# Transversals and Trades in Latin Squares.

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September 5, 2016

Transversals work with Ji Lijun - Suzhou University

# Introduction: Latin squares

## Definition

A *latin square* of order $n$ is an $n \times n$ array of cells filled with entries from $\{0, \ldots, n-1\}$ such that each row and each column contain each symbol precisely once.

| | | | |
|---|---|---|---|
| 2 | 1 | 3 | 0 |
| 3 | 2 | 0 | 1 |
| 1 | 0 | 2 | 3 |
| 0 | 3 | 1 | 2 |

# 36 officers problem

The thirty-six officers asks if it is possible to arrange six regiments consisting of six officers each of different ranks in a $6 \times 6$ square so that no rank or regiment will be repeated in any row or column.

# Mutually Orthogonal Latin Squares

## Definition

A pair of latin squares $A = [a_{ij}], B = [b_{ij}]$ of order $n$ are *orthogonal mates* if each of the $(a_{ij}, b_{ij})$ are distinct.

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 2 | 0 |
| 2 | 0 | 1 |

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 2 | 0 | 1 |
| 1 | 2 | 0 |

# Transversals

## Definition

A *transversal* of a latin square of order $n$, $L$, is a set of $n$ cells such that the set of cells contains a cell from each row of $L$, a cell from each column of $L$, and such that each symbol appears in precisely one cell of the transversal.

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | **3** | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | **6** | 0 |
| **2** | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | **4** | 5 | 6 | 0 | 1 | 2 |
| 4 | 5 | 6 | 0 | **1** | 2 | 3 |
| 5 | 6 | **0** | 1 | 2 | 3 | 4 |
| 6 | 0 | 1 | 2 | 3 | 4 | **5** |

# Transversals: MOLS

**Theorem**

*A latin square has an orthogonal mate if and only if it has a decomposition into disjoint transversals.*

# Transversals: MOLS

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 3 |
| 3 | 4 | 0 | 2 | 1 |
| 4 | 0 | 1 | 3 | 2 |
| 1 | 2 | 3 | 0 | 4 |
| 2 | 3 | 4 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |

# Transversals: MOLS

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 3 |
| 3 | 4 | 0 | 2 | 1 |
| 4 | 0 | 1 | 3 | 2 |
| 1 | 2 | 3 | 0 | 4 |
| 2 | 3 | 4 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| 0 | | | | |
| | | | | 0 |
| | | | 0 | |
| | 0 | | | |
| | | 0 | | |

# Transversals: MOLS

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 3 |
| 3 | 4 | 0 | 2 | 1 |
| 4 | 0 | 1 | 3 | 2 |
| 1 | 2 | 3 | 0 | 4 |
| 2 | 3 | 4 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| 0 | 1 | | | |
| | | | 1 | 0 |
| 1 | | | 0 | |
| | 0 | 1 | | |
| | | 0 | | 1 |

# Transversals: MOLS

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 3 |
| 3 | 4 | 0 | 2 | 1 |
| 4 | 0 | 1 | 3 | 2 |
| 1 | 2 | 3 | 0 | 4 |
| 2 | 3 | 4 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 3 |
| 2 | 3 | 4 | 1 | 0 |
| 1 | 2 | 3 | 0 | 4 |
| 4 | 0 | 1 | 3 | 2 |
| 3 | 4 | 0 | 2 | 1 |

# Transversals: Questions

- Q Minimum/maximum number of transversals in any latin square of a given order
- Q Largest possible partial transversal in a latin square

# Transversals: $B_n$

## Definition

A back circulant Latin square of order $n$, $B_n$, is the Cayley table of addition modulo $n$ with the borders removed.

$B_6 =$

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

# Transversals: Equivalences

A transversal of $B_n$ is equivalent to:

1. a diagonally cyclic latin square of order $n$;
2. a complete mapping of the cyclic group of order $n$;
3. an orthomorphism of the cyclic group of order $n$;
4. a magic juggling sequences of period $n$; and
5. a placements of $n$ non-attacking semi-queens on an $n \times n$ toroidal chessboard.

# Transversals: $B_n$

### Theorem

*(Donovan & Cooper, 1996) There exists a critical set of a latin square of size* $(n^2 - n)/2$.

### Theorem

*(Cavenagh & Wanless, 2009) If* $n$ *is a sufficiently large integer then there exists a latin square of order* $n$ *that has at least* $(3.246)^n$ *transversals.*

# Transversals

**Theorem**

*(Cavenagh & Wanless, 2009) For $n \neq 5$ an odd integer, there exists two transversals in $B_n$ of intersection size $t$, for $t \in \{0, \ldots, n\} \setminus \{n-2, n-1\}$*

# $\mu$-way Transversals

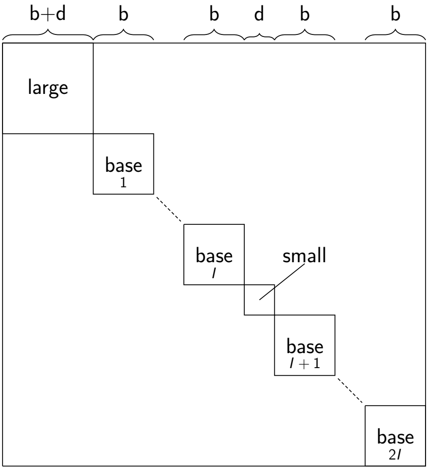
Consider a set of $\mu$ transversals of $B_n$, $T_1, \ldots, T_\mu$, such that there exists a set $S$ with $T_i \cap T_j = S$ for all $1 \leq i < j \leq \mu$.

The $\mu$-way transversal intersection spectrum for $B_n$ is the set of possible interesections sizes $|S|$ of $\mu$ transversals.

# Construction: Basic idea

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 |
| 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 |
| 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 |
| 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 |
| 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

# Construction: Shape



# Construction: Shape

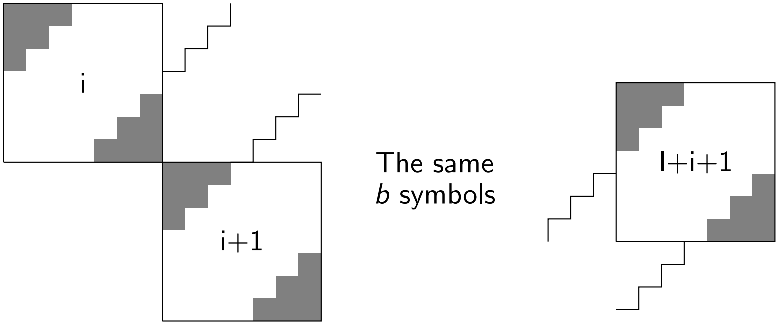



Figure: We choose partial transversal such that the symbols not used between the $i$th and $(i+1)$th base blocks are used in the $(l+i+1)$th base block.

# Construction: Base blocks

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | $4^*$ |
| 1 | 2 | $3^*$ | 4 | 5 |
| $2^*$ | 3 | 4 | 5 | 6 |
| 3 | 4 | 5 | $6^*$ | 7 |
| 4 | $5^*$ | 6 | 7 | 8 |

# Construction: Base blocks

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | $4^*$ |
| 1 | 2 | $3^*$ | 4 | 5 |
| $2^*$ | 3 | 4 | 5 | 6 |
| 3 | 4 | 5 | $6^*$ | 7 |
| 4 | $5^*$ | 6 | 7 | 8 |

# Construction: Other blocks

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4* | 5 |
| 1 | 2 | 3* | 4 | 5 | 6 |
| 2* | 3 | 4 | 5 | 6 | 7 |
| 3 | 4 | 5 | 6 | 7 | 8* |
| 4 | 5 | 6 | 7* | 8 | 9 |
| 5 | 6* | 7 | 8 | 9 | 10 |

| |
|---|
| 5* |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

# Results

For odd $n$, there are integers $d, d'$ with $0 \leq d < 9$ and $0 \leq d' < 11$ such that $n = 18l + 9 + 2d$ and $n = 22l' + 11 + 2d'$.

## Theorem (M.)

*For odd $n \geq 33$, there exists three transversals of $B_n$ of intersection size $t$ for $t \in \{min(d' + 3, d), \ldots, n\} \setminus \{n - 5, \ldots, n - 1\}$ except, perhaps, when:*

- $n = 51$ *and* $t = 29$,
- $n = 53$ *and* $t = 30$.

# Results

**Theorem (M.)**

*For odd $n \geq 33$, there exists four transversals of $B_n$ of intersection size $t$ for $t \in \{min(d'+3, d), \dots, n\} \setminus \{n-7, \dots, n-1\}$, except, perhaps, when:*

- $33 \leq n \leq 43$ *and* $t \in [d'+10, d'+11] \cup [2d'+18, 2d'+21]$,
- $45 \leq n \leq 53$ *and* $t \in [d'-1, d'+2] \cup [d'+10, d'+11] \cup [d'+18, d'+20]$,
- $63 \leq n \leq 75$ *and* $t \in [d+7, d+8]$.

# Partial latin squares

## Definition

A *partial latin square*, $L$, of order $n$ is an $n \times n$ array with cells either empty or filled with elements from the set $\{0, 1, \ldots, n-1\}$, such that each row and each column contains each element at most once.

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 3 | • | 2 |
| 0 | 2 | 1 | • | • |
| • | 4 | 2 | 1 | 3 |
| • | 1 | 0 | • | 4 |
| 4 | 3 | • | 0 | 1 |

The volume of a partial latin square is the number of filled cells.

# Latin trades

Example:

| | | | |
|---|---|---|---|
| 2 | 1 | 3 | 0 |
| 3 | 2 | 0 | 1 |
| 1 | 0 | 2 | 3 |
| 0 | 3 | 1 | 2 |

| | | | |
|---|---|---|---|
| | | | |
| 3 | | | 1 |
| 1 | | | 3 |
| | | | |

| | | | |
|---|---|---|---|
| | | | |
| 1 | | | 3 |
| 3 | | | 1 |
| | | | |

# Latin trades

Example:

| | | | |
|---|---|---|---|
| 0 | 2 | | 1 |
| 2 | 1 | 3 | |
| | 3 | 2 | 0 |
| 1 | | 0 | 3 |

| | | | |
|---|---|---|---|
| 2 | 1 | | 0 |
| 1 | 3 | 2 | |
| | 2 | 0 | 3 |
| 0 | | 3 | 1 |

# Latin trades

## Definition

A $\mu$-way latin trade of order $n$ and volume $s$ is a set of $\mu$ partial latin squares $L_1, \ldots, L_\mu$ of order $n$ and volume $s$ such that:

1. Each partial latin square occupy the same filled cells;
2. Any filled cell is filled differently in each of the partial latin squares; and
3. Each row and each column of the partial latin squares are set-wise the same.

# $k$-homogeneous partial latin squares

## Definition

A partial latin square is *$k$-homogeneous* if it has exactly $k$ filled cells in each row and in each column, and each element appears in the partial latin square exactly $k$ times.

| | | | | | |
|---|---|---|---|---|---|
| 2 | • | 1 | 5 | 3 | 4 |
| • | 3 | 0 | 4 | 5 | 2 |
| 1 | 4 | 5 | • | 0 | 3 |
| 4 | 5 | • | 1 | 2 | 0 |
| 5 | 0 | 3 | 2 | 1 | • |
| 3 | 2 | 4 | 0 | • | 1 |

# $\mu$-way $k$-homogeneous latin trades

## Definition

A $\mu$-way latin trade is *$k$-homogeneous* if any, and hence all, of the partial latin squares that define it are $k$-homogeneous.

| | | | | |
|---|---|---|---|---|
| 4 | 3 | 2 | 1 | |
| | 0 | 4 | 3 | 2 |
| 3 | | 1 | 0 | 4 |
| 0 | 4 | | 2 | 1 |
| 2 | 1 | 0 | | 3 |

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 4 | 2 | |
| | 4 | 2 | 0 | 3 |
| 4 | | 0 | 3 | 1 |
| 2 | 0 | | 1 | 4 |
| 0 | 3 | 1 | | 2 |

| | | | | |
|---|---|---|---|---|
| 2 | 4 | 1 | 3 | |
| | 3 | 0 | 2 | 4 |
| 0 | | 4 | 1 | 3 |
| 4 | 1 | | 0 | 2 |
| 3 | 0 | 2 | | 1 |

# Spectrum of $\mu$-way latin trades

**Theorem**

*There exists 2 latin squares of order $n$ that intersect in $s$ cells, for $s \in \{0, \dots, n^2 - 6\} \cup \{n^2 - 4, n^2\}$ and $n \geq 4$. (Fu 1980)*

**Theorem**

*There exists 3 latin squares of order $n$ that all intersect in the same $s$ cells, for $s \in \{0, \dots, n^2 - 15\} \cup \{n^2 - 12, n^2 - 9, n^2\}$ and $n \geq 8$. (Adams-Billington-Bryant 2002)*

# Spectrum of $\mu$-way latin trades

**Theorem**

*There exists a 2-way latin trade of order n and volume $s \in \{6, \ldots, n^2\} \cup \{0, 4\}$, for $n \geq 4$. (Fu 1980)*

**Theorem**

*There exists a 3-way latin trade of order n and volume $s \in \{15, \ldots, n^2\} \cup \{0, 9, 12\}$, for $n \geq 8$. (Adams-Billington-Bryant 2002)*

# Spectrum of 2-way $k$-homogeneous latin trades

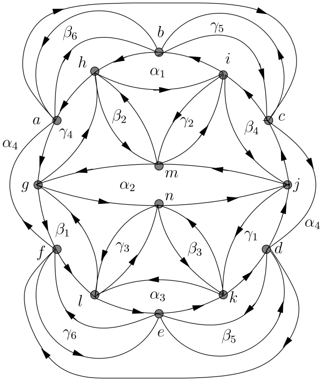
### Theorem

*There exists a 2-way 3-homogeneous latin trade of order $3m$, for $m \geq 3$. (Cavenagh-Donovan-Drapal 2003)*

### Theorem

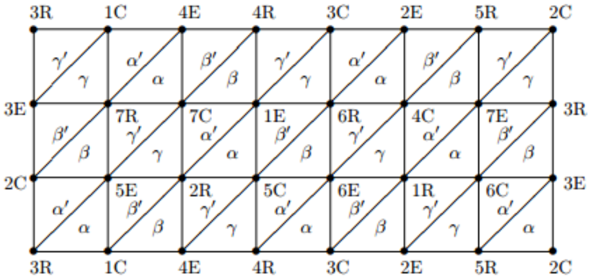
*There exists a 2-way 4-homogeneous latin trade of order $4m$, for $m \geq 3$. (Cavenagh-Donovan-Drapal 2003)*

# Latin trades and geometry




A. Drápal. *Geometrical structure and construction of latin trades.* ADV GEOM., 9(3):311-348, 2009.

# Latin trades and geometry




Grannell M.J., Griggs T.S., Knor M., *Biembeddings of symmetric configurations and 3-homogeneous latin trades.* Comment. Math. Univ. Carolin. 49:411420, 2008.

# Spectrum of 2-way $k$-homogeneous latin trades

**Theorem**

*There exists a 2-way $k$-homogeneous latin trade of order $n$, for $3 \leq k \leq 8$ and $n \geq k$. (Bean-Bidkhori-Khosravi-Mahmoodian 2005)*

**Theorem**

*There exists a 2-way $k$-homogeneous latin trade of order $n$, for $3 \leq k \leq 37$ and $n \geq k$. (BehroozBagheri-Mahmoodian 2011)*

# Spectrum of 2-way $k$-homogeneous latin trades

**Theorem**

*(Cavenagh & Wanless, 2009) There exists a 2-way $k$-homogeneous latin trade of order $n$, for all $k \geq 3$ and $n \geq k$.*

# Spectrum of 3-way $k$-homogeneous latin trades

## Theorem

*There exists a 3-way $k$-homogeneous latin trades of order $n$, for $n \geq k$ when:*

- $k = 3$ *and* $3 \mid n$;
- $k = 4$ *and* $n \neq 6, 7, 11$;
- $5 \leq k \leq 13$;
- $k = 15$;
- $k \geq 4$ *and* $n \geq k^2$;
- $5 \mid n$, *except possibly for* $n = 30$; *and*
- $7 \mid n$, *except for* $k = 4$ *and* $n = 7$, *and possibly for* $n = 42$;

*(BagheriGh-Donovan-Mahmoodian 2012)*

# Application: $\mu$-way $k$-homogeneous Latin trades

**Theorem**

*A set of $\mu$ transversals of $B_n$ with intersection size $t$ defines a $\mu$-way $(n-t)$-homogeneous Latin trade of order $n$*

# Construction from large sets of idempotent latin squares

**Theorem**

*There exists a $\mu$-way $(n-1)$-homogeneous latin trade of order $n$ for each $1 \leq \mu \leq n-2$.*

# Packing construction

### Theorem

*Suppose there exists a $\mu$-way latin trade of volume $s$ and of order $\lambda$. For every $n = \lambda(\lambda + a) + b$, where $0 < b < \lambda$, $a \geq b + 1$, and $gcd(n, \lambda) = 1$, there exists a $\mu$-way $s$-homogeneous latin trade of order $n$.*

# Packing construction

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 3 | | 1 |
| 2 | 3 | |

| | | |
|---|---|---|
| 2 | 3 | 1 |
| 1 | | 3 |
| 3 | 2 | |

This is a 2-way latin trade with order $\lambda = 3$ and volume $s = 8$.

Take $b = 1$ and $a = 2$, and so our construction will yield a 2-way 8-homogeneous latin trade of order $n$, where
$n = \lambda(\lambda + a) + b = 16$.

# Packing construction

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 8 | 12 | | | 1 | | 9 | | | 2 | 6 | | | | |
| 12 | | 4 | | | 13 | 1 | | | | | 15 | 3 | 7 | | |
| 8 | 12 | | | | | 10 | 14 | 2 | | | 7 | | 15 | | |
| | 5 | 9 | 13 | | | 2 | | 10 | | | 3 | 7 | | | |
| | 13 | | 5 | | | 14 | 2 | | | | | 16 | 4 | 8 | |
| | 9 | 13 | | | | | 11 | 15 | 3 | | | 8 | | 16 | |
| | | 6 | 10 | 14 | | | 3 | | 11 | | | 4 | 8 | | |
| | | 14 | | 6 | | | 15 | 3 | | | | | 1 | 5 | 9 |
| | | 10 | 14 | | | | | 12 | 16 | 4 | | | 9 | | 1 |
| | | | 7 | 11 | 15 | | | 4 | | 12 | | | 5 | 9 | |
| 10 | | | 15 | | 7 | | | 16 | 4 | | | | | 2 | 6 |
| 2 | | | 11 | 15 | | | | | 13 | 1 | 5 | | | 10 | |
| | | | | 8 | 12 | 16 | | | 5 | | 13 | | | 6 | 10 |
| 7 | 11 | | | 16 | | 8 | | | 1 | 5 | | | | | 3 |
| | 3 | | | 12 | 16 | | | | | 14 | 2 | 6 | | | 11 |
| 11 | | | | | 9 | 13 | 1 | | | 6 | | 14 | | | 7 |

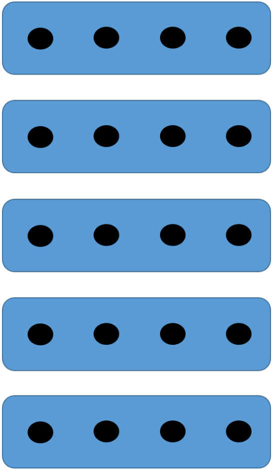
# Packing construction

**Theorem**

*For $\lambda \geq 3$, there exists a 3-way $k$-homogeneous latin trade of order $n$ for $n = \lambda(\lambda + a) + b$, where $0 < b < \lambda$, $\gcd(\lambda, b) = 1$, and $a \geq b + 1$, and:*

- $k \in \{0, 9\}$, *for* $\lambda = 3$;
- $k \in \{0, 9, 12, 15, 16\}$, *for* $\lambda = 4$;
- $k \in \{0, 9, 12, 15, 16\}$ *or* $18 \leq k \leq 25$, *for* $\lambda = 5$;
- $k \in \{0, 9, 12\}$ *or* $15 \leq k \leq \lambda^2$, *for* $\lambda \geq 6$.

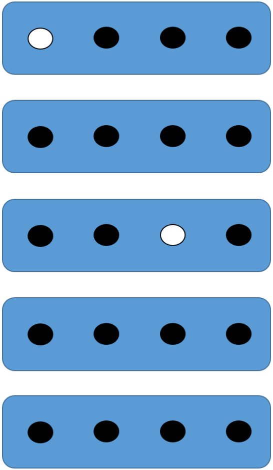
# Construction via RPBs



Transversal Design $TD(\alpha, n)$

- $n$ points in each group
- $\alpha$ groups

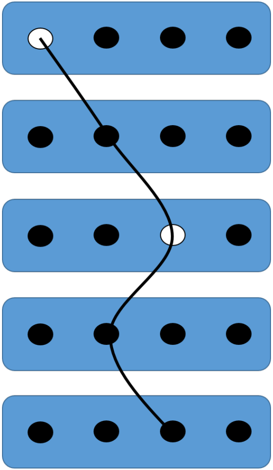
# Construction via RPBs



Transversal Design $TD(\alpha, n)$

- $n$ points in each group
- $\alpha$ groups

# Construction via RPBs



Transversal Design $TD(\alpha, n)$

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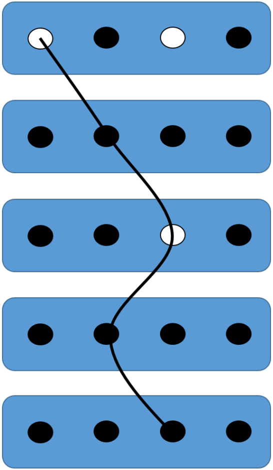
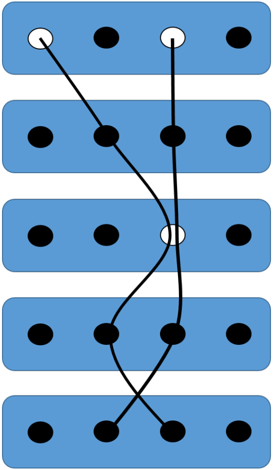
# Construction via RPBs



Transversal Design $TD(\alpha, n)$

- $n$ points in each group
- $\alpha$ groups

# Construction via RPBs



Transversal Design $TD(\alpha, n)$

- $n$ points in each group
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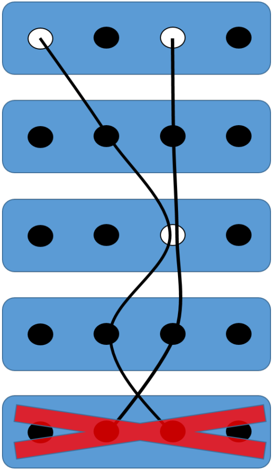
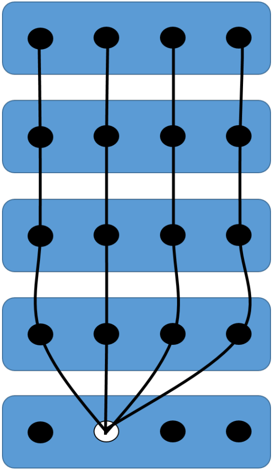
# Construction via RPBs



Transversal Design $TD(\alpha, n)$

- $n$ points in each group
- $\alpha$ groups

# Construction via RPBs

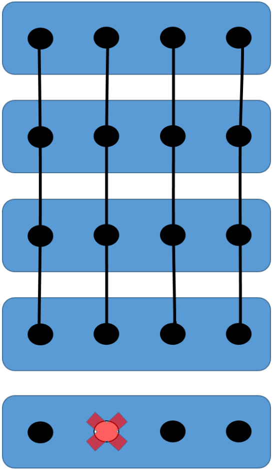


Resolvable Transversal Design
$RTD(\alpha - 1, n)$

- $n$ points in each group
- $\alpha - 1$ groups

Obtainable from a $TD(\alpha, n)$

# Construction via RPBs

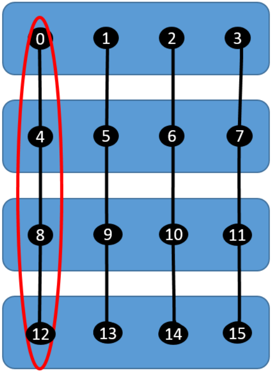


Resolvable Transversal Design
$RTD(\alpha - 1, n)$

- $n$ points in each group
- $\alpha - 1$ groups

Obtainable from a $TD(\alpha, n)$

# Construction via RPBs




$B \in R_i$

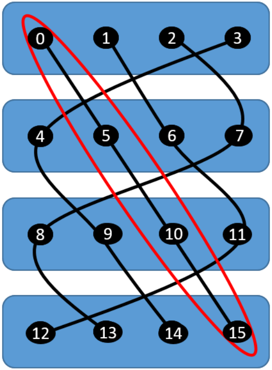
$(\alpha - 1)$-homogeneous latin trade of order $\alpha$, with the diagonal empty.

| | | | |
|---|---|---|---|
| • | 2 | 3 | 1 |
| 3 | • | 0 | 2 |
| 1 | 3 | • | 0 |
| 2 | 0 | 1 | • |

$B = \{0, 4, 8, 12\}$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | 8 | | | | 12 | | | | 4 | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| 12 | | | | | | | | 0 | | | | 8 | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| 4 | | | | 12 | | | | | | | | 0 | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| 8 | | | | 0 | | | | 4 | | | | | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |

# Construction via RPBs




$C \in R_j$

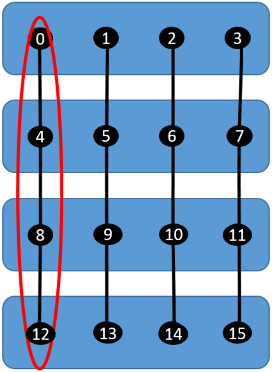
$(\alpha - 2)$-homogeneous latin trade of order $\alpha$, with the diagonal empty.

| | | | |
|---|---|---|---|
| • | • | 0 | 1 |
| • | • | 1 | 0 |
| 2 | 3 | • | • |
| 3 | 2 | • | • |

$C = \{0, 5, 10, 15\}$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X | | | | 8 | | | | 12 | | 0 | | 4 | | | 5 |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| 12 | | | | | | | | 0 | | | | 8 | | | |
| | | | | | | | | | | 5 | | | | | 01 |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| 4 | | | | 12 | | | | | | | | 0 | | | |
| | | | | | | | | | | | | | | | |
| 10 | | | | | 15 | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| 8 | | | | 0 | | | | 4 | | | | | | | |
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |
| 15 | | | | | 10 | | | | | | | | | | |

# Construction via RPBs




$B \in R_i$

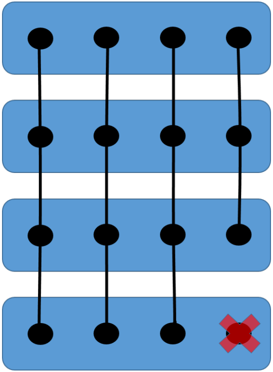
$(\alpha - 1)$-homogeneous latin trade of order $\alpha$, with the diagonal empty.

| | | | |
|---|---|---|---|
| • | 2 | 3 | 1 |
| 3 | • | 0 | 2 |
| 1 | 3 | • | 0 |
| 2 | 0 | 1 | • |

| | | | |
|---|---|---|---|
| • | 3 | 1 | 2 |
| 2 | • | 3 | 0 |
| 3 | 0 | • | 1 |
| 1 | 2 | 0 | • |

$B = \{0, 4, 8, 12\}$

# Construction via RPBs



$B \in R_i$

$(|B| - d_i)$-homogeneous latin trade of order $|B|$, with the diagonal empty.

# Construction via RPBs

**Lemma**

*If there exists 3-way $k$-homogeneous latin trades of order $n$ for $4 \leq k \leq n$ and $2^{r-2} - 4 \leq n \leq 2^r$, then there exists 3-way $k$-homogeneous latin trades of order $n$ for $4 \leq k \leq n$ and $2^{2r-2} \leq n \leq 2^{2r}$.*

# Results

**Theorem**

*There exists a 3-way $k$-homogeneous latin trade of order $n$ for $4 \leq k \leq n$ except, perhaps, for those values in the following Table:*

| $n$ | $k$ such that existence is unknown or known not to exist |
|---|---|
| 6 | 4 |
| 7 | 4 |
| 11 | 4 |
| 22 | $17, 19$ |
| 26 | $17, 19, 21, 23$ |
| 34 | $19, 21, 23, 25, 27, 29, 31$ |
| 37 | $33, 35$ |
| 38 | $21, 23, 25, 27, 29, 31, 33, 35$ |
| 41 | $35 \leq k \leq 39$ |
| 43 | $36 \leq k \leq 41$ |
| 46 | $27 \leq k \leq 41$ such that $k$ is odd |
| 58 | $31 \leq k \leq 55$ such that $k$ is odd |
| 59 | $55 \leq k \leq 57$ |
| 62 | $33 \leq k \leq 49$ such that $k$ is odd |
| 74 | $39 \leq k \leq 63$ such that $k$ is odd |
| 82 | $43 \leq k \leq 65$ such that $k$ is odd |
| 86 | $51 \leq k \leq 67$ such that $k$ is odd |
| 94 | $51, 53$ |

# References

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2. B. Bagheri Gh, D. Donovan, and E.S. Mahmoodian. On the existence of 3-way k-homogeneous latin trades. Discrete Mathematics, 312(24):3473-3481, 2012.
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