Random Walks and Electric Resistance on Distance-Regular Graphs

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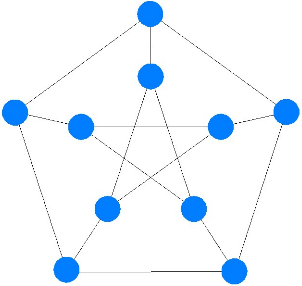
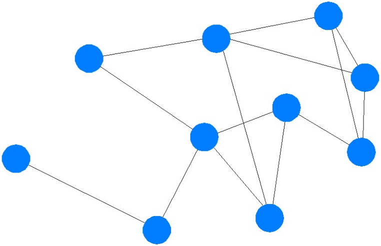
Graphs

A *graph* is a set of vertices V (can be taken to be $\{1, 2, \dots, n\}$) and edges E, where each edge is an element of $V \times V$. We assume all graphs in this talk are *simple*, which means that $(a, a) \notin E$ and E has no repeated elements, and *undirected*, which means that (a, b) and (b, a) represent the same edge.



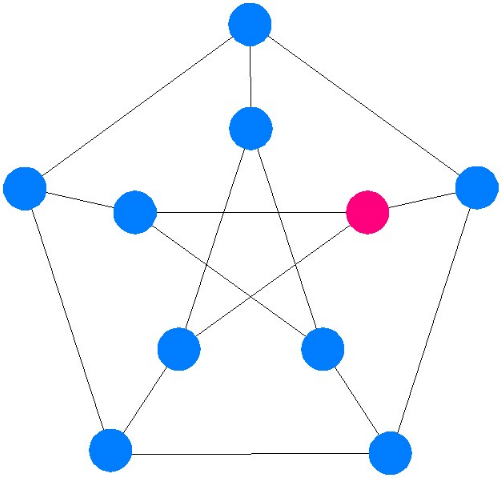


Random Walks

The *degree* of a vertex of a graph is the number of edges containing that vertex. A random walk is a process in which a walker moves on the vertices of a graph, at each stage moving to the adjacent vertices with probability $1/d$, where d is the degree of the current vertex. Formally, a random walk is a random process X_n with independent increments on the vertices of the graph, with conditional probabilities

$$P(X_{n+1} = b | X_n = a) = \begin{cases} \frac{1}{\deg(a)} & \text{if } a \sim b \\ 0 & \text{if } a \nsim b\,. \end{cases} \tag{1}$$

Example:



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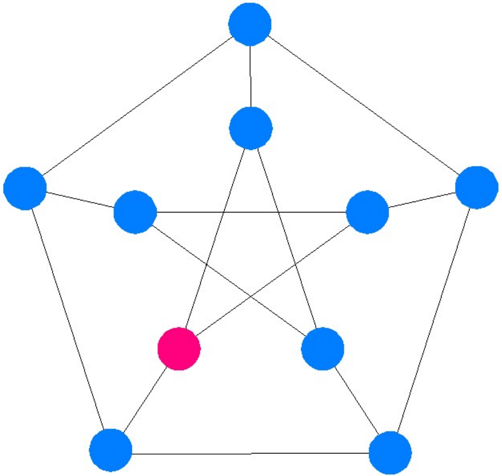

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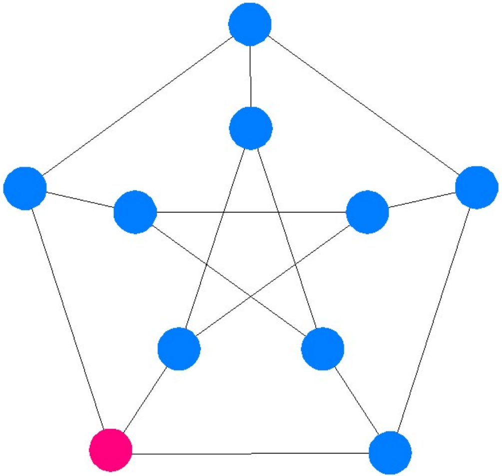

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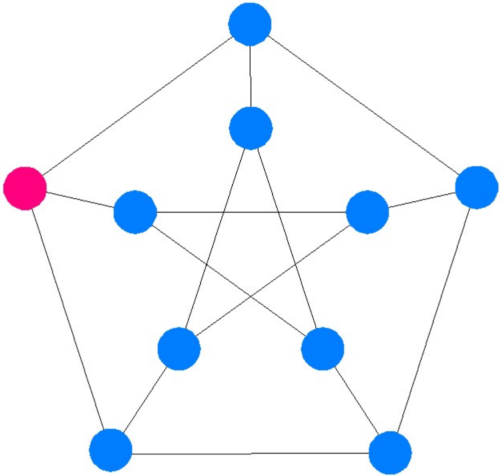

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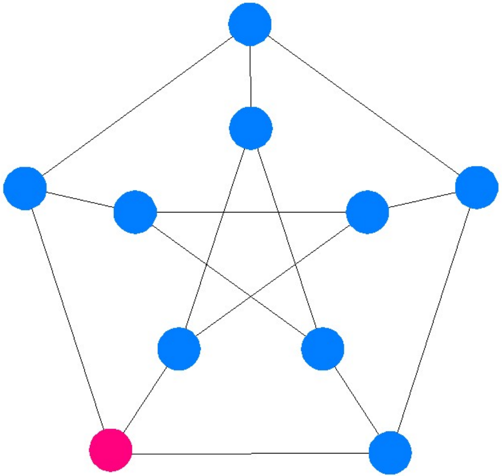

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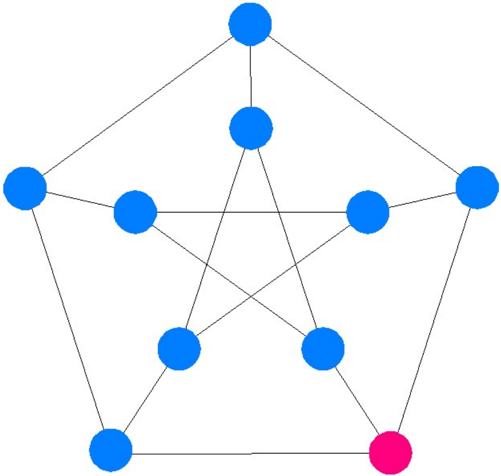
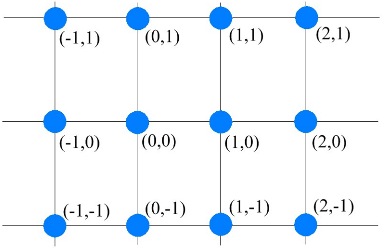


Some interesting questions regarding random walks

1. On finite graphs:
   - Let V_1, V_0 be subsets of the vertex set V. Starting from point a, what is the probability that we hit set V_1 before set V_0?
   - What is the expected amount of time until the entire graph is covered?
   - Does the random walk approach some stable distribution as we let it go forever?
2. On infinite graphs:
   - Starting from a point a, is there a nonzero probability that the random walk will never return?
   - If the walk must return, what is the expected return time?
   - What is the expected distance from the origin at any time?

Integer Lattice

We will consider the integer lattice in n dimensions. This is the infinite graph whose vertices are the set of elements of $\mathbb{Z}^n$, with the edge set defined by $\{a_1, \ldots, a_n\} \sim \{b_1, \ldots, b_n\}$ if there exists j such that $|a_j - b_j| = 1$ and $a_i = b_i$ for $i \neq j$. For example, we have the 2-dimensional lattice:



Recurrence vs. transience on $\mathbb{Z}^n$

Probably the most fundamental question regarding random walk on $\mathbb{Z}^n$ is the question of *recurrence*: Must a random walk on $\mathbb{Z}^n$ return to its starting point with probability 1?

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We will prove this fact using the concept of electric resistance.

Electric resistance on a graph

Suppose that a graph is taken to represent an electric circuit, where each edge has unit resistance. We imagine that we attach one pole of a battery to a vertex z_0, and the other pole to another vertex z_1, so that z_0 is at voltage 0 and z_1 is at voltage 1. All other points z receive a voltage $V(z)$, which can be calculated using Ohm's Law and Kirchoff's Current Law.

Ohm's Law: Voltage is equal to current times resistance.

$$V = IR \tag{2}$$

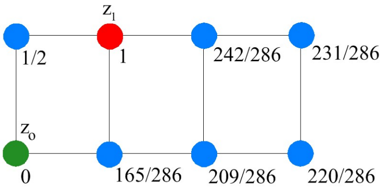
Kirchoff's Current Law: The sum of the currents entering and leaving any point other than v_0 and v_1 is 0.

Harmonic functions on a graph

As a consequence of Ohm's and Kirchoff's Laws, the voltage function on G is *harmonic*. That is, for $v \neq v_0, v_1$, We have

$$V(z) = \frac{1}{\deg(z)} \sum_{x \sim z} V(x) \tag{3}$$

The following is an example:



Connection with random walks

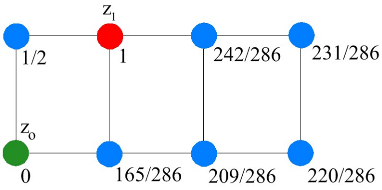
Let $g(z) = P_z(v_1 \text{before} v_0)$ denote the probability that a random walk, started at z, strikes v_1 before hitting v_0. Random walk has no memory, so

$$P_z(v_1 \text{ before } v_0) = \frac{1}{d} P_{x_1}(z_1 \text{ before } z_0) + \ldots + \frac{1}{d} P_{x_d}(z_1 \text{ before } z_0) \tag{4}$$

where $x_1, \ldots, x_d$ are the points adjacent to z. This is the same definition as before, so $g(z)$ is a harmonic function as well on $G - \{z_0, z_1\}$. Given boundary values and a finite graph, there is exactly one possible harmonic function, so we see that $g(z)$ is equal to $V(z)$.

Effective resistance

Due to Ohm's Law, $V = IR$, the amount of current flowing between two adjacent vertices is given by the difference in voltage. We can therefore measure the amount of current flowing from z_0 to z_1 by summing the voltages of vertices adjacent to z_0. The reciprocal of the amount of current flowing is called the *effective resistance*, and is a metric on the graph. In the example below, the effective resistance between z_0 and z_1 is $286/308$.

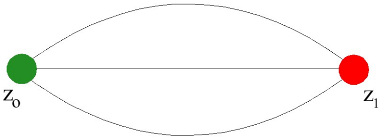


Intuition I

Resistances in series add. The effective resistance between z_0 and z_1 is 3.

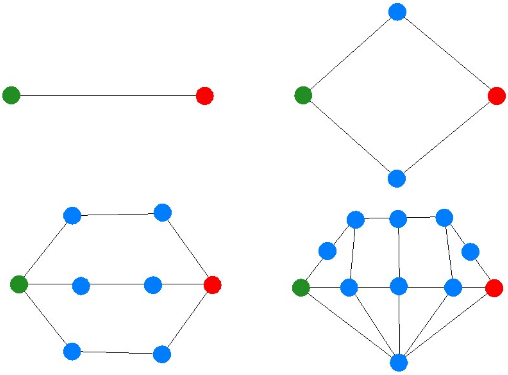
Intuition II

Resistances in parallel satisfy $\frac{1}{R} = \sum_i \frac{1}{R_i}$. The effective resistance between z_0 and z_1 is 1/3.



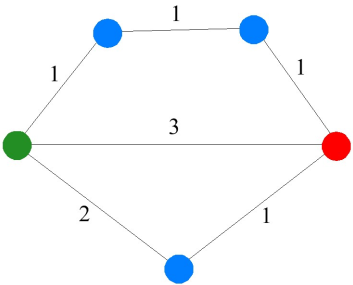
Intuition III

The following graphs all have effective resistances of 1 between the red and green vertices.



Variable resistances

We may also place resistances other than 1 on each edge, to create a different problem. The same rules apply, so that the resistance between the red and green vertices below is 1.



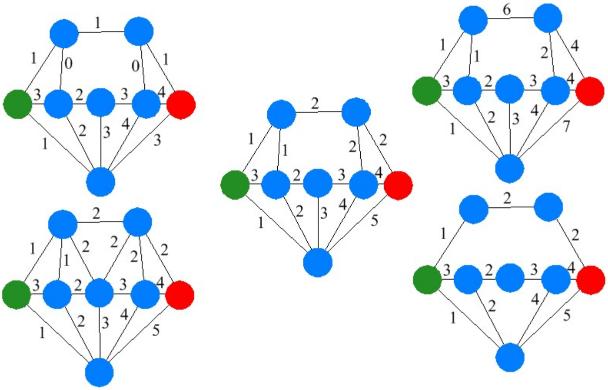
Rayleigh's monotonicity law

Rayleigh's monotonicity law: If the resistances of one or more edges in a graph are increased, the resistance in the new graph between any pair of points must be at least the resistance between the pair in the old graph.

Of course, this also implies the reverse, that if the resistances of one or more edges in a graph are decreased, the resistance in the new graph between any pair of points must be at most the resistance between the pair in the old graph.

Rayleigh’s monotonicity law

Suppose the middle graph below is our original graph. Then the resistance between the red and green vertices is less in the two graphs on the left and more in the two graphs on the right.



Recurrence vs. transience on infinite graphs.

In light of what has come before, we can determine whether a random walk is recurrent or transient by choosing a set F_n of vertices far from the origin which separate the origin from infinity. We can then find the harmonic function $g_n(x)$ which is 0 at the origin and 1 on F_n. The probability that a random walk will return to the origin before hitting F_n will then be given by

$$\frac{1}{\deg O} \sum_{x \sim O} g_n(x) \tag{5}$$

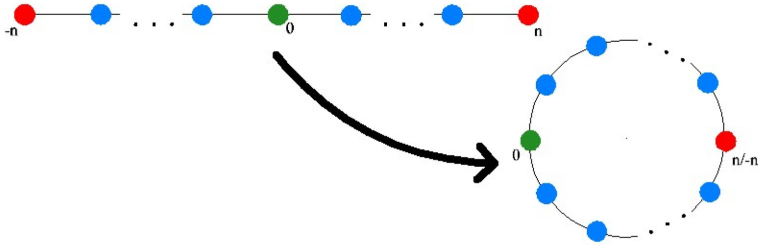
By Ohm's law this is equal to

$$\frac{1}{(\deg O)(\text{resistance between O and } F_n)} \tag{6}$$

So if the resistance between O and F_n is finite, the random walk is transient, but if it is infinite, the random walk is recurrent.

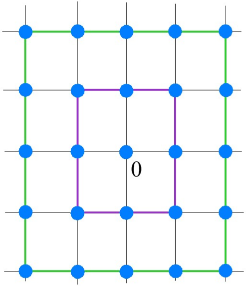
Recurrence in $\mathbb{Z}^1$

Let F_n be the set $\{n, n\}$. The resistance between 0 and F_n is given by $\frac{n}{2}$, as is illustrated by the picture below. This clearly $\longrightarrow \infty$ as $n \longrightarrow \infty$, so the walk is recurrent.



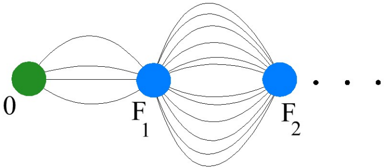
Recurrence in $\mathbb{Z}^2$

Let F_n be the set $\{(x, y) : |x| = n \text{ or } |y| = n\}$. The sets F_1 and F_2 are shown below in purple and green. Let us note that, by Rayleigh's Monotonicity Theorem, the resistance between 0 and F_n will be greater than that obtained by considering Z_2 with each set F_n "shorted out". That is, each edge with both endpoints lying in the same F_n is given a resistance 0.



Recurrence in $\mathbb{Z}^2$

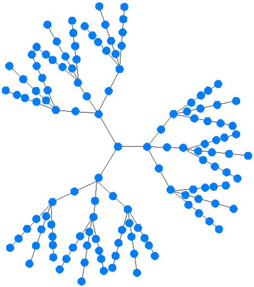
The number of edges between F_{n-1} and F_n is given by $4(2n-1)$. The graph with the 0 resistances is therefore equivalent to the following graph:



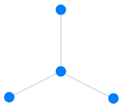
We see that the resistance between 0 and F_N is $\sum_{n=1}^{N} \frac{1}{4(2n-1)}$. This sum diverges as $n \longrightarrow \infty$, so we conclude that, as with $\mathbb{Z}^1$, random walk is recurrent on $\mathbb{Z}^2$.

Transience in $\mathbb{Z}^3$

The situation in $\mathbb{Z}^3$ is more difficult. We must show that the resistance between 0 and any distant set is always bounded. It suffices to show that there is an infinite, connected subgraph of $\mathbb{Z}^3$ in which the resistance between any two points is bounded by a constant. It can be shown that the following tree, with certain vertices identified, can be embedded in $\mathbb{Z}^3$. The resistance of this tree can be calculated to be finite.



Embedding the tree in $\mathbb{Z}^3$



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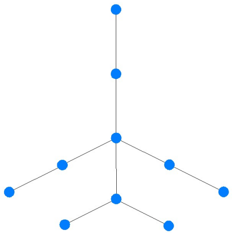

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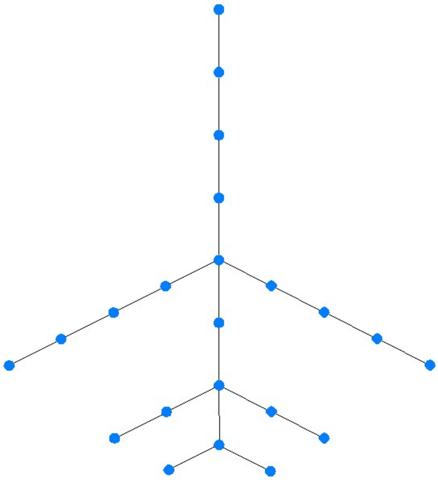
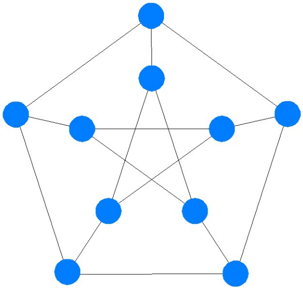


Distance-transitive graphs

Let $d(x, z)$ denote the length of the shortest path between x and z within G. A graph is *distance-transitive* if, whenever there are points X_1, z_1, x_2, z_2 such that $d(x_1, z_1) = d(x_2, z_2)$, there is an automorphism γ of G such that $\gamma(x_1) = x_2$, $\gamma(z_1) = z_2$. An automorphism is a bijection from the vertex set of G to itself, with the property that $\gamma(u) \sim \gamma v$ if, and only if, $u \sim v$.



Distance-regular graphs

A graph is *distance-regular* if, for any points x and z within G, the sizes of the following sets depends only on $d(x, z)$:

$$B = \{v | d(x, v) = d(x, z) - 1\} \bigcap \{v | d(z, v) = 1\}$$
$$A = \{v | d(x, v) = d(x, z)\} \bigcap \{v | d(z, v) = 1\}$$
$$C = \{v | d(x, v) = d(x, z) + 1\} \bigcap \{v | d(z, v) = 1\}$$

If a graph is distance transitive, it is distance regular. If $d(x, z) = i$, we define b_i, a_i, and c_i to be the sizes of the three sets above.

Distance-regular graphs - Examples

Hamming graphs: Let d, q be positive integers. The vertex set is elements of $\{1, 2, \ldots, q\}^d$. Two vertices are adjacent if they differ in exactly one component. These graphs have found applications in computer science.

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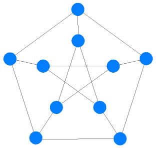
Petersen graph:



Random Walks and Electric Resistance on Distance-regular graphs

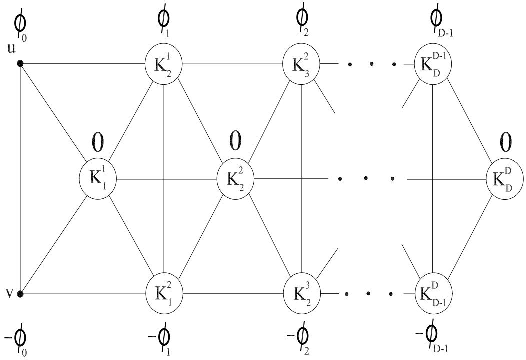
Distance-regular graphs are a very natural class of graphs upon which to study random walks and electric resistance. This is because, unlike most finite graphs, it is possible to explicitly construct harmonic functions on a D-R graph. Suppose G is a D-R graph with vertex set V, n vertices, diameter D, valency k, and intersection array $(b_0, \ldots, b_{D-1}; c_1, \ldots, c_D)$. For $0 \leq i \leq D-1$ define the numbers ϕ_i recursively by

$$\phi_0 = n - 1 \tag{7}$$
$$\phi_i = \frac{c_i \phi_{i-1} - k}{b_i}$$

We refer to the ϕ_i's as *Biggs potentials*. It can be shown that the ϕ_i's form a strictly decreasing positive sequence.

Harmonic functions on Distance-regular graphs I

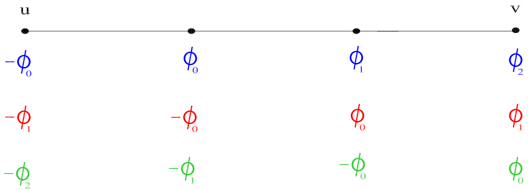
With the ϕ_i's defined recursively as in the previous frame, if u and v are adjacent define the following function:



It turns out that this function is harmonic on $V - \{u, v\}$!

Harmonic functions on Distance-regular graphs II

If u and v are not adjacent we can still construct a non-constant function harmonic on $V - \{u, v\}$. For example, it $d(u, v) = 3$ add the following functions:



The resulting function is harmonic on $V - \{u, v\}$.

Resistance between vertices on Distance-regular graphs

Using these functions constructed by the potentials ϕ_i, the resistance between two vertices of distance j can be explicitly calculated. It is given by

$$\frac{2\sum_{0\leq i<j}\phi_i}{nk} \tag{8}$$

It is an interesting question, first posed by Biggs, as to whether this quantity can be bounded for all D-R graphs.

Biggs' conjecture

Let R_i be the effective resistance between any two points of distance i in a distance-regular(or distance-transitive) graph G. Biggs conjectured that $R_d < 2R_1$, where d is the diameter of G(i.e., largest possible distance between two points). R_d is the maximal possible resistance, so this says that the resistance between any pair of points in the graph is bounded by double the smallest possible value.

Biggs' conjecture is true- worst offenders

This conjecture is now proved, and the extremal graphs have been identified.

Name	$\frac{R_D}{R_1} = \frac{\phi_0+\ldots+\phi_{D-1}}{\phi_0}$
Biggs-Smith Graph	1.930693
Foster Graph	1.896067
Flag graph of $GH(2,2)$	1.882979
Tutte's 12-Cage	1.872

The proof can be found in the paper "A Conjecture of Biggs Concerning the Resistance of a Distance-Regular Graph" by Markowsky and Koolen, appearing in the Electronic Journal of Combinatorics.

Consequences

1. Numerous statements regarding random walks are immediate, all of which are essentially equivalent to "random walks move rapidly through distance-regular graphs".

2. Certain intersection arrays are ruled out. We don't have a really good example of one that is ruled out by this theorem which is difficult to rule out by other, more well-known theorems on D-R graphs.

An extension of Biggs' conjecture

We(Markowsky, Koolen, Park) have proved

Theorem

$$\phi_2 + \ldots + \phi_{D-1} \leq \phi_1 \quad (9)$$

Except in for a small number of known cases, it can be shown that

$$\frac{\phi_1}{\phi_0} < \frac{2}{k} \quad (10)$$

so that

$$\frac{R_D}{R_1} < 1 + \frac{4}{k} \quad (11)$$

Current and future work

It is tempting to hope that $\phi_{m+1} + \ldots + \phi_{D-1} \leq \phi_m$ for all m. However, the Biggs-Smith graph, with intersection array $(3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3)$, yields the following potentials.

$$\phi_0 = 101, \phi_1 = 49, \phi_2 = 23, \phi_3 = 10, \phi_4 = 7, \phi_5 = 4, \phi_6 = 1 \tag{12}$$

Note that $\phi_4 + \phi_5 + \phi_6 > \phi_3$. Nonetheless, we have proved

Theorem

For any $m \geq 0$,

$$\phi_{m+1} + \ldots + \phi_{D-1} < (3m + 3)\phi_m \tag{13}$$

We conjecture that the $(3m + 3)$ can be replaced by a universal constant.

Current and future work, continued

Let π_0 be a probability distribution on the vertices of a graph. After undergoing a step of random walk π_0 is transformed into a new probability density, π_1. Repeating gives a sequence $\pi_2, \pi_3, \pi_4, \ldots$. It is known that if the graph is not bipartite, then π_n converges to a steady state distribution π_∞ as $n \longrightarrow \infty$. The rate at which $\pi_n \longrightarrow \pi_\infty$ is the *speed of mixing*. The theorems we have proved should show that random walk is rapidly mixing on distance-regular graphs, but so far we have not been able to get as good results as we think should be possible.

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- J. Koolen, G. Markowsky, J. Park *On electric resistances for distance-regular graphs*, preprint, should be on arxiv.org soon.