Analytic representations of large discrete structures

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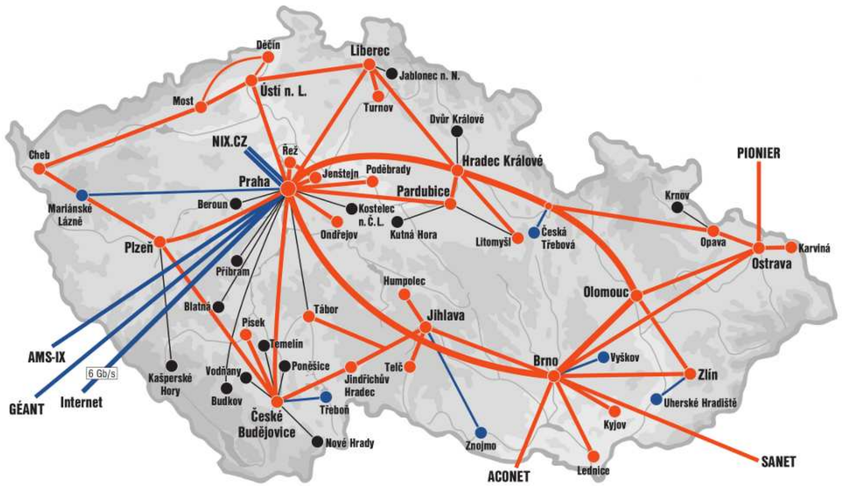
European Research Council
Established by the European Commission

8/8/2019

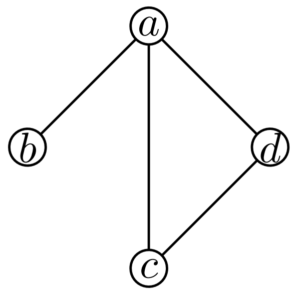
Networks

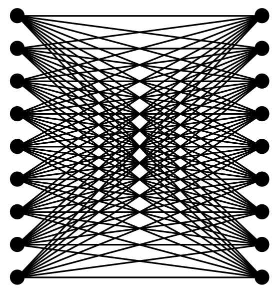
- nodes joined by links
- cities with roads
- public transportation
- internet connections
- facebook friendships



Graphs

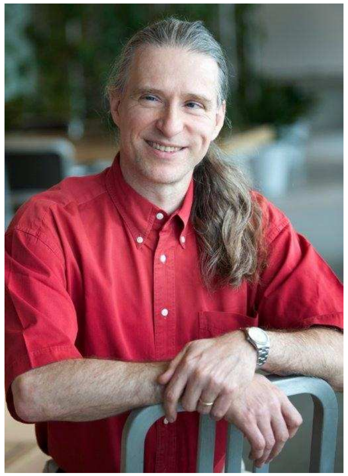
- graph is a pair (V, E)
  V are nodes (vertices) and $E \subseteq \binom{V}{2}$ are edges
- example: $V = \{a, b, c, d\}$ and $E = \{ab, ac, ad, cd\}$
- algorithmic graph theory
  shortest path, travelling salesperson problem, etc.





Large graphs

- The story starts in Microsoft Research around 2005...
- How can we represent and analyze large networks?



Borgs

Chayes

Lovász

What is the goal?

- representation of a large graph

  mathematical object "having" the same key parameters

- sampling

  generate a graph with the same key parameters

- robustness of the model

  the key parameters mutually interplay

  suitable to infer unobserved properties

Sparse vs. dense regime

- tools different depending on the density
- sparse graphs
  constant number of edges per node (on average)
  computer science – most real world networks
- dense graphs
  positive proportion of node pairs are edges
  mathematics – extremal graph theory

Sparse regime

- poor local vs. global interaction
- Benjamini-Schramm convergence
- right convergence
- partition convergence
- local-global convergence
- large deviation convergence

Dense regime

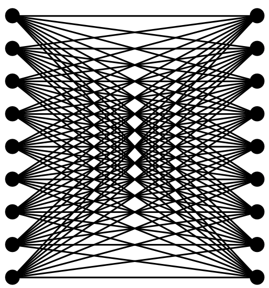
- Why should we care?

  extremal graph theory

- Mantel's Theorem (1907):

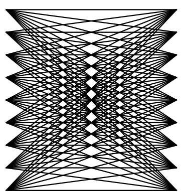
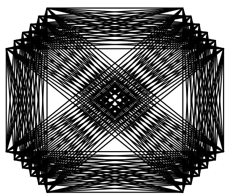
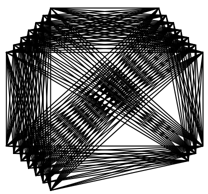
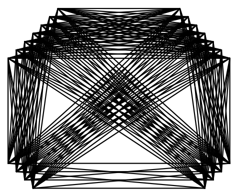
  maximum number of edges in a triangle-free graph

  important and difficult generalizations



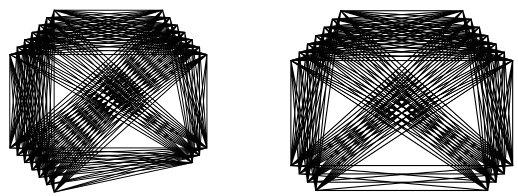
Edge vs. Triangle Problem

- Minimum density of K_3 for a specific edge-density
- determined by Razborov (2008), $K_{\alpha n,\dots,\alpha n,(1-k\alpha)n}$
- extensions by Nikiforov (2011) and Reiher (2016) for K_ℓ
- Pikhurko and Razborov (2017) gave extremal examples generally not unique, can be made unique by $\overline{K_{2,1}} = 0$

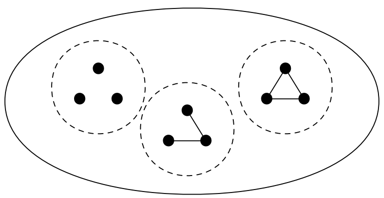
More general phenomenon?

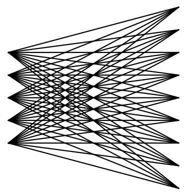
- Conjecture (Lovász 2008, Lovász and Szegedy 2011)
  Every finite feasible set $H_i = d_i$, $i = 1, \dots, k$,
  can be extended to a finite feasible set
  with an asymptotically unique structure.
- Theorem (Grzesik, K., Lovász Jr.): FALSE

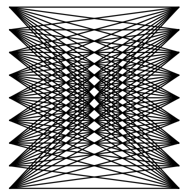


Convergent graph sequence

- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
- examples: K_n, $K_{\alpha n,n}$, blow ups $G[K_n]$ Erdős-Rényi random graphs $G_{n,p}$, planar graphs
- extendable to other discrete structures





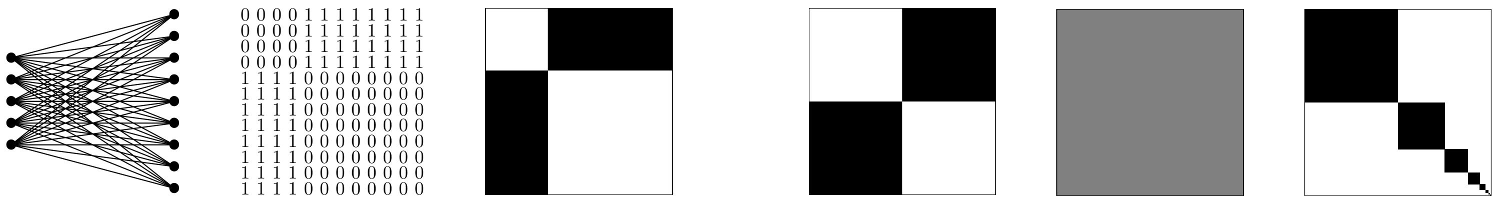


Regularity method

- developed by Endré Szemerédi
  Abel Prize 2012
- Regularity Decompositions
  every graph can be split into finitely many quasirandom pieces
- interplay between local and global
- property testing algorithms
- existence of arithmetic progressions in dense subsets of integers

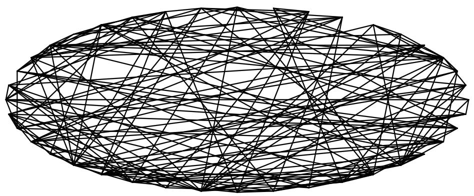
Graphons

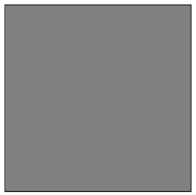
- a graph can be described by the adjacency matrix
  rows/columns ≈ nodes, 0/1 ≈ adjacencies
- take a “continuous” adjacency matrix
  regularity decompositions, martingale convergence
- (local) graph densities ⇒ global structure
- in general, all graph densities required



Quasirandom graphs

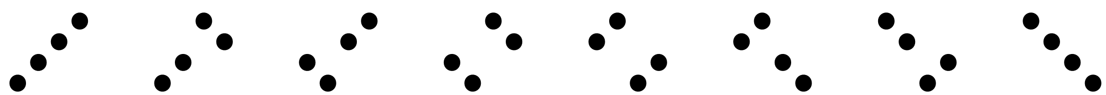
- Thomason, and Chung, Graham and Wilson (1980's)
  When does a graph look like a random graph?
  Erdős-Rényi model $G_{n,p}$: include each edge with prob. p
- a sequence G_i is quasirandom if $d(H, G_i) \approx d(H, G_{n,p})$
  $\Leftrightarrow$ G_i converges to the constant graphon W_p
  $\Leftrightarrow$ uniform edge density, cut sizes, spectral properties
  $\Leftrightarrow$ $h(K_2, G_i) \to p$ and $h(C_4, G_i) \to p^4$





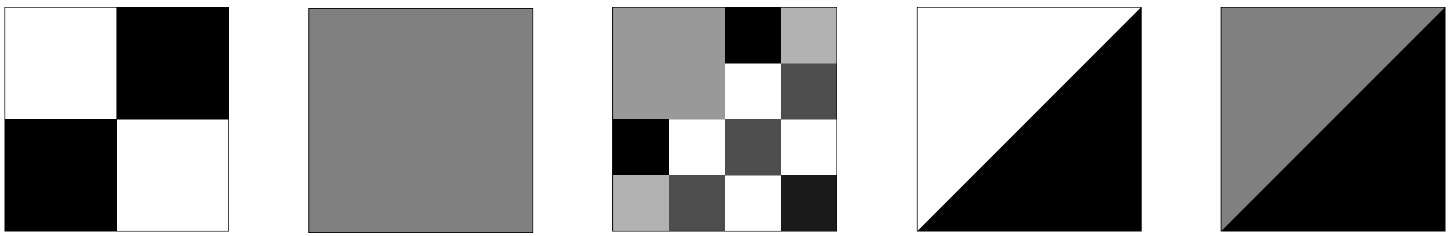
Quasirandom permutations

- When does a permutation look random?
- n-point permutation: ordering of numbers $1, \ldots, n$
  pattern: $4\underline{53}21\underline{6} \rightarrow 213$ $\quad$ $4\underline{53}2\underline{1}6 \rightarrow 321$
- Question (Graham): The same finite phenomenon?
  Theorem (K., Pikhurko, 2013): yes, 4-point patterns
- Theorem (Chan, K., Noel, Pehova, Sharifzadeh, Volec)
  quasirandom $\Leftrightarrow$ minimizer of pattern sum
  classification of all such sets of 4-point permutations



Finitely forcible graphons

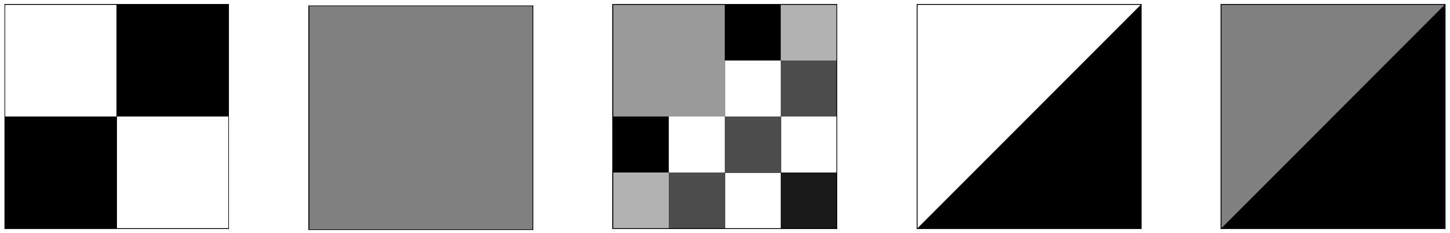
- finitely forcible graphon
  determined by densities of finitely many graphs
- edge density $= 1/2$ and triangle density $= 0$
  quasirandom graphon, step graphons
- every finitely forcible graphon is a unique solution of an extremal graph theory problem



Extremal combinatorics

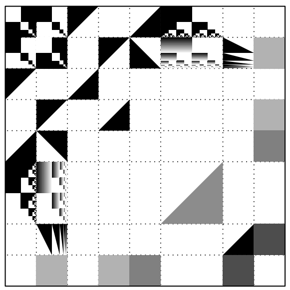
- Conjecture (Lovász and Szegedy, 2011):
  Every extremal graph theory problem
  has a finitely forcible optimal solution.

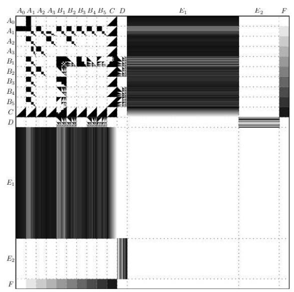
- extremal graph theory problem →
  finitely forcible optimal solution →
  "simple structure" gives new bounds on old problems

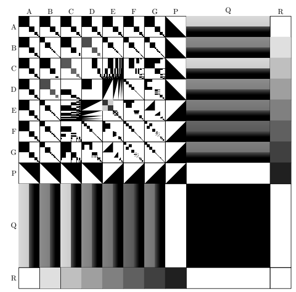


Complex finite graphons

- Conjectures (Lovász and Szegedy):
  The space $T(W)$ of a finitely forcible W is compact.
  The space $T(W)$ has finite dimension.
- disproved and stronger results in a series of papers coauthored by Cooper, Glebov, Kaiser, Klimošová, Noel and Volec

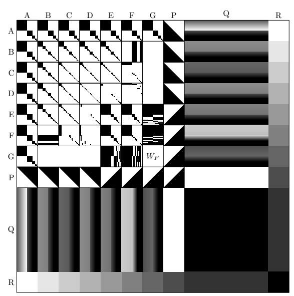


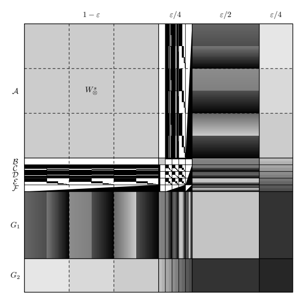




Universal constructions

- Theorem (Cooper, K., Martins)
  For every graphon W, there exists
  a finitely forcibly graphon W_0 such that
  $W_0(x/13, y/13) = W(x, y)$ for every $(x, y) \in [0, 1]^2$.
- Theorem (K., Lovász Jr., Noel, Sosnovec)
  $1/13$ can be replaced by $1 - \varepsilon$.

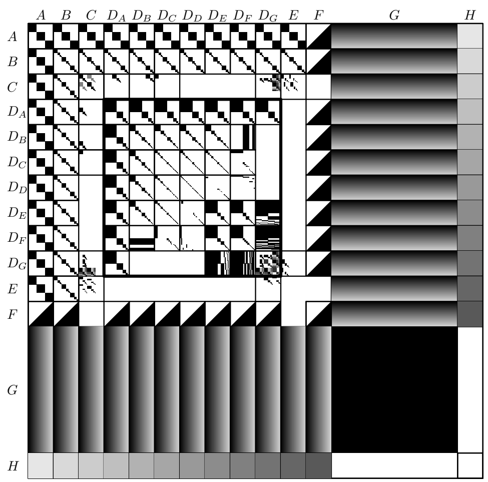




No finitely forcible optima

- Theorem (Grzesik, K., Lovász Jr.)
  There are extremal graph problems
  with no finitely forcible optimal solution.

- extensions to other parameters possible



- blackbox universal construction
- variable parts of graphon
- analysis of densities
- Implicit function theorem

What next?

- bridge dense and sparse settings
  Backhausz and Szegedy 2018

- sampling in the sparse setting
  Aldous-Lyons conjecture, soficity of groups

- relation between approaches in the sparse setting

Thank you for your attention!