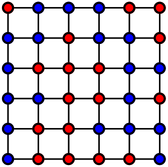

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SW transitions given by $\mu(\sigma_{t+1}|A_{t+1})\mu(A_{t+1}|\sigma_t)$ where $\mu(A, \sigma)$ is Edwards-Sokal coupling of Potts and Fortuin-Kasteleyn models

SW process on complete graph

On K_n:

- Potts model has transition at $\beta = \lambda_c/n$ with $\lambda_c = \Theta(1)$
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- Potts model has transition at $\beta = \lambda_c/n$ with $\lambda_c = \Theta(1)$
  - Continuous transition for $q = 2$ (Ising)
  - Discontinuous transition for $q \geq 3$
- Potts energy depends only on magnetization $s(\sigma)$

$$-\beta H(\sigma) = \frac{\lambda}{2}\, n\, s(\sigma) \cdot s(\sigma) + \text{ constant}$$

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- Let $\mathrm{SW}_n(\lambda, q) =$ SW process on K_n with parameters λ and q

  Given $\sigma_t \in [q]^n$, $\mathrm{SW}_n(\lambda, q)$ chooses σ_{t+1} as follows:
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Note: edge probability in $\mathcal{G}(\sigma_t^{-1}(i), \lambda/n)$ is $\lambda/n = s^i(\sigma_t)\lambda/|\sigma_t^{-1}(i)|$

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If $q = 2$ *then* $\mathrm{SW}_n(\lambda, q)$ *has mixing time*

$$t_{\mathrm{mix}} = O(\sqrt{n})$$

for all $\lambda \notin (\lambda_c - \delta, \lambda_c + \delta)$ *with* $\delta\sqrt{\log n} \to \infty$ *as* $n \to \infty$*.*

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If $q = 2$ *then* $\mathrm{SW}_n(\lambda, q)$ *has mixing time*

$$t_{\mathrm{mix}} = \begin{cases} \Theta(1) & \lambda < \lambda_c \\ \Theta(n^{1/4}) & \lambda = \lambda_c \\ \Theta(\log n) & \lambda > \lambda_c \end{cases}$$

- **Ray, Tamayo, & Klein** (1989) conjectured $n^{1/4}$ at λ_c

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Theorem (Cuff, Ding, Louidor, Lubetzky, Peres, Sly 2012)

If $q \geq 3$ then the single-site Glauber process for the Potts model has

$$t_{\mathrm{mix}} = \begin{cases} \Theta(n \log n) & \lambda < \lambda_o(q) \\ \Theta(n^{4/3}) & \lambda = \lambda_o(q) \\ \exp(\Omega(n)) & \lambda > \lambda_o(q) \end{cases}$$

*where $\lambda_o(q) < \lambda_c(q)$, so torpid mixing begins **before** transition*

Magnetization distribution

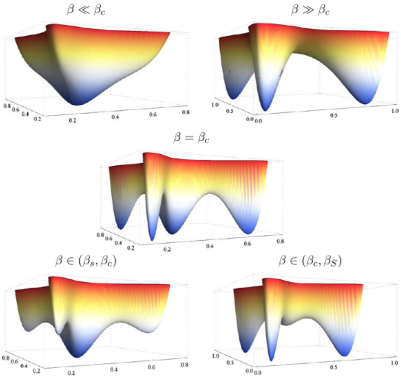

Figure: From Cuff *et. al* 2012

Large n distribution of $s(\sigma)$ known explicitly:

$$-\frac{1}{n}\log \mathbb{P}(s(\sigma) = a) \sim \phi_\lambda(a) - \inf_{a\in\Delta^{q-1}} \phi_\lambda(a)$$

$$\phi_\lambda(a) = \sum_{i=1}^{q}\left(a_i \log a_i - \frac{1}{2}\lambda\, a_i^2\right)$$

Minima of ϕ_λ correspond either to:

- **disordered** state: $s^i = 1/q$ for all $i \in [q]$
- **ordered** states: $s^i = \alpha > 1/q$ and $s^j = \frac{1-\alpha}{q-1}$ for $j \neq i$

$\lambda_{\mathrm{o}}(q) := \inf\{\lambda \geq 0 :$ there exist ordered local minima of $\phi_\lambda\}$,

$\lambda_{\mathrm{d}}(q) := \sup\{\lambda \geq 0 :$ the disordered state locally minimizes $\phi_\lambda\}$.

Complete picture for $\mathrm{SW}_n(\lambda, q)$ with $q \geq 3$

Theorem (Lin & G. 2013)

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- Gore & Jerrum's torpid mixing result extends to a non-trivial interval $(\lambda_{\mathrm{o}}(q), \lambda_{\mathrm{d}}(q))$ containing $\lambda_{\mathrm{c}}(q)$
- Nothing special happens at $\lambda_{\mathrm{c}}(q)$
- Non-trivial scaling arises at $\lambda_{\mathrm{o}}(q)$
- Low and high temperature same as Ising case

Sketch of Proof

- If $Y_{t+1} := s^1_{t+1} - \mathbb{E}[s^1_{t+1}|\mathcal{F}_t]$ then

$$s^1_{t+1} \approx s^1_t + D(s^1_t) + Y_{t+1} \qquad (*)$$

where

$$D_{\lambda,q}(x) := \theta(\lambda x)(1 - 1/q)x + 1/q - x$$

  - $\theta(\lambda)\, n = \mathbb{E}$(size of giant component) in Erdös-Renyi $\mathcal{G}(n, \lambda/n)$
  - $\mathcal{F}_t$ is the σ-algebra generated by $\{\sigma_s : s \leq t\}$
  - $(Y_t)_{t\geq 0}$ is a sequence of martingale increments with respect to $\mathcal{F}_t$
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- Coupling arguments reduce mixing time to hitting time of s^1_t

Swendsen-Wang drift

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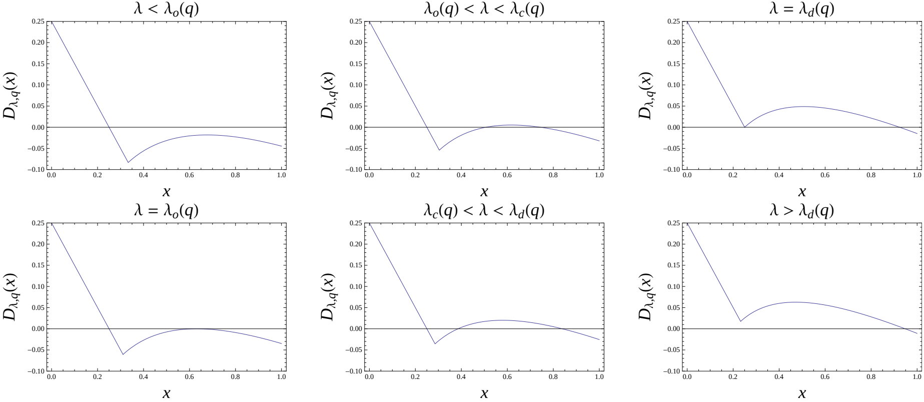

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Hitting times for stochastic difference equations

Lemma (Lin & G. 2013)

Consider the family of processes defined for each $n \in \mathbb{N}$ *by*

$$X_{t+1} = X_t + n^{-\gamma} D(X_t) + Y_{t+1}^{(n)}$$

- D *has a unique root at* x_*
- $D(x_* + h) = c|h|^k + \Theta(h^{k+1})$ *for some* $c > 0$ *and* $k \geq 2$.
- $\mathrm{var}(Y_{t+1}^{(n)}|\mathcal{F}_t) = \Theta(n^{-\nu})$ *for some* $\nu > 0$
- $M_t^{(n)} = \sum_{s=1}^{t} Y_s^{(n)}$ *is a martingale for each* $n \in \mathbb{N}$
- *Some technical assumptions which are true for SW and Glauber*

Define

$$\tau(a, b) = \inf\{t \geq 0 : X_t \geq b | X_0 = a\}$$

Let $a \in (x_* - \epsilon, x_*]$. *For each sufficiently small* $\alpha > 0$

$$\mathbb{P}\left(\tau(a, x_* + \epsilon) \leq \alpha\, n^{\frac{2\gamma+\nu(k-1)}{k+1}}\right) = \Omega(1),$$

$$\mathbb{P}\left(\tau(a, x_* + \epsilon) > \alpha\, n^{\frac{2\gamma+\nu(k-1)}{k+1}}\right) = \Omega(1).$$

Scaling Exponents for SW and Glauber

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- $q \geq 3$ for $\lambda = \lambda_{\rm o}$ has $\gamma = 0$, $k = 2, \nu = 1$, so it takes $\Theta(n^{1/3})$ time to traverse the non-stationary root

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Note that:

- At $\lambda_o(q)$, $t_{\text{mix}}^{(\text{SW})} = \Theta(n^{1/3}) = \frac{1}{n} t_{\text{mix}}^{(\text{Glb})}$
- At $\lambda_d(q)$, $t_{\text{mix}}^{(\text{SW})} = \Theta(\log n)$ while $\frac{1}{n} t_{\text{mix}}^{(\text{Censored Glb})} \stackrel{?}{=} \Theta(n^{1/3})$

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- Can one say anything for the Glauber chain for the Fortuin-Kasteleyn model?