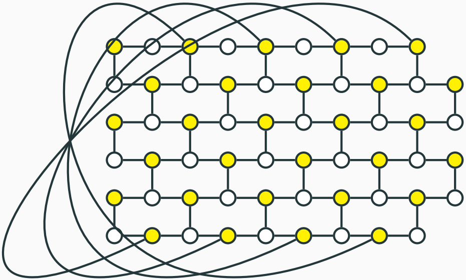
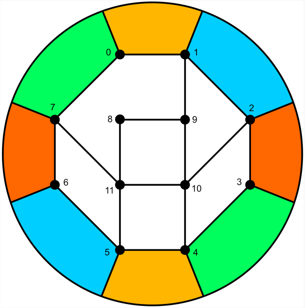
Stable Sets in Graphs with Bounded Odd Cycle Packing Number

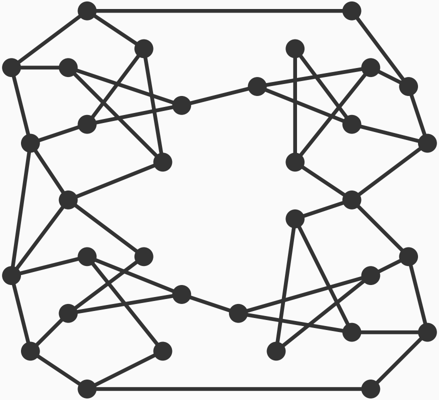




Tony Huynh (Monash) joint with

Michele Conforti, Samuel Fiorini, Gwenaël Joret, and Stefan Weltge

Maximum Weight Stable Set



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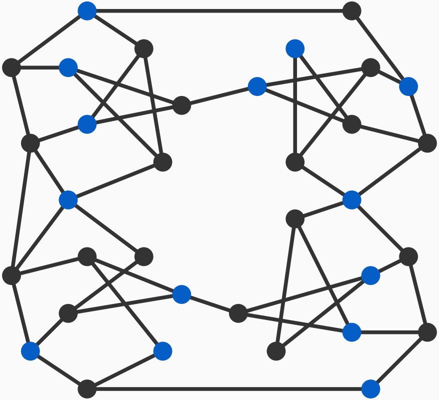

Maximum Weight Stable Set

Problem

Given a graph G and $w : V(G) \to \mathbb{R}_{\geqslant 0}$, compute a maximum weight stable set **(MWSS)** *of G.*

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Theorem

For every $\epsilon > 0$, it is NP*-hard to approximate maximum stable set within a factor of $n^{1-\epsilon}$.*

Theorem

MWSS *can be solved on bipartite graphs in polynomial time.*

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$$\begin{array}{ll} \max & \sum_{v \in V(G)} w(v)x_v \\ \text{s.t.} & x_u + x_v \leqslant 1 \quad \forall uv \in E(G) \\ & x \in \{0,1\}^{V(G)} \end{array} \quad \equiv \quad \begin{array}{ll} \max & \sum_{v \in V(G)} w(v)x_v \\ \text{s.t.} & Mx \leqslant \mathbf{1} \\ & x \in \{0,1\}^{V(G)} \end{array}$$

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If G is **bipartite**, then M is a **totally unimodular** matrix.

Conjecture

Fix $k \in \mathbb{N}$. Integer Linear Programming can be solved in strongly polynomial time when all subdeterminants of the constraint matrix are in $\{-k, \ldots, k\}$.

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True for $k = 2$. Bimodular Integer Programming can be solved in strongly polynomial time.

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Open for $k \geqslant 3$.

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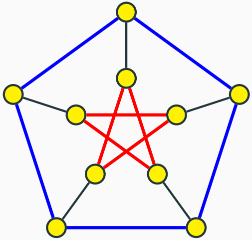

$$M = \begin{pmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

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MWSS *can be solved in polynomial time in graphs without two vertex-disjoint odd cycles.*

Conjecture

Fix $k \in \mathbb{N}$. **MWSS** *can be solved in polynomial time in graphs without k vertex-disjoint odd cycles.*

Polynomial Time Approximation Schemes

Theorem (Bock, Faenza, Moldenhauer, Ruiz-Vargas ’14)

For every **fixed** $k \in \mathbb{N}$, **MWSS** *on graphs with* $\mathrm{OCP}(G) \leqslant k$ *has a PTAS.*

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Theorem (Tazari '10)

For every **fixed** $k \in \mathbb{N}$, **MWSS** *and Minimum Vertex Cover on graphs with* $\mathrm{OCP}(G) \leqslant k$ *has a PTAS.*

Definition

A polytope $Q \subseteq \mathbb{R}^p$ is an *extension* of a polytope $P \subseteq \mathbb{R}^d$ if there exists an affine map $\pi : \mathbb{R}^p \to \mathbb{R}^d$ with $\pi(Q) = P$. The *extension complexity* of P, denoted $\mathrm{xc}(P)$, is the minimum number of facets of any extension of P.

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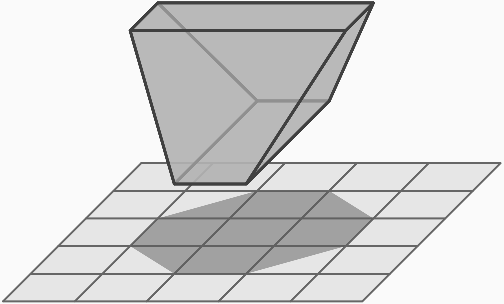

Theorem (Edmonds '71)

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Let $G = (V, E)$ be a graph. Then $x \in \mathbb{T}(G)$ if and only if

- $x \geqslant \mathbf{0}$,
- $x(E) = |V| - 1$,
- $x(E[U]) \leqslant |U| - 1$, for all non-empty $U \subseteq V$.

Spanning Tree Polytope

Theorem (Wong '80 and Martin '91)

For every connected graph $G = (V, E)$, $\mathrm{xc}(\mathbb{T}(G)) = O(|V| \cdot |E|)$.

Theorem (Fiorini, Massar, Pokutta, Tiwary, and de Wolf '12)

There is no extended formulation of $\mathbb{TSP}_n$ *of polynomial size.*

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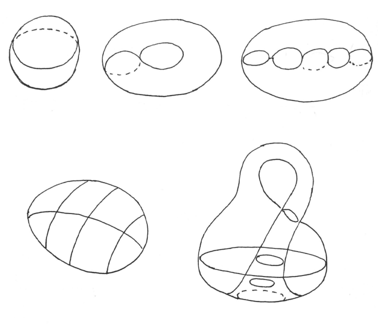
Theorem (Rothvoß '14)

The extension complexity of $\mathbb{M}(K_n)$ *is exponential in n.*

Classification of Surfaces:

- **orientable** $\cong$ sphere with h handles $= \mathbb{S}_h$
- **non-orientable** $\cong$ sphere with c cross-caps $= \mathbb{N}_c$



Euler genus:

- $g(\mathbb{S}_h) = 2h$
- $g(\mathbb{N}_c) = c$

Our Main Results

Theorem (Conforti, Fiorini, H, Weltge '19)

If $\mathrm{OCP}(G) \leqslant 1$ *then* $\mathrm{STAB}(G)$ *has a size-*$O(n^2)$ *extended formulation.*

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Theorem (Conforti, Fiorini, H, Joret, Weltge '19)

Fix $k, g \in \mathbb{N}$. *Then for every graph* G *with* $\mathrm{OCP}(G) \leqslant k$ *and* **Euler genus** $\leqslant g$, **MWSS** *can be solved in polynomial time and* $\mathrm{STAB}(G)$ *has a polynomial-size extended formulation.*

$\mathsf{OCP} = 1$ Graphs

Theorem (Lovász)

Let G be a 4-connected graph. Then $\mathsf{OCP}(G) \leqslant 1$ iff

- *$G - X$ is bipartite for some $X \subseteq V(G)$ with $|X| \leqslant 3$*
- *G has a* **nice embedding** *in the projective plane*

Nicely Embedded Graphs

Definition

Let G be a graph embedded in a surface $\mathbb{S}$. A cycle of G is 1-*sided* if it has a neighborhood that is a **Möbius strip**, and 2-*sided* if it has a neighborhood that is a **cylinder**.

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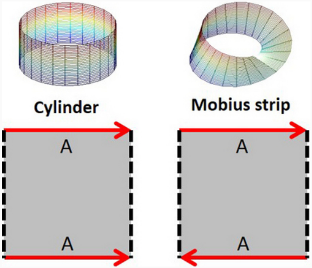

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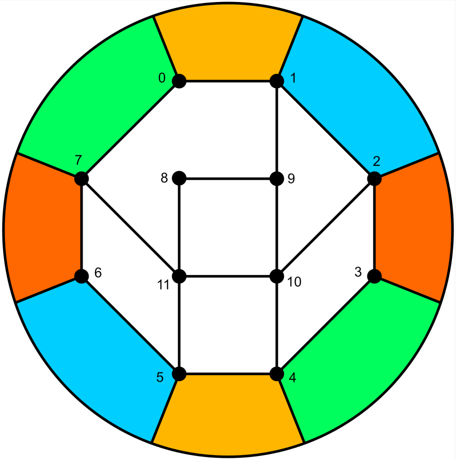
Lemma

If G is nicely embedded on a surface of Euler genus k, then $\mathrm{OCP}(G) \leqslant k$.

Nicely Embedded Graphs



Theorem (Erdős and Pósa, '65)

Every graph has one of the following:

- k *vertex-disjoint cycles;*
- *a feedback vertex set of size* $O(k \log k)$.

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The Erdős-Pósa Property does not hold for odd cycles.

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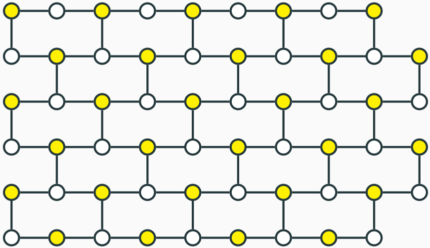

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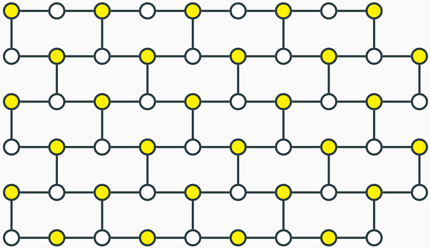

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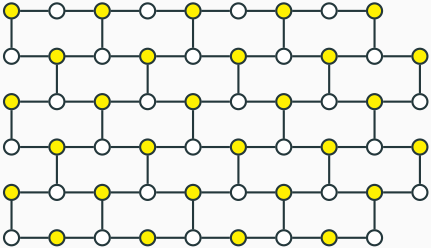

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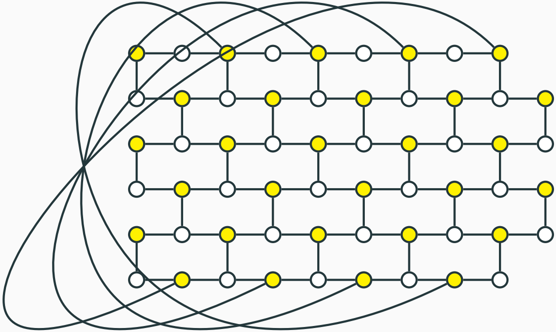

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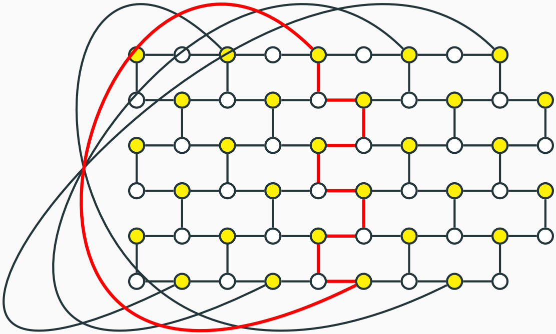

Theorem (CFHJW)

There exists a computable function $f(g, k)$ such that for all graphs G embedded in a surface with Euler genus g and with no $k + 1$ node-disjoint 2-sided odd cycles, there exists $X \subseteq V(G)$ with $|X| \leqslant f(g, k)$ such that $G - X$ does not contain a 2-sided odd cycle. Furthermore, there is such a set X of size at most $19^{g+1} \cdot k$ if the surface is orientable.

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Theorem (Kawarabayashi and Nakamoto '07)

Odd cycles satisfy the Erdős-Pósa property in graphs embedded in a fixed orientable surface

Dropping Nonnegativity Constraints

Let $P(G) = \text{conv}\{x \in \mathbb{Z}^{V(G)} \mid Mx \leq \mathbf{1}\}$.

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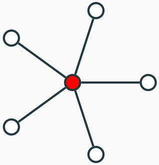
Theorem

For every graph G we have $\text{STAB}(G) = P(G) \cap [0,1]^{V(G)}$.

- **Node space:** $x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$



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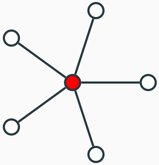
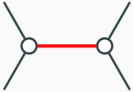


- **Slack space:** $y_{uv} = \begin{cases} 1 & \text{if } u, v \notin S \\ 0 & \text{otherwise} \end{cases}$

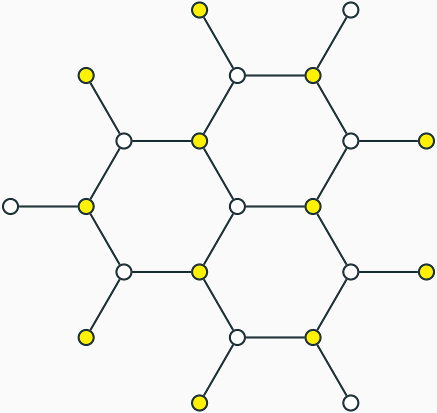


The Dual Graph

Theorem

If G is a nicely embedded, then the edges of the dual graph G^* can be oriented such in the local cyclic order of the edges incident to each dual node f, the edges alternatively leave and enter f. We call such an orietation alternating and denote it by D.

Slack Space



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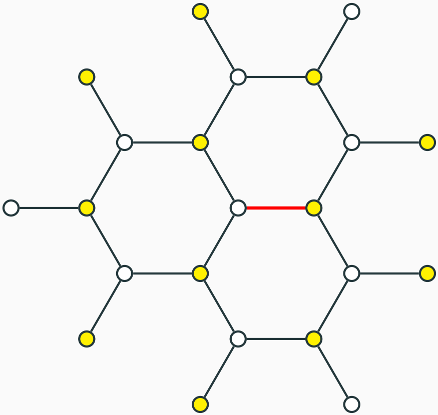

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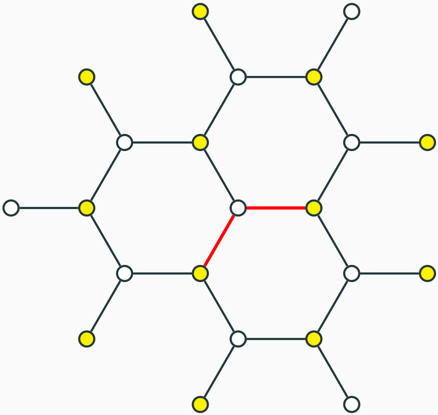

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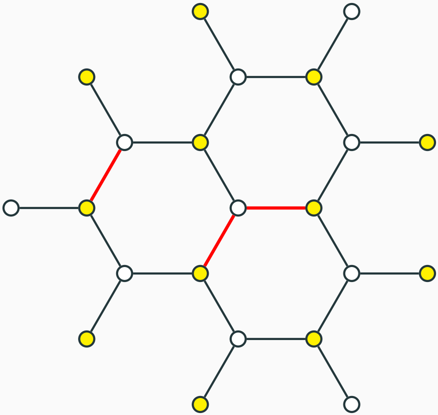

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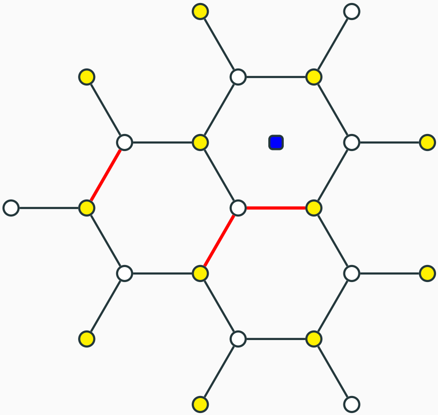

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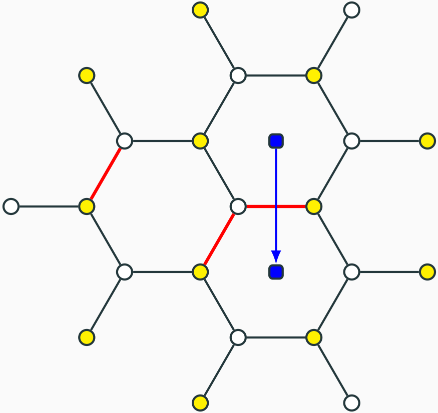

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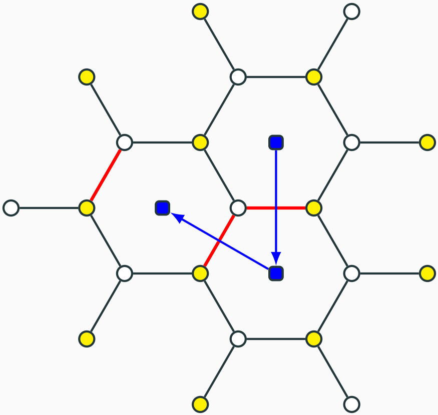

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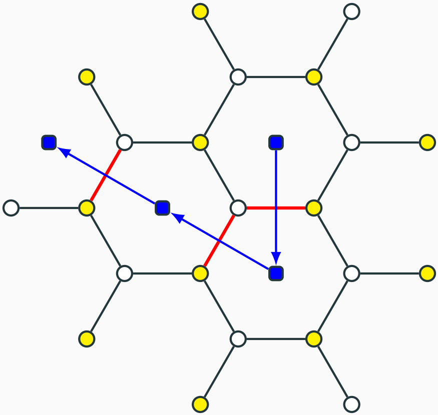

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- So $y_{uv} = 1 - x_u - x_v$ for all edges $uv \in E(G)$
- *Slack map* $\sigma : \mathbb{R}^{V(G)} \to \mathbb{R}^{E(G)} : x \mapsto y = \sigma(x) := \mathbf{1} - Mx$

Lemma

The image $\sigma(\mathbb{R}^{V(G)})$ of the slack map is the linear subspace of $\mathbb{R}^{E(G)}$ defined by

$$\sum_{i=1}^{2k}(-1)^{i-1}y_{e_i} = 0 \quad \forall \textit{even cycles } C = (e_1, e_2, \ldots, e_{2k})$$

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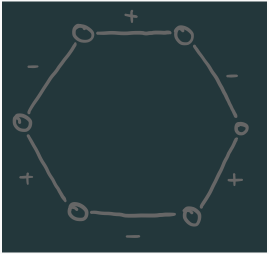

Observation

By **Euler's formula**, in $\mathbb{S} \cong \mathbb{N}_g$,

$$|E(G)| - |V(G)| = (|V(G^*)| - 1) + (g - 1)$$

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Observation

If G is nicely embedded in $\mathbb{S}$ then,

$$\sigma(\mathbb{R}^{V(G)}) = \{\text{circulations in } G^* \text{ subject to } g-1 \textbf{ extra} \text{ constraints}\}$$

Definition

Two integer circulations y, y' in G^* are *homologous* if $y - y'$ is a sum of **facial** circulations.

Homology

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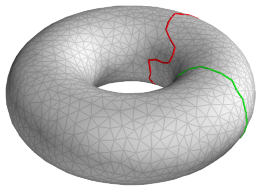

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Fact

$H_1(\mathbb{N}_g; \mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}^{g-1}$

$$\begin{aligned} \max \quad & \sum_{v \in V(G)} w(v) x_v \\ \text{s.t.} \quad & Mx \leqslant \mathbf{1} \\ & x \geqslant \mathbf{0} \\ & x \in \mathbb{Z}^{V(G)} \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \min \quad & \sum_{e \in E(G)} c(e) y_e \\ \text{s.t.} \quad & y \text{ circulation in } G^* \\ & [y] = (1, 0, \ldots, 0) \\ & y \geqslant \mathbf{0} \\ & y \in \mathbb{Z}^{E(G)} \end{aligned}$$

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where $c \in \mathbb{R}_+^{E(G)}$ is such that $c(\delta(v)) = w(v)$ for all $v \in V(G)$

Theorem (Chambers, Erickson, Nayyeri '10)

Given a graph G embedded on a surface of Euler genus g, a cost function $c : E(G) \to \mathbb{R}$, and a circulation $\theta : E(G) \to \mathbb{R}$, a minimum-cost circulation homologous to θ can be computed in time $g^{O(g)} n^{3/2}$.

Theorem (Chambers, Erickson, Nayyeri '10)

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Theorem (Malnič and Mohar '92)

Suppose G is embedded in a surface $\mathbb{S}$ with Euler genus $g \geqslant 1$. If $C_1, \ldots, C_\ell$ are vertex-disjoint directed cycles in G **whose homology classes are mutually distinct**, *then $\ell \leqslant 6g$.*

Theorem (Conforti, Fiorini, H, Weltge '19)

If $\mathrm{OCP}(G) \leqslant 1$ *then* $\mathrm{STAB}(G)$ *has a size-*$O(n^2)$ *extended formulation.*

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Thank you!