Extending Fisher's inequality to coverings

Daniel Horsley (Monash University, Australia)

Introduction 1

Designs and Fisher's inequality

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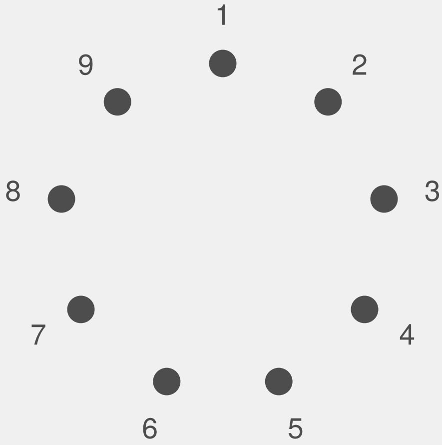

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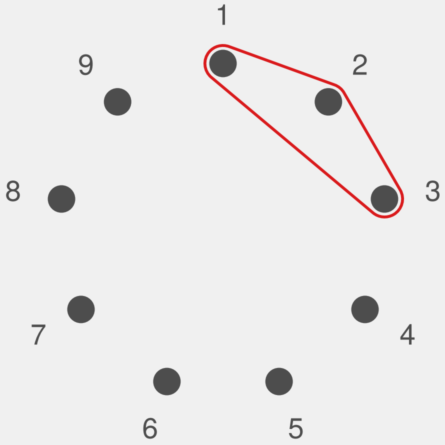

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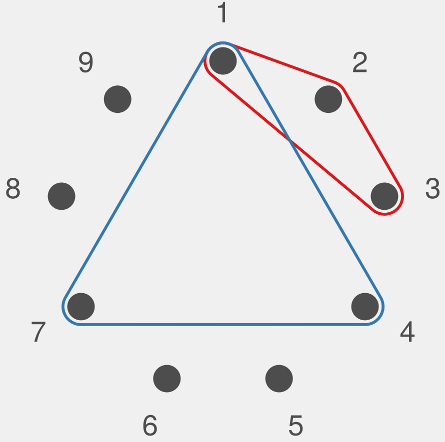

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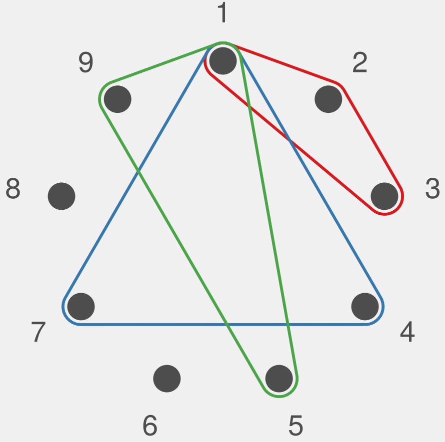

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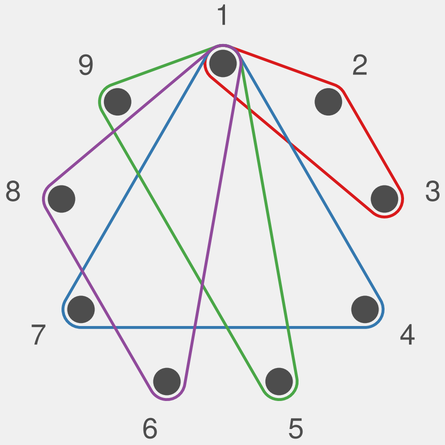

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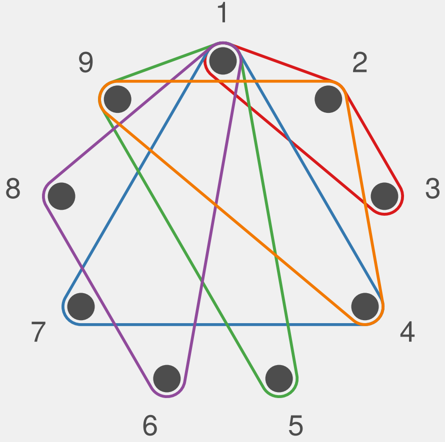

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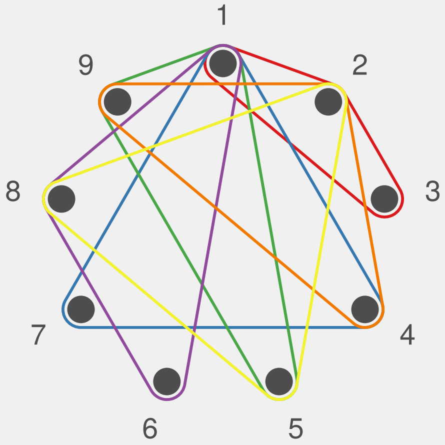

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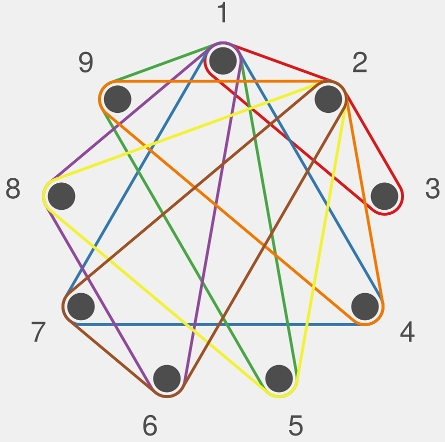

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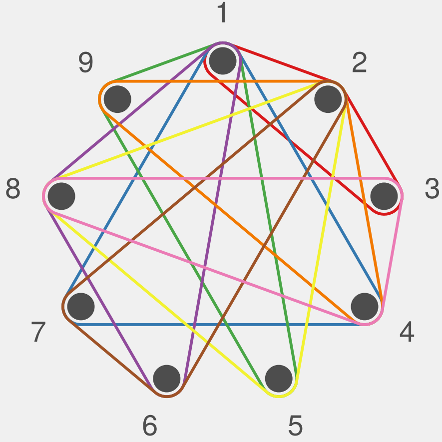

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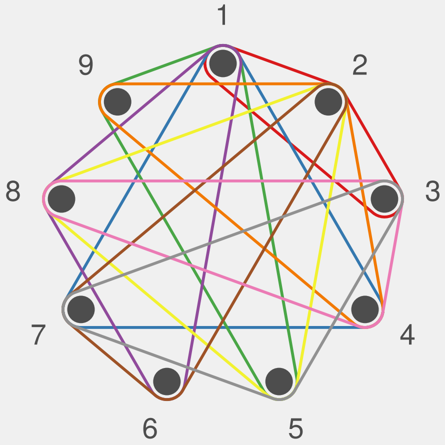

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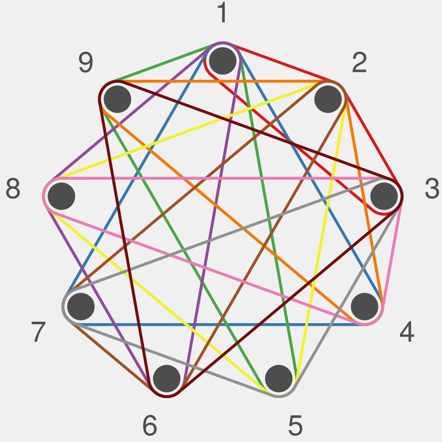

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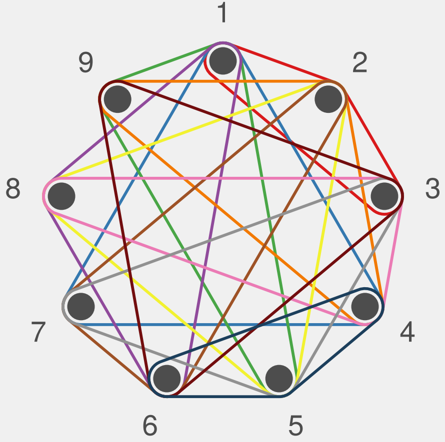

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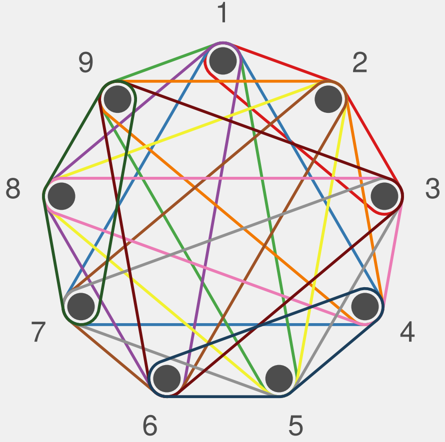

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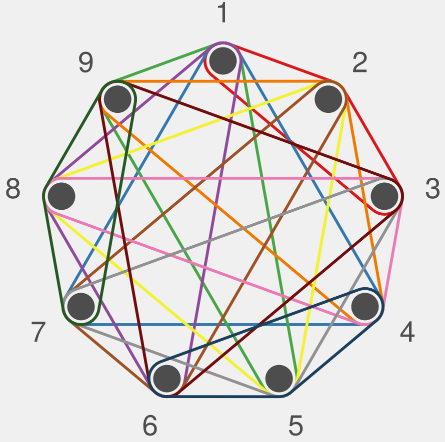

A $(9, 3, 1)$-design with 12 blocks

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Obvious necessary conditions: If there exists a (v, k, λ)-design then

(1) $r = \dfrac{\lambda(v-1)}{k-1}$ is an integer;

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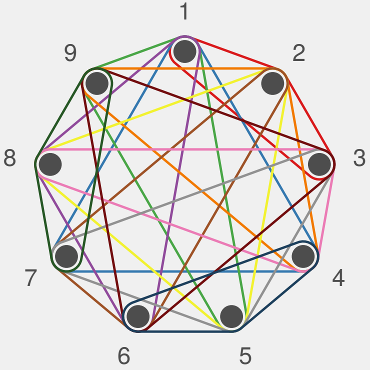

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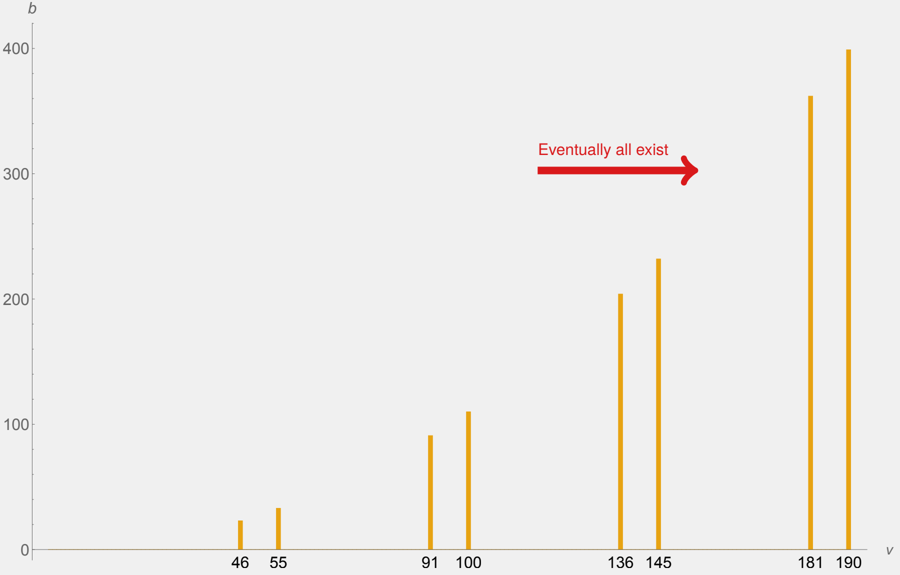

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Symmetric designs have $v = \frac{k(k-1)}{\lambda} + 1$ (or $b = v$ or $r = k$).

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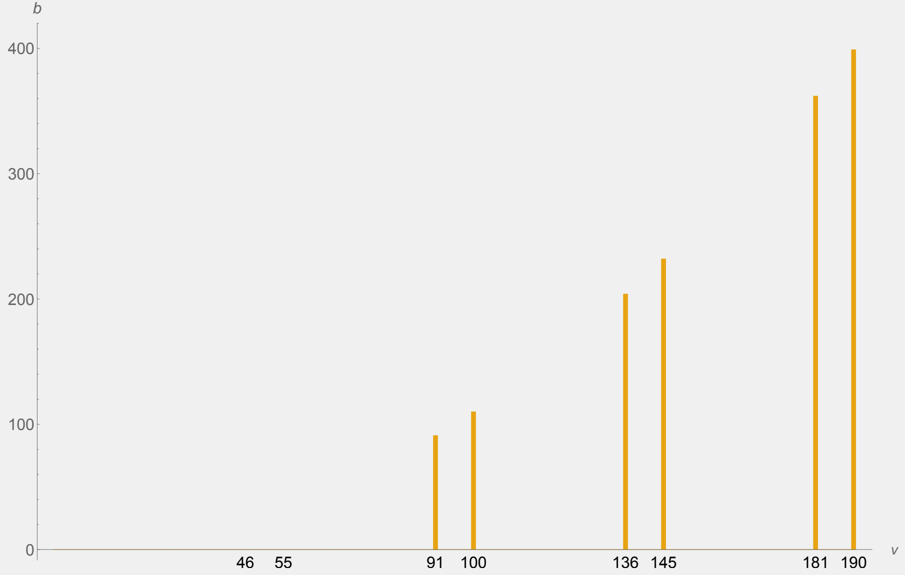

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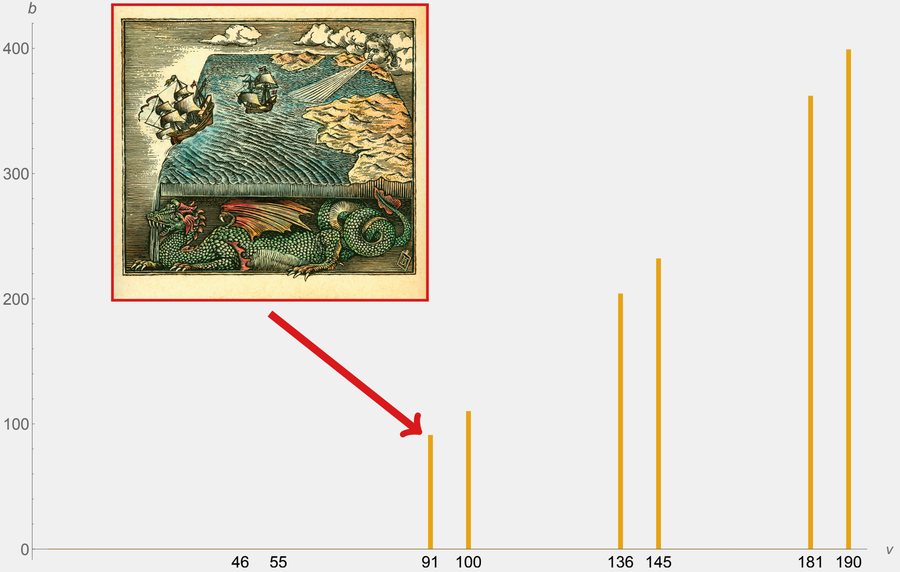

Incidence matrix arithmetic

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Consider the incidence matrix of our $(9, 3, 1)$-design.

$$
\begin{array}{cc}
 & \text{12 blocks} \\
\text{9 points} & \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
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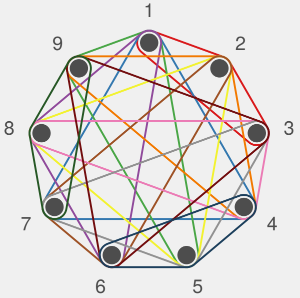

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$$AA^T = \begin{pmatrix} 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

In general, $z_{xx} = r$ and $z_{xy} = \lambda$.

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Introduction 2

Coverings and the Schönheim bound

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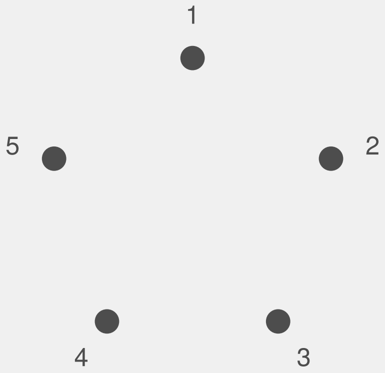

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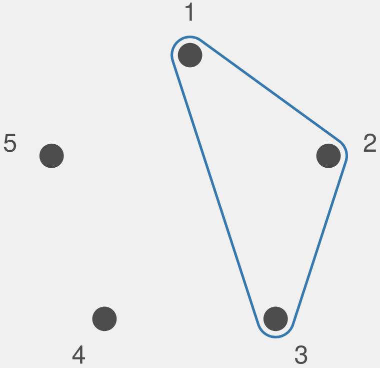

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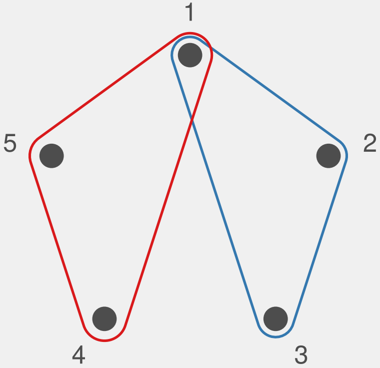

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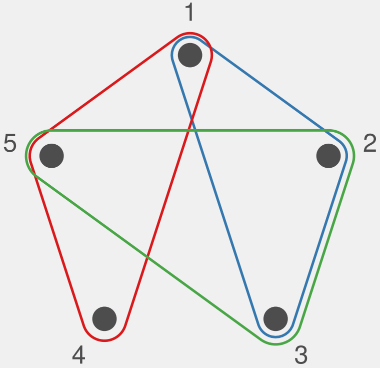

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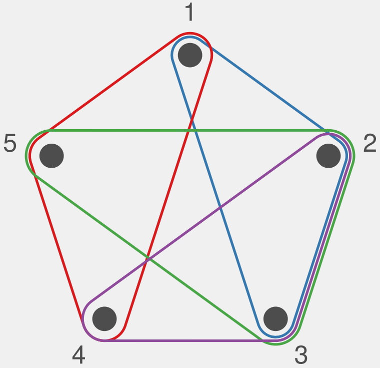

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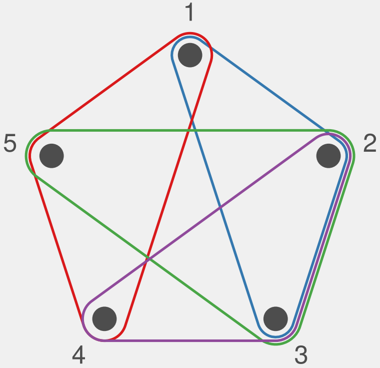

A $(5, 3, 1)$-covering with 4 blocks.

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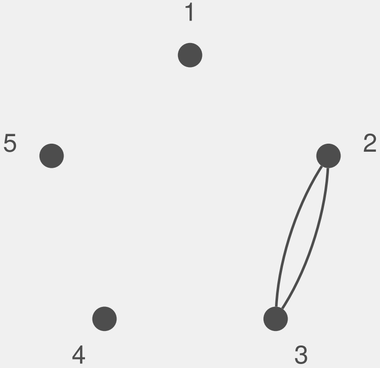

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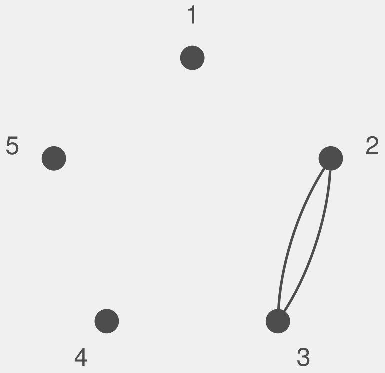

The *excess* of the covering.

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Schönheim bound: $C_\lambda(v, k) \geqslant \left\lceil \frac{rv}{k} \right\rceil$.

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Tweak: We can improve the Schönheim bound by 1 if

- $\lambda(v-1) \equiv 0 \pmod{k-1}$; and
- $\lambda v(v-1) \equiv 1 \pmod{k}$.

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For the rest of this talk "the Schönheim bound" includes this tweak.

Possible $(v, 10, 1)$-designs

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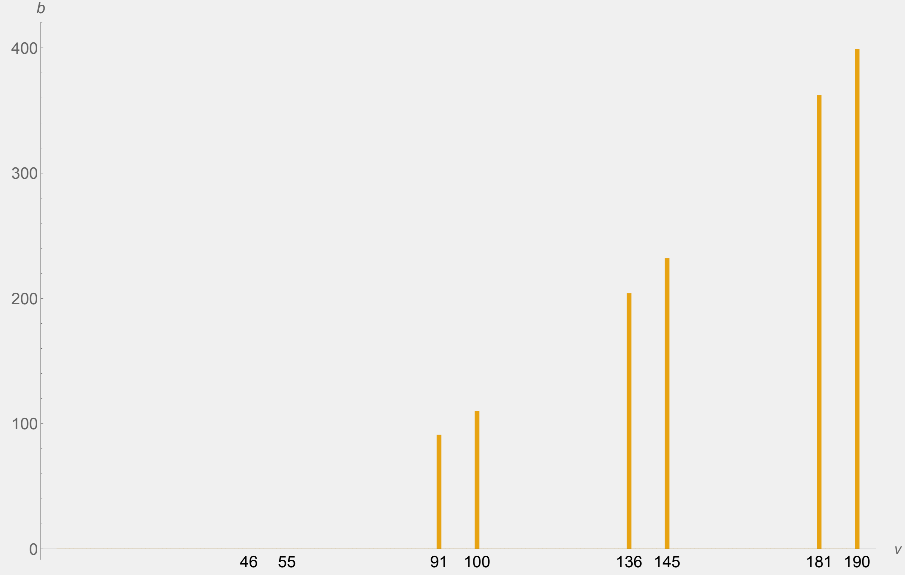
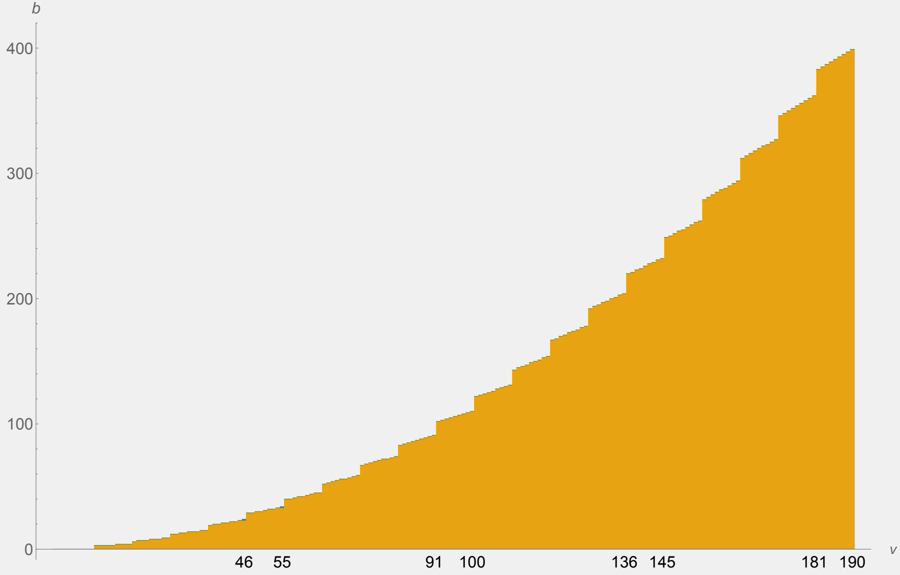


Bounds on $(v, 10, 1)$-coverings



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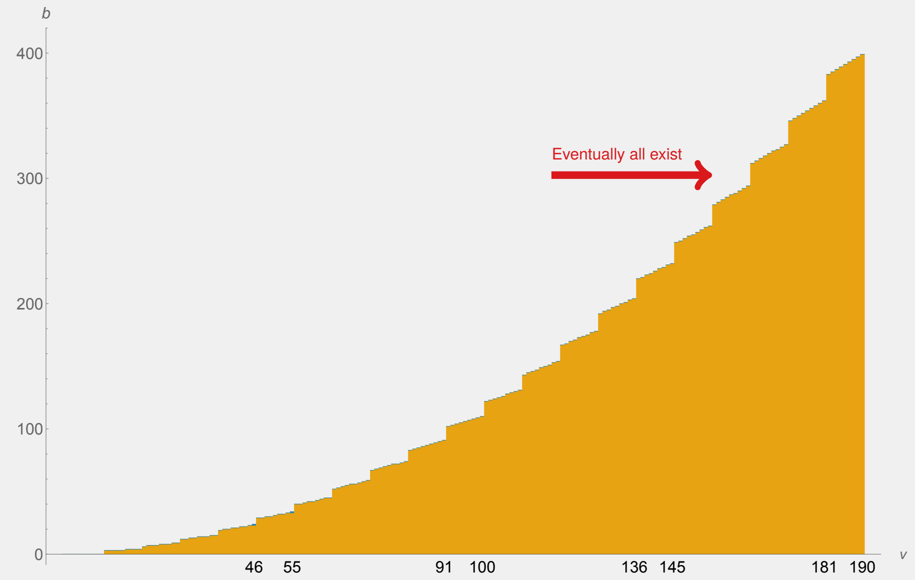

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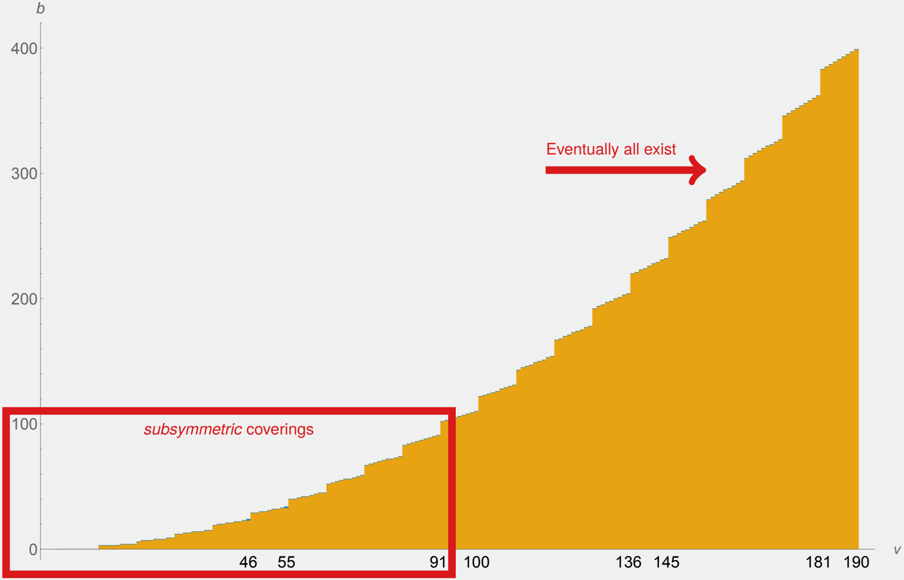

Possible subsymmetric $(v, 10, 1)$-coverings

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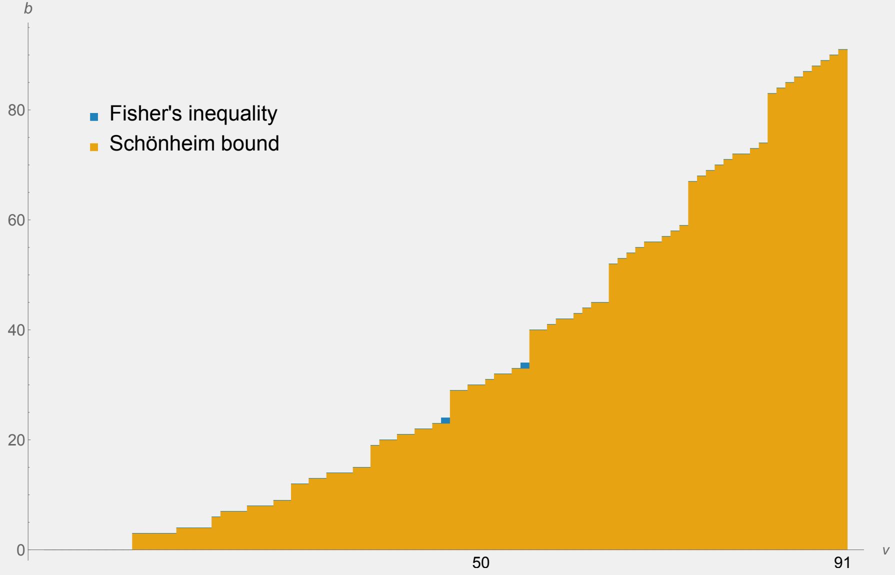

Manifesto

Subsymmetric coverings are particularly interesting because there are no analogous designs.

We should investigate the value of $C_\lambda(v,k)$ for subsymmetric (v,k,λ).

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Subsymmetric coverings are particularly interesting because there are no analogous designs.

We should investigate the value of $C_\lambda(v, k)$ for subsymmetric (v, k, λ).

This talk

- Fisher's inequality itself improves on the Schönheim bound for certain (very special) subsymmetric parameter sets.
- I've generalised Bose's proof to improve on the Schönheim bound for a much wider variety of subsymmetric parameter sets.
- In some cases this yields exact covering numbers.

Other work

Other work

Other results also improve on the classical bounds for subsymmetric coverings.

Fisher (1940):
There do not exist subsymmetric coverings with empty excesses.

Bose and Connor (1952):
Certain subsymmetric coverings with 1-regular excesses do not exist.

Todorov (1989):
Some general bounds on subsymmetric coverings.

Bryant, Buchanan, Horsley, Maenhaut and Scharaschkin (2011):
Certain subsymmetric coverings with 2-regular excesses do not exist.

Various:
Exact covering numbers are known when

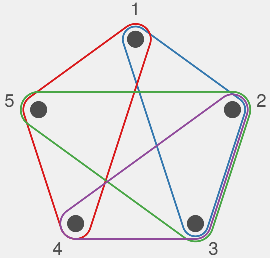
- $k \in \{3, 4\}$
- $\lambda = 1$ and $v \leqslant \frac{13}{4}k$.

Part 1

A simple new bound

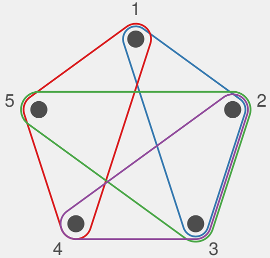
Excesses and incidence matrices

Excesses and incidence matrices



A $(5, 3, 1)$-covering with 4 blocks.

Excesses and incidence matrices



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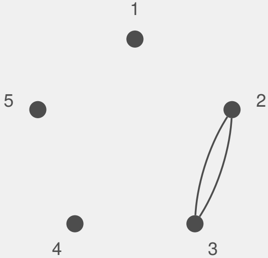
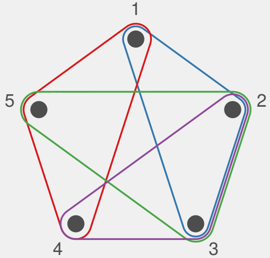


The *excess* of the covering.

Excesses and incidence matrices



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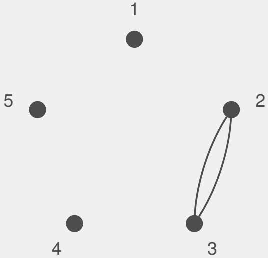
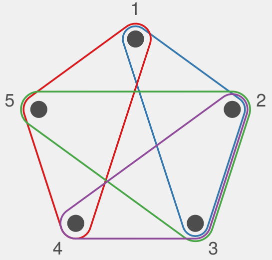


The *excess* of the covering.

$$\begin{array}{cc} & \text{4 blocks} \\ \text{5 points} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

The *incidence matrix* of the covering.

Excesses and incidence matrices



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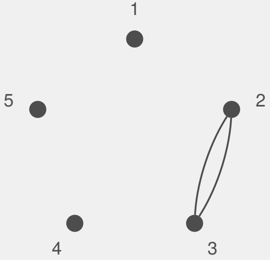

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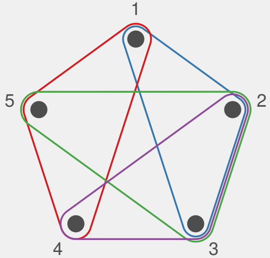

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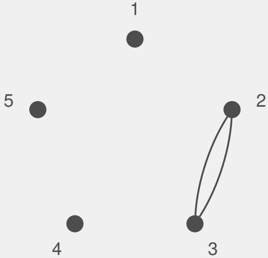

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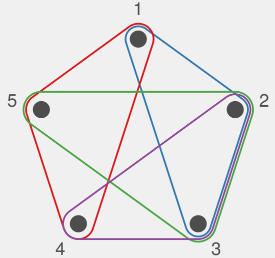
The *incidence matrix* of the covering.

$$AA^T = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 1 & 1 \\ 1 & 3 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Excesses and incidence matrices



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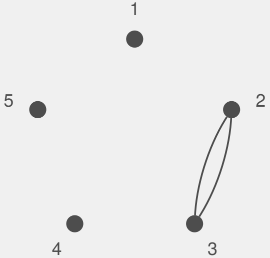

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$$z_{xx} = r_x,\ \ z_{xy} = \lambda + \mu_E(xy)$$

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- Let A be the 176×153 incidence matrix of a $(176, 15, 1)$-covering with exactly 153 blocks.
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- So $AA^T - J$ is 176×176, symmetric, and looks like

$$\begin{pmatrix} \geqslant 13 & & & & & & \\ & \ddots & & & & E & \\ & & \geqslant 13 & & & & \\ & & & 12 & & & \\ & E & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 12 \end{pmatrix} \quad \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \leqslant 7 \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} \geqslant 169 \text{ rows} \end{array}$$

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E & \boxed{\begin{matrix}12 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 12\end{matrix}}
\end{array}\right)
&
\begin{array}{l}
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We need $d < r - \lambda$ for this idea to work.

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- generalises Fisher's inequality.

Bounds for subsymmetric $(v, 10, 1)$-coverings

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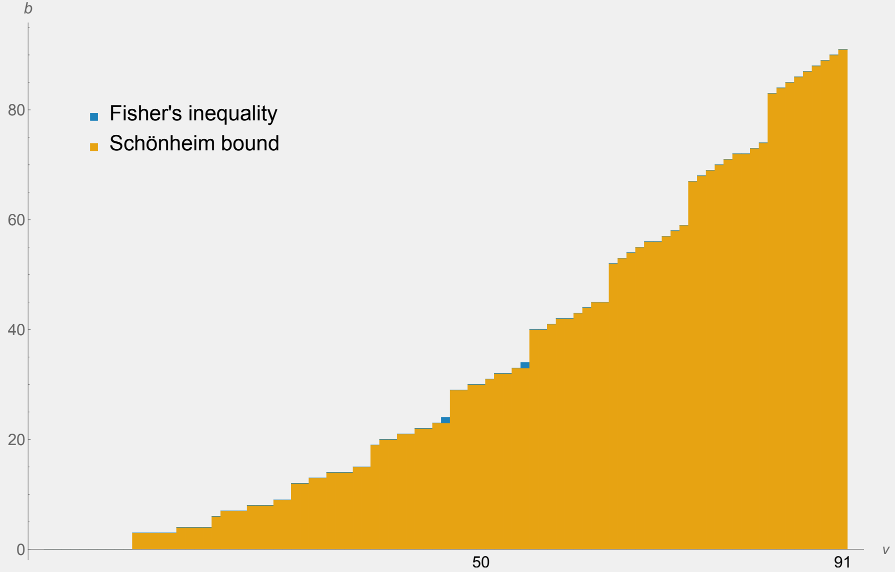

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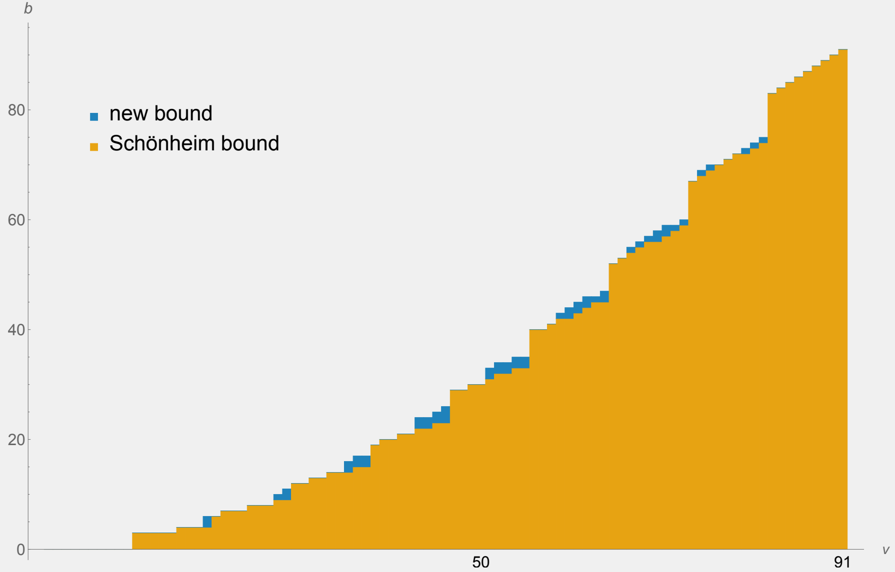

Bounds for subsymmetric $(v, 15, 1)$-coverings

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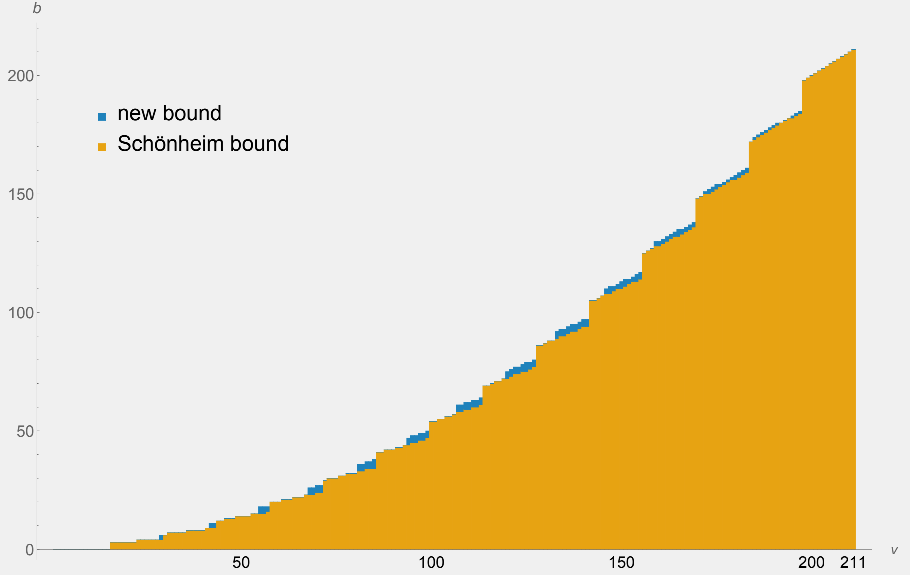

Part 2

Extending this idea

A new bound when $d \geqslant r - \lambda$

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- Let A be the incidence matrix of a $(79, 15, 1)$-covering with 32 blocks.
- Then $AA^T - J$ is 79×79, symmetric, and looks like

$$\left(\begin{array}{cccccc} \geqslant 6 & & & & & \\ & \ddots & & & E & \\ & & \geqslant 6 & & & \\ & & & 5 & & \\ & E & & & \ddots & \\ & & & & & 5 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \geqslant 20 \\ \vdots \\ \geqslant 20 \end{array}\right\} \leqslant 6 \text{ rows} \\ \left.\begin{array}{c} 6 \\ \vdots \\ \vdots \\ 6 \end{array}\right\} \geqslant 73 \text{ rows} \end{array}$$

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- If there is a 33×33 symmetric submatrix that is diagonally dominant, then we can obtain a contradiction as before.

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- Such a submatrix corresponds to a set of 33 vertices in the excess that induces a subgraph with maximum degree less than 5.

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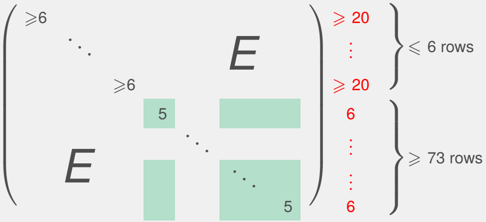

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- If there is a 33×33 symmetric submatrix that is diagonally dominant, then we can obtain a contradiction as before.
- Such a submatrix corresponds to a set of 33 vertices in the excess that induces a subgraph with maximum degree less than 5.
- A result of Caro and Tuza guarantees such a *5-independent set* in any multigraph with degree sequence $[20^6, 6^{73}]$.

Improving the $d < r - \lambda$ bound

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- Sometimes this same idea can improve our original $d < r - \lambda$ bound.

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- It can help to weight the columns of $AA^T - J$. For example:

$$
\begin{array}{c}
\left(\begin{array}{cccccc}
\geqslant 13 & & & & & \\
& \ddots & & & E & \\
& & \geqslant 13 & & & \\
& & & 12 & & \\
& E & & & \ddots & \\
& & & & & 12
\end{array}\right)
\\
\hphantom{\geqslant 13 \;\; \ddots \;\; \geqslant 13} \underbrace{\hphantom{12 \quad \ddots \quad 12}}_{\times\left(\frac{7}{12}+\epsilon\right)}
\end{array}
\begin{array}{c}
\left.\begin{array}{c} \geqslant 21 \\ \vdots \\ \geqslant 21 \end{array}\right\} \leqslant 22 \text{ rows} \\
\left.\begin{array}{c} 7 \\ \vdots \\ \vdots \\ 7 \end{array}\right\} \geqslant 154 \text{ rows}
\end{array}
$$

Improving the $d < r - \lambda$ bound

- Sometimes this same idea can improve our original $d < r - \lambda$ bound.
- It can help to weight the columns of $AA^T - J$. For example:

$$\left(\begin{array}{cccccc}
\geqslant 13 & & & & & \\
& \ddots & & & E & \\
& & \geqslant 13 & & & \\
& & & 12 & & \\
& E & & & \ddots & \\
& & & & & 12
\end{array}\right)
\begin{array}{c}
\geqslant 21 \\ \vdots \\ \geqslant 21 \\ 7 \\ \vdots \\ 7
\end{array}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \leqslant 22 \text{ rows} \\
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \geqslant 154 \text{ rows}
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$$\underbrace{\hphantom{12 \quad \ddots \quad 12}}_{\times\left(\frac{7}{12}+\epsilon\right)}$$

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 & \ddots & & & E & \\
 & & \geqslant 13 & & & \\
 & & & 12 & & \\
 & E & & & \ddots & \\
 & & & & & 12
\end{array}\right)
\begin{array}{c}
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\end{array}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} \leqslant 22 \text{ rows} \\
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\begin{matrix}\geqslant 13 & & \\ & \ddots & \\ & & \geqslant 13\end{matrix} & E \\
E & \underbrace{\begin{matrix}12 & & \\ & \ddots & \\ & & 12\end{matrix}}_{\times\left(\frac{7}{12}+\epsilon\right)}
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$$

- We then use an edge-weighted version of the excess.
- An easy extension of the Caro-Tuza result covers edge-weighted multigraphs.

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- For $d \geqslant r - \lambda$ we can find infinite families of improvements over the Schönheim bound.
- For $d < r - \lambda$ we can find infinite families of improvements over our simple bound.

Bounds for subsymmetric $(v, 15, 1)$-coverings

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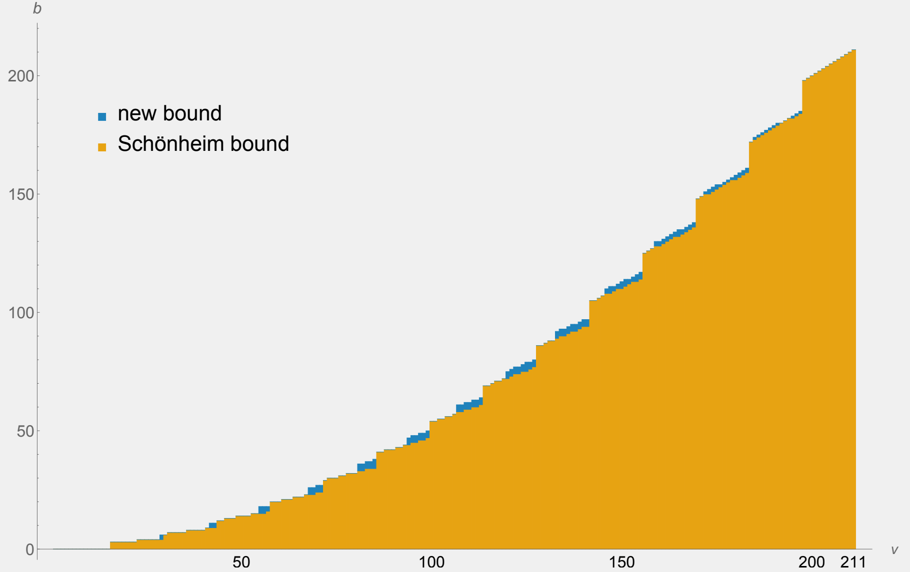

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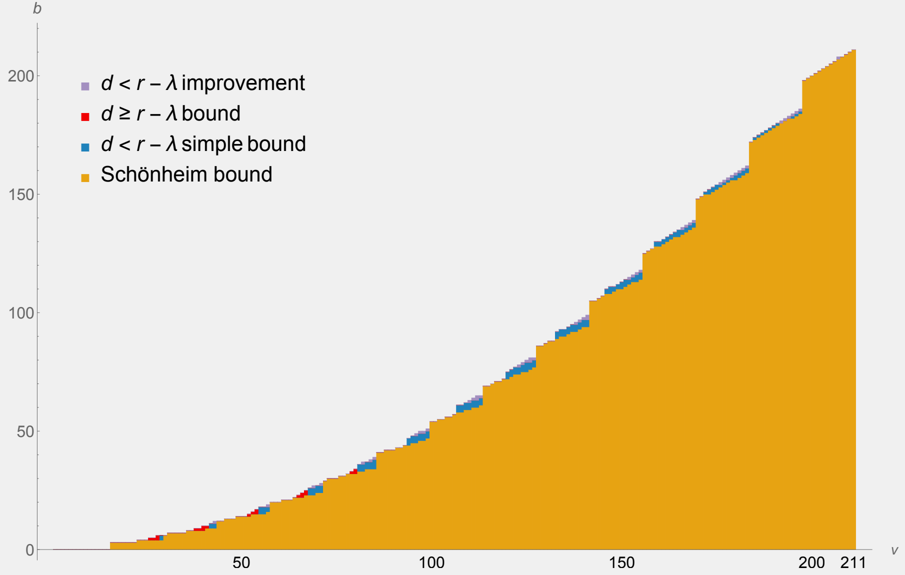
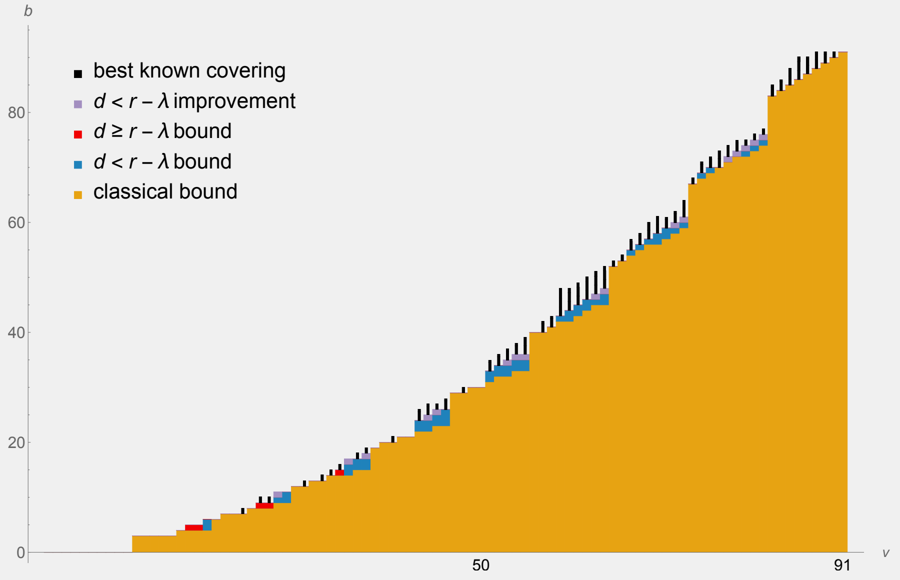


Part 3

Upper bounds and exact covering numbers

Bounds for subsymmetric $(v, 10, 1)$-coverings

Bounds for subsymmetric $(v, 10, 1)$-coverings



A construction for coverings

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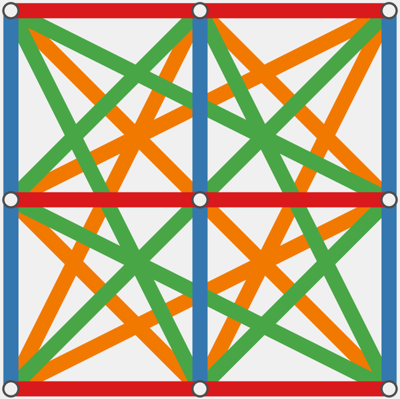
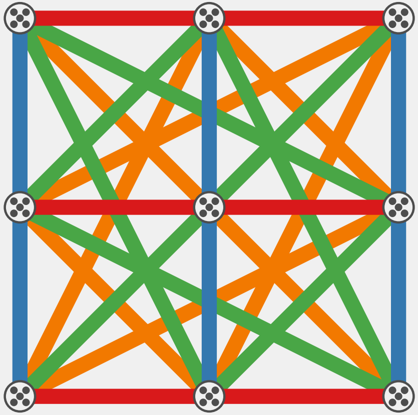


A construction for coverings



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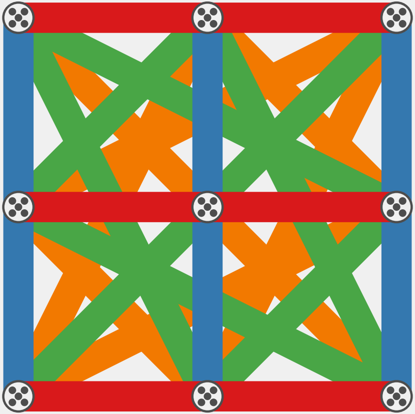
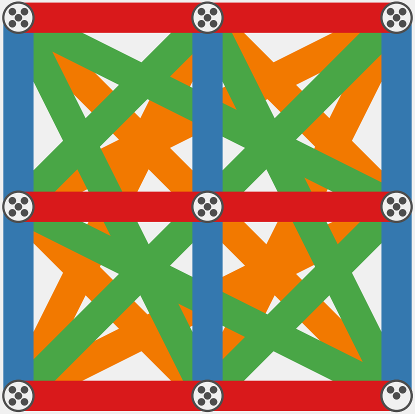


A construction for coverings



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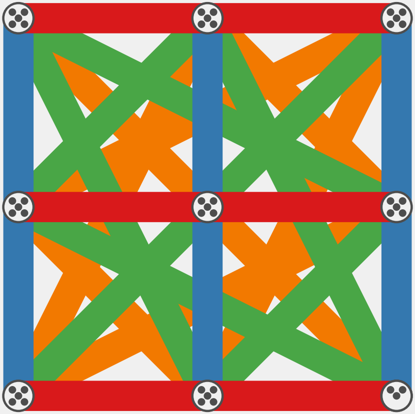

Coverings constructed like this sometimes meet our new bounds. We get new infinite families of covering numbers.

Bounds for $(v, 48, 1)$-coverings ($v \leqslant 250$)

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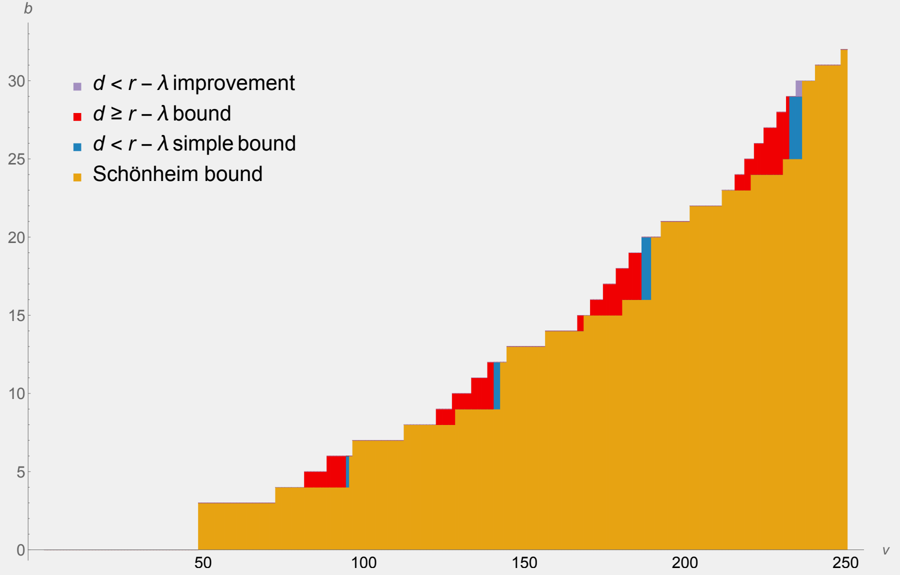
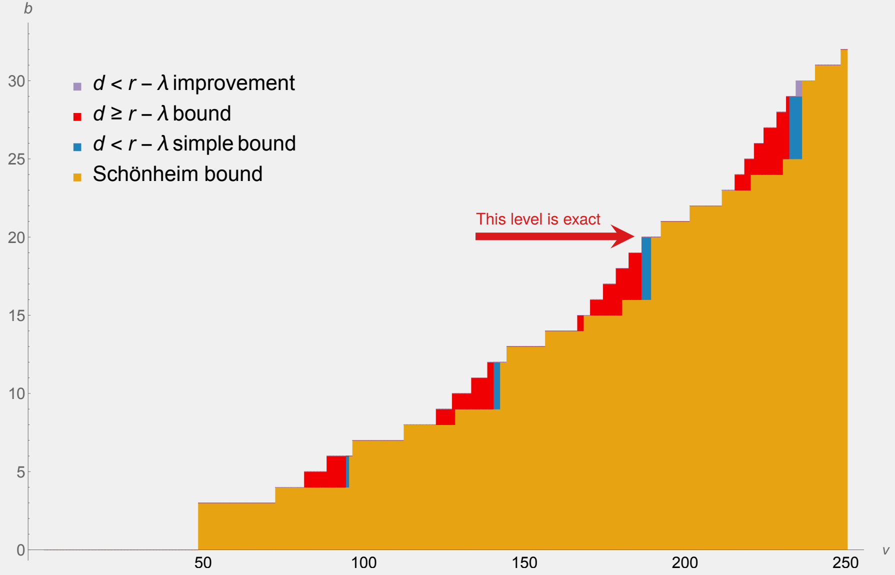


Bounds for $(v, 48, 1)$-coverings ($v \leqslant 250$)



Conclusion

Some final things

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For $\lambda = 1$ these results are weaker than the *second Johnson bound*.

They're still of interest for $\lambda \geqslant 2$, however.

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- Better results on m-independent sets in multigraphs translate immediately to improved bounds.
- With Francetić, Herke and Singh I'm working on a procedural bound for the size of an m-independent set and on special cases where $d = r - \lambda$.

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Symmetric coverings:

- I've looked at these with Bryant, Buchanan, Maenhaut and Scharaschkin and with Francetić and Herke.

Thanks.