The Cheeger constant for distance-regular graphs

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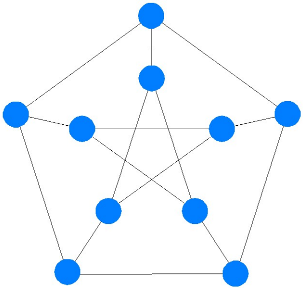
August 9, 2019

Graphs

A *graph* is a set of vertices V (can be taken to be $\{1, 2, \ldots, n\}$) and edges E, where each edge is an element of $V \times V$. We assume all graphs in this talk are *simple*, which means that $(a, a) \notin E$ and E has no repeated elements, and *undirected*, which means that (a, b) and (b, a) represent the same edge.

Distance-transitive graphs

Let $d(x, z)$ denote the length of the shortest path between x and z within G. A graph is *distance-transitive* if, whenever there are points X_1, z_1, x_2, z_2 such that $d(x_1, z_1) = d(x_2, z_2)$, there is an automorphism γ of G such that $\gamma(x_1) = x_2$, $\gamma(z_1) = z_2$. An automorphism is a bijection from the vertex set of G to itself, with the property that $\gamma(u) \sim \gamma(v)$ if, and only if, $u \sim v$.



Distance-regular graphs

A graph is *distance-regular* if, for any points x and z within G, the sizes of the following sets depends only on $d(x,z)$:

$$B = \{v|d(x,v) = d(x,z) - 1\} \bigcap \{v|d(z,v) = 1\}$$
$$A = \{v|d(x,v) = d(x,z)\} \bigcap \{v|d(z,v) = 1\}$$
$$C = \{v|d(x,v) = d(x,z) + 1\} \bigcap \{v|d(z,v) = 1\}$$

If a graph is distance transitive, it is distance regular. If $d(x,z) = i$, we define b_i, a_i, and c_i to be the sizes of the three sets above.

Distance-regular graphs - Examples

Hamming graphs: Let d, q be positive integers. The vertex set is elements of $\{1, 2, \ldots, q\}^d$. Two vertices are adjacent if they differ in exactly one component. This family of graphs includes the hypercubes.

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The Hamming and Johnson graphs are important in coding theory. In particular Hamming graphs arise naturally in the study of error-correcting codes.

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Grassman Graphs: Let V be a vector space of dimension n over the finite field F with q elements (q a prime power). The vertices are the subspaces of dimension t (over F) of V and such two vertices are adjacent if their intersection is a vector space of dimension $t-1$. Grassman graphs are relevant to certain questions in quantum physics.

Cheeger constant

The *Cheeger constant* h_G of a graph G is a prominent measure of the connectivity of G, and is defined as

$$h_G = \inf \left\{ \frac{E[S, S^c]}{\text{vol}(S)} | S \subset V(G) \text{ with } |S| \leq \frac{|V(G)|}{2} \right\}, \quad (1)$$

where $V(G)$ is the vertex set of G, vol(S) is the sum of the valencies of the vertices in S, S^c is the complement of S in $V(G)$, $|S|$ is the number of vertices in S, and for any sets A, B we use $E[A, B]$ to denote the number of edges in G which connect a point in A with a point in B.

Graph eigenvalues

For a distance-regular graph G of diameter D, we will write $k = \theta_0 > \theta_1 > \ldots > \theta_D$ to describe the eigenvalues of the adjacency matrix A of G, and refer to $\theta_0, \ldots, \theta_D$ as simply the *eigenvalues of G*.

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The connection between eigenvalues and the Cheeger constant

Theorem

Let λ_1 be the smallest positive eigenvalue of the Laplacian matrix of G. Then

$$\frac{\lambda_1}{2} \leq h_G \leq \sqrt{\lambda_1(2 - \lambda_1)}. \tag{2}$$

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The lower bound can be proved by eigenvalue interlacing.

Our conjecture

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Suppose G is a distance-regular graph, and λ_1 is the smallest positive eigenvalue of the Laplacian matrix of G. Then

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$$\frac{\lambda_1}{2} \leq h_G \leq \lambda_1. \tag{3}$$

We do not have a complete proof of this, but we have proved it in a number of cases.

The following are the major known families of infinite families of distance-regular graphs.

- Johnson graphs.
- Hamming graphs.
- Doob graphs.
- Halved *n*-cubes.
- Folded *n*-cubes.
- Folded halved 2*n*-cubes.
- Odd graphs.
- Doubled odd graphs.
- Grassmann graphs.
- Twisted Grassmann graphs.
- Doubled Grassman graphs.
- Bilinear forms graphs.
- Alternating forms graphs.
- Hermitian forms graphs.
- Quadratic forms graph
- Dual polar graphs and Hemmeter graphs.
- Half dual polar and Ustimenko graphs.

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We have proved the conjecture for all of these, except for the doubled Grassman graphs with $q = 2$ or 3.

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Lemma: If G is a strongly regular graph, then $h_G \leq \max(\frac{b_1}{k+1}, \frac{c_2}{k})$.

Graphs with small valency

The distance-regular graphs with valencies 3 and 4 are completely classified, and we have verified the conjecture for all of them except the following two graphs:

- The flag graph of $GH(2,2)$, with intersection array $\{4,2,2,2,2,2;1,1,1,1,1,2\}$. Here $\lambda_1 = \frac{3-\sqrt{6}}{4} \approx .138$.
- The incidence graph of $GH(3,3)$, with intersection array $\{4,3,3,3,3,3;1,1,1,1,1,4\}$. Here $\lambda_1 = \frac{1}{4}$.

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The question remains open for these.

Other classes

We also proved the conjecture for several other classes of distance-regular graphs. These are

- Diameter 3 and bipartite or antipodal.
- Incidence graphs of generalized quadrangles $GQ(q, q)$ with $q \neq 3, 4$.
- Incidence graphs of generalized hexagons $GH(q, q)$ with $q \neq 3, 4, 5, 7, 8, 9$.

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