Locating arrays and disjoint partitions

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Joint work with Charles Colbourn and Bingli Fan.

Covering arrays

A fault detection problem

A fault detection problem

WhatsApp text formatting rolls out, including bold, italics and strikethrough in messages

by: KRIS CARLON
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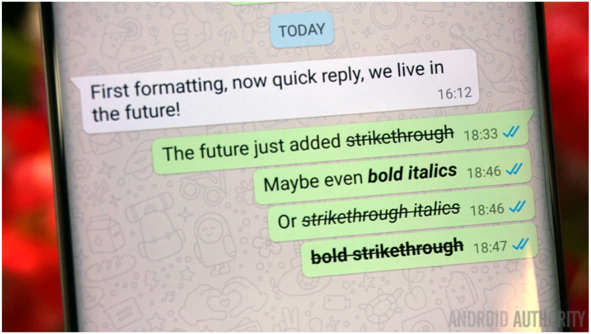

A fault detection problem

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Suppose WhatsApp wants to test its new features:

- B **bold**
- I *italic*
- U underline
- abc ~~strikethrough~~
- A subscript

A fault detection problem

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Testing every possible combination of these would require 32 tests:

B	I	U	abc	A	
0	0	0	0	0	test
0	0	0	0	1	test
0	0	0	1	0	~~test~~
⋮	⋮	⋮	⋮	⋮	⋮
1	1	1	1	0	***~~test~~***
1	1	1	1	1	***~~test~~***

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Assumption: Faults are caused by interactions of at most two settings.

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Examples:

abc
0

A *1-way interaction*

B	U
1	0

A *2-way interaction*

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~~abc~~
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A *2-way interaction*

How many tests do we need now?

Covering arrays

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0	0	1	0	0
0	1	0	1	1
1	0	0	1	0
1	1	1	0	1
0	1	0	0	0
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A *covering array* with $N = 6$, $k = 5$ and $v = 2$

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Reveals if there's a faulty 1- or 2-way interaction. (The *strength* is 2.)

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This approach becomes vital when k is big. For $k = 3000$ and $v = 2$, there exists a strength 2 covering array with 15 rows.

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The covering array problem

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- A piece of software has k parameters; each can take one of v values.
- We know faults are caused only by t-way interactions.
- We can test the software on any assignment of values to parameters and obtain a pass or a fail.
- We wish to prespecify a schedule of N tests after which we will be able to determine if a fault exists (for N as small as possible).

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We can give a solution to this problem as a strength t covering array.

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- The symbols in any column of a covering array can be permuted.

- Covering arrays with strength 1 are trivial. For example,

B	*I*	U	~~abc~~	A
0	0	0	0	0
1	1	1	1	1

Locating arrays

Locating faults

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Locating faults

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0	1	0	1	1
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Locating faults

Suppose

B	U
1	0

is faulty.

B	I	U	abc	A	
0	0	1	0	0	pass
0	1	0	1	1	pass
1	0	0	1	0	fail
1	1	1	0	1	pass
0	1	0	0	0	pass
0	0	1	1	1	pass

Locating faults

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B	I	U	abc	A	
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0	1	0	1	1	pass
1	0	0	1	0	fail
1	1	1	0	1	pass
0	1	0	0	0	pass
0	0	1	1	1	pass

We would get the same pass/fail pattern if

I	U
0	0

were faulty.

Locating faults

Suppose (B = 1, U = 0) is faulty.

B	*I*	U	~~abc~~	A	
0	0	1	0	0	pass
0	1	0	1	1	pass
1	0	0	1	0	fail
1	1	1	0	1	pass
0	1	0	0	0	pass
0	0	1	1	1	pass

We would get the same pass/fail pattern if (I = 0, U = 0) were faulty.

A *locating array* is a covering array which also allows faults to be identified.

Locating faults

Suppose $\frac{\mathbf{B}\quad \underline{\text{U}}}{1\quad 0}$ is faulty.

B	*I*	U	~~abc~~	A	
0	0	1	0	0	pass
0	1	0	1	1	pass
1	0	0	1	0	fail
1	1	1	0	1	pass
0	1	0	0	0	pass
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We would get the same pass/fail pattern if $\frac{\textit{I}\quad \underline{\text{U}}}{0\quad 0}$ were faulty.

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Locating arrays with strength 1 are *not* trivial.

Locating faults

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0	1	0	1	1	pass
1	0	0	1	0	fail
1	1	1	0	1	pass
0	1	0	0	0	pass
0	0	1	1	1	pass

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Locating arrays with strength 1 are *not* trivial.
In fact, this talk will be about locating arrays with strength 1.

Our problem

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- A piece of software has k parameters; each can take one of v values.
- We know the software is faulty on *at most one 1-way interaction*.
- We can test the software on any assignment of values to parameters and obtain a pass or a fail.
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We give a solution to this problem as a $(\overline{1}, 1)$-*locating array*.

Locating arrays

Locating arrays

Not a $(\overline{1}, 1)$-locating array with $N = 6$, $k = 9$ and $v = 3$:

$$\begin{matrix} 1 & 3 & 3 & 3 & 2 & 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 3 & 3 & 3 & 1 & 2 & 3 \\ 3 & 2 & 2 & 1 & 3 & 3 & 3 & 1 & 3 \\ 3 & 3 & 2 & 2 & 1 & 3 & 2 & 3 & 1 \\ 3 & 3 & 3 & 2 & 2 & 1 & 3 & 2 & 2 \end{matrix}$$

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1	3	3	3	2	2	1	3	2
2	1	3	3	3	2	2	1	1
2	2	1	3	3	3	1	2	3
3	2	2	1	3	3	3	1	3
3	3	2	2	1	3	2	3	1
3	3	3	2	2	1	3	2	2

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2	2	1	3	3	3	1	2	3
3	2	2	1	3	3	3	1	3
3	3	2	2	1	3	2	3	1
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2	1	3	3	3	2	2	1	3
2	2	1	3	3	3	1	2	1
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Given k and v, we want to minimise N.

Our problem: Given N and v, find the maximum number of columns in a $(\overline{1}, 1)$-locating array with N rows and v symbols.

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Similar problems have been considered in the combinatorial group testing literature, but not this one exactly.

Disjoint set partitions

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Disjoint set partitions

1	3	3	3	2	2	1	3	2
2	1	3	3	3	2	2	1	3
2	2	1	3	3	3	1	2	1
3	2	2	1	3	3	3	1	2
3	3	2	2	1	3	2	3	1
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Disjoint set partitions

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Equivalently:

$\{1\}$	$\{2,3\}$	$\{4,5,6\}$
$\{2\}$	$\{3,4\}$	$\{1,5,6\}$
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$\{6\}$	$\{1,2\}$	$\{3,4,5\}$
$\{1,3\}$	$\{2,5\}$	$\{4,6\}$
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Disjoint set partitions

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Disjoint set partitions

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Fact

A $(\overline{1}, 1)$-locating array with N rows, k columns and v symbols is equivalent to k partitions of $\{1, \ldots, N\}$, each with v nonempty classes, such that no two of the kv classes are equal.

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Given N and v, find the maximum number of disjoint v-partitions of $\{1, \ldots, N\}$.

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Similar problems have been considered in the set systems literature, but not this one exactly.

Shapes

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Shapes

partition			shape
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Our problem (re-rephrased)
Given N and v, find the maximum size of an admissible family of v-shapes.

Maximal families

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Close to $\binom{38}{5}$ 5s, and fewer than $\binom{38}{6}$ 6s are used.

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Too many 6s.

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where $r = 2\binom{41}{1} + 3\binom{41}{2} + 4\binom{41}{3} + 5\binom{41}{4}$.

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Exactly $\binom{41}{5}$ 5s, close to $\binom{41}{6}$ 6s, and fewer than $\binom{41}{7}$ 7s are used.

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Theorem

Let N and v be integers such that $2 \leqslant v \leqslant N$. The maximum number of disjoint v-partitions of $\{1, \ldots, N\}$ is

$$\left\lfloor \frac{1}{d} \sum_{\substack{i=f-d+2 \\ i \geqslant 1}}^{f} (f+1-i) \binom{N}{i} \right\rfloor + \sum_{i=1}^{f-d+1} \binom{N}{i},$$

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For fixed v, we have $N = \frac{v}{v \log v - (v-1)\log(v-1)} \log k + O(\log \log k)$.

Future work

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- Sperner partition systems (Meagher, Moura, Stevens).

That’s all.