Courcelle's Theorem, tree automata, hypergraphs, and matroids

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Courcelle's Theorem

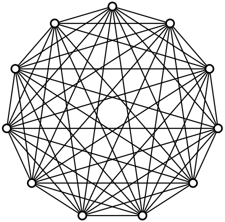
Courcelle's Theorem (1990)

Let φ be a sentence in MS_2, the monadic second-order logic of graphs.

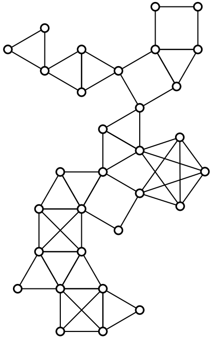
Let $\mathcal{G}$ be a class of graphs with bounded tree-width.

There is a polynomial-time algorithm that tests graphs in $\mathcal{G}$ and decides whether they satisfy φ.

Tree-width



High tree-width



Low tree-width

Monadic second-order logic

$$
\begin{aligned}
&\exists E_1\ \forall v\ \exists e_1\ \exists e_2\ (e_1 \in E_1 \wedge e_2 \in E_1 \wedge e_1 \neq e_2 \\
&\quad \wedge \mathrm{inc}(v, e_1) \wedge \mathrm{inc}(v, e_2) \\
&\qquad \wedge \forall e_3\ (e_3 \in E_1 \wedge e_3 \neq e_1 \wedge e_3 \neq e_2 \rightarrow \neg\, \mathrm{inc}(v, e_3))) \\
&\qquad\quad \wedge \forall V_1\ \forall V_2 \\
&\qquad\qquad (\exists v_1\ \exists v_2\ (v_1 \in V_1 \wedge v_2 \in V_2) \\
&\qquad\qquad\quad \wedge \forall v\ (v \in V_1 \vee v \in V_2 \wedge \neg(v \in V_1 \wedge v \in V_2))) \\
&\qquad\qquad\qquad \rightarrow \\
&\qquad\qquad\qquad\quad (\exists e\ (e \in E_1 \wedge \exists v_1\ \exists v_2 \\
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\end{aligned}
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"There exists a set of edges, E_1, such that every vertex is incident with exactly two edges in E_1, and whenever (V_1, V_2) is a partition of the vertices, there is an edge in E_1 that is incident with vertices in both V_1 and V_2."

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\begin{aligned}
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In other words, the graph is hamiltonian.

Courcelle's Theorem

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Let $\mathcal{G}$ be a class of graphs with bounded tree-width.

There is a polynomial-time algorithm that tests graphs in $\mathcal{G}$ and decides whether they satisfy φ.

Courcelle's Theorem means that we can test intractable properties efficiently, if we limit the structural complexity of the input graph.

Matroids

A matroid is a structured hypergraph (set-system).

Definition

A matroid is a pair, $(E, \mathcal{I})$, where E is a finite set (the ground set), and $\mathcal{I}$ is a family of subsets (the independent sets), satisfying:

- $\emptyset \in \mathcal{I}$,
- $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$ implies $I_2 \in \mathcal{I}$,
- $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$ implies there exists $e \in I_1 - I_2$ such that $I_2 \cup \{e\} \in \mathcal{I}$.

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Definition

A maximal independent set is a basis. A subset of E that is not independent is dependent. A minimal dependent set is a circuit.

Graphic matroids

Example

Let G be a graph with edge set E. Then

$$(E, \{I \subseteq E \colon G[I] \text{ does not contain a cycle}\})$$

is a graphic matroid.

The bases of a graphic matroid are maximal forests. The circuits are cycles in the graph.

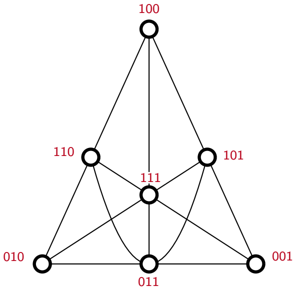
Representable matroids

Example

Let $\mathbb{F}$ be a field. Let E be a finite subset of the vector space $\mathbb{F}^n$. Then

$$(E, \{I \subseteq E \colon I \text{ is linearly independent}\})$$

is an $\mathbb{F}$-representable matroid.



Matroids and the Robertson-Seymour project

Statements about graphs are sometimes special cases of more general statements about matroids.

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Geelen, Gerards, and Whittle have now established a qualitative structural description of any proper minor-closed class of $\mathbb{F}$-representable matroids, when $\mathbb{F}$ is a finite field.

We can deduce that in any infinite collection of $\mathbb{F}$-representable matroids, one is isomorphic to a minor of another.

Hliněný's Theorem

Hliněný's Theorem (2006)

Let φ be a sentence in MS_0, the monadic second-order logic of matroids.

Let $\mathcal{M}$ be a class of $\mathbb{F}$-representable matroids with bounded branch-width, where $\mathbb{F}$ is a finite field.

There is a polynomial-time algorithm that tests matroids in $\mathcal{M}$ and decides whether they satisfy φ.

Monadic second-order logic

$$\forall X_1 \forall X_2 \ (\mathsf{Ind}(X_1) \wedge X_2 \subseteq X_1 \rightarrow \mathsf{Ind}(X_2))$$

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$$\forall X_1 \forall X_2 \; (\mathrm{Ind}(X_1) \wedge X_2 \subseteq X_1 \rightarrow \mathrm{Ind}(X_2))$$

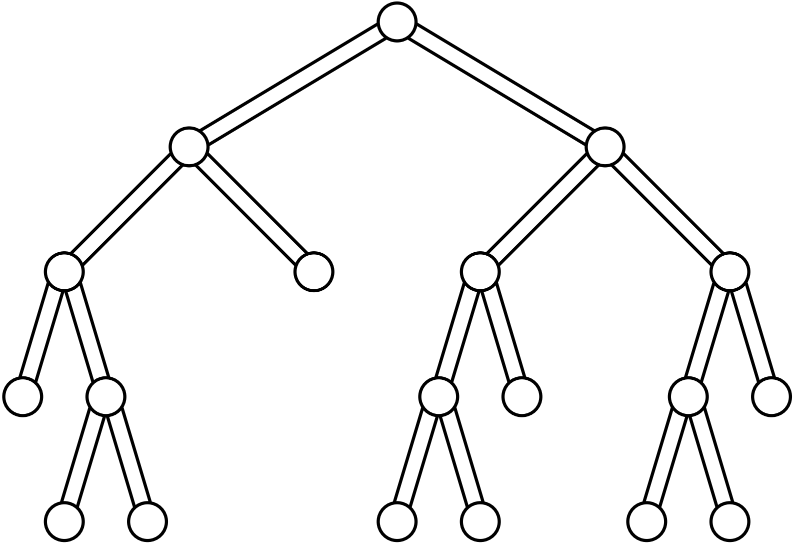
This expresses the second matroid axiom: every subset of an independent set is independent.

Tree automata

A tree automaton consists of a set of states (colours)

and a distinguished subset of accepting states.

Tree automata



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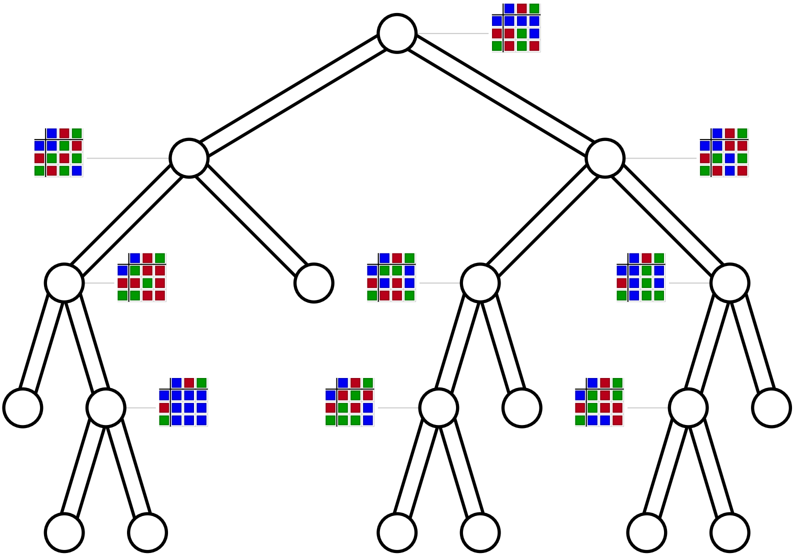

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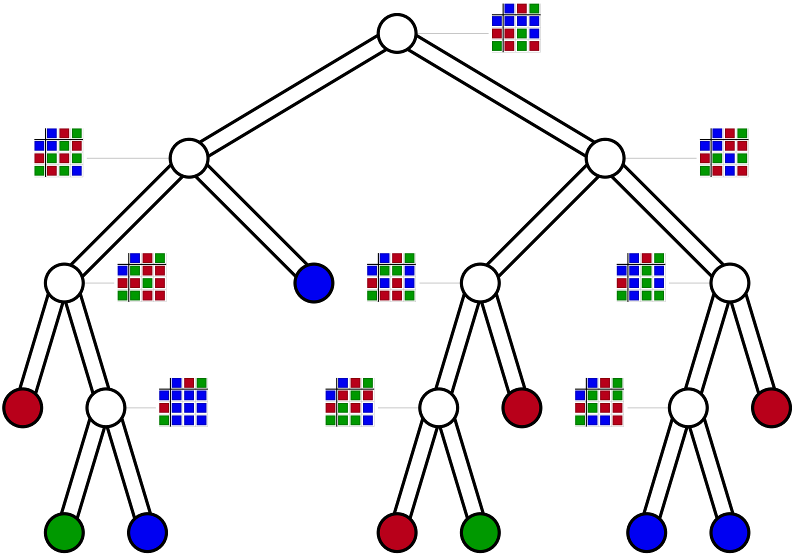

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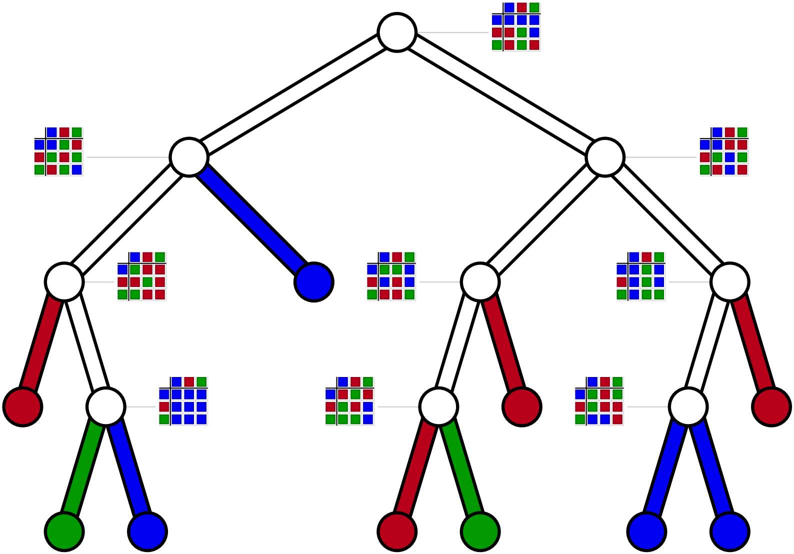

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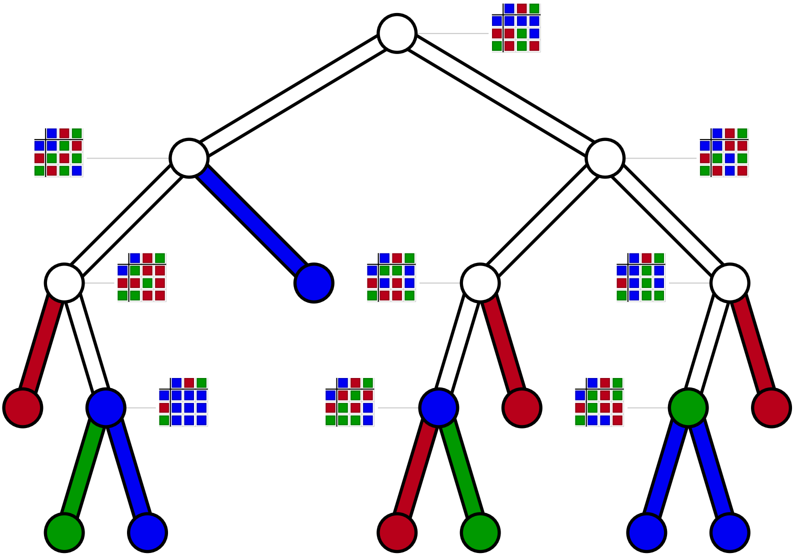

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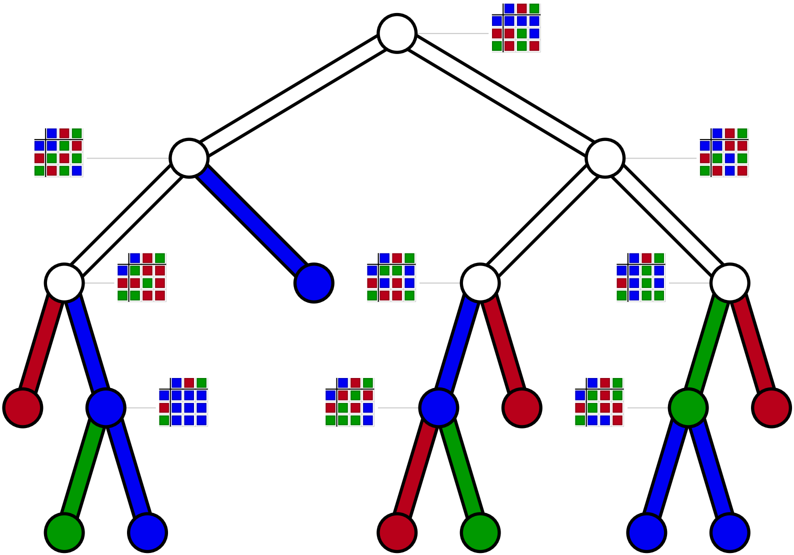

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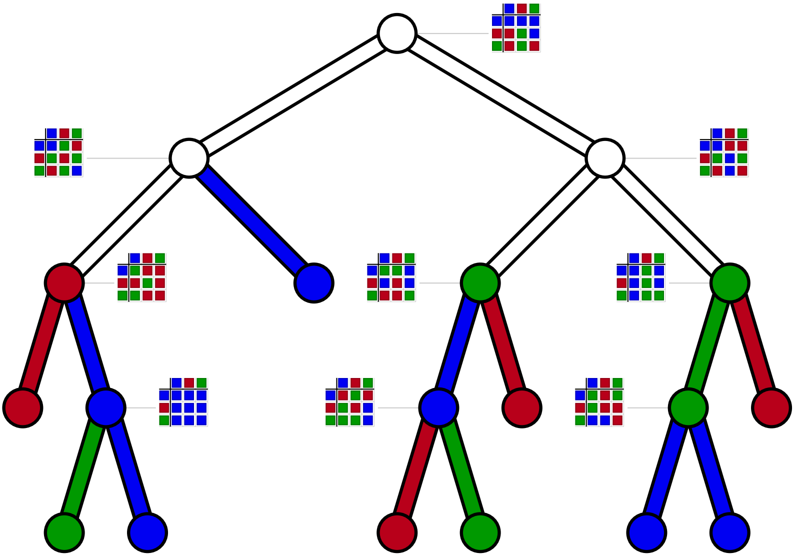

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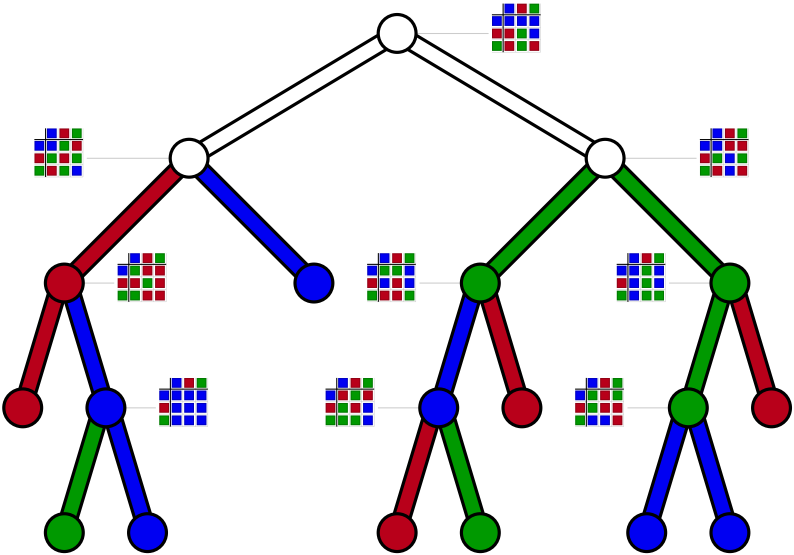

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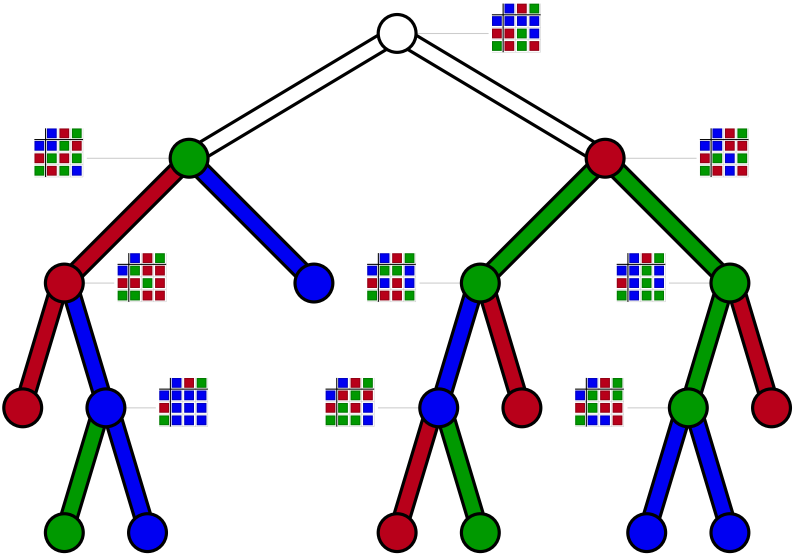

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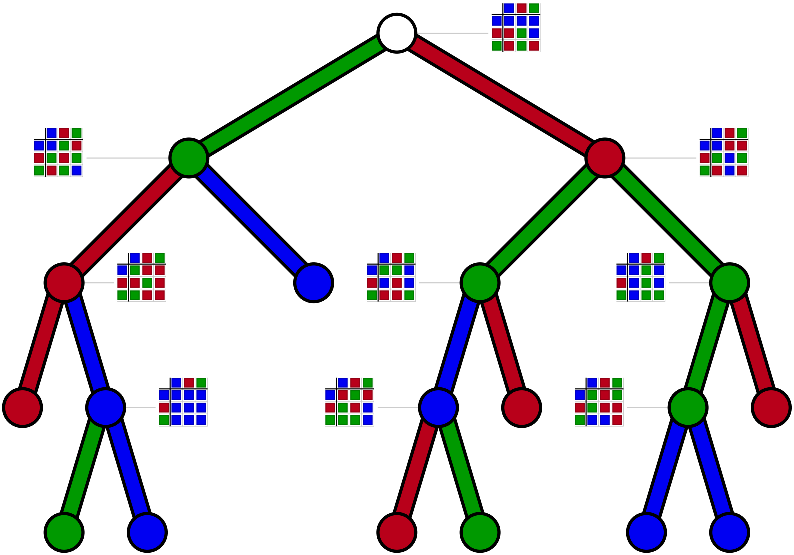

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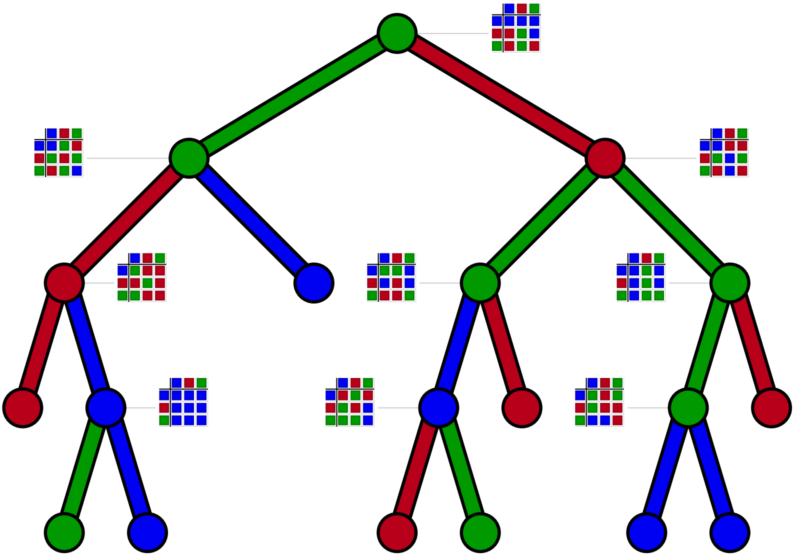

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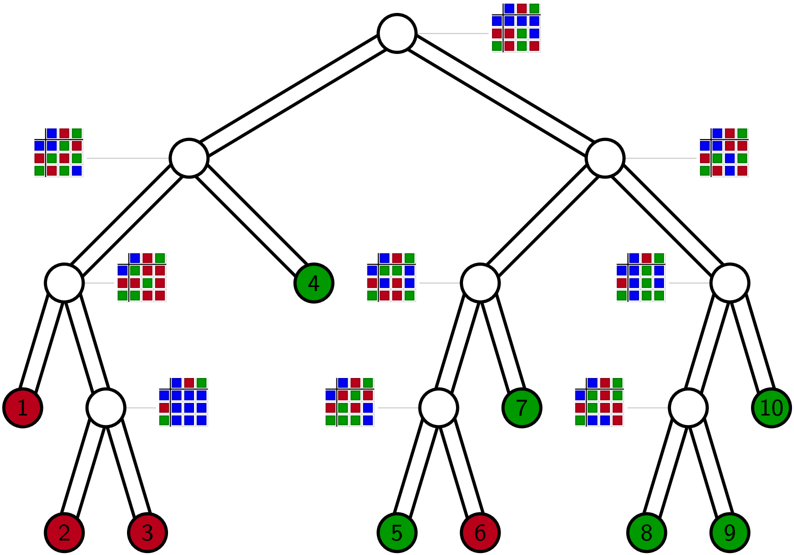

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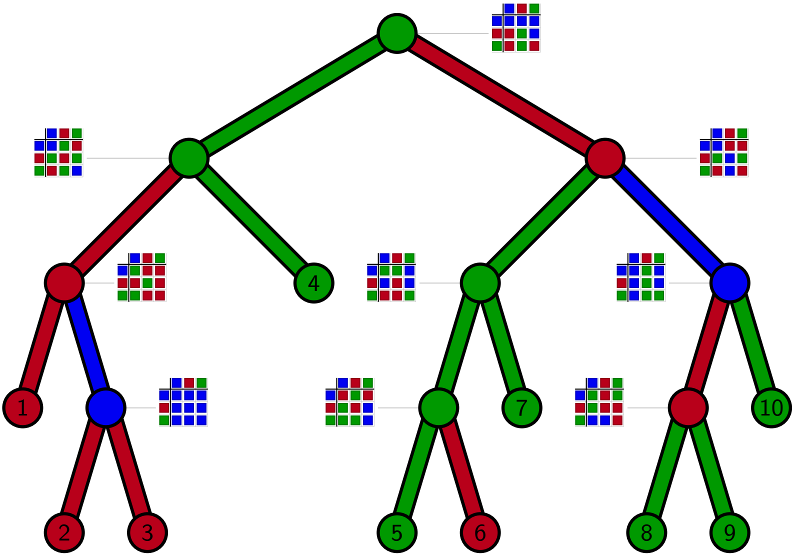

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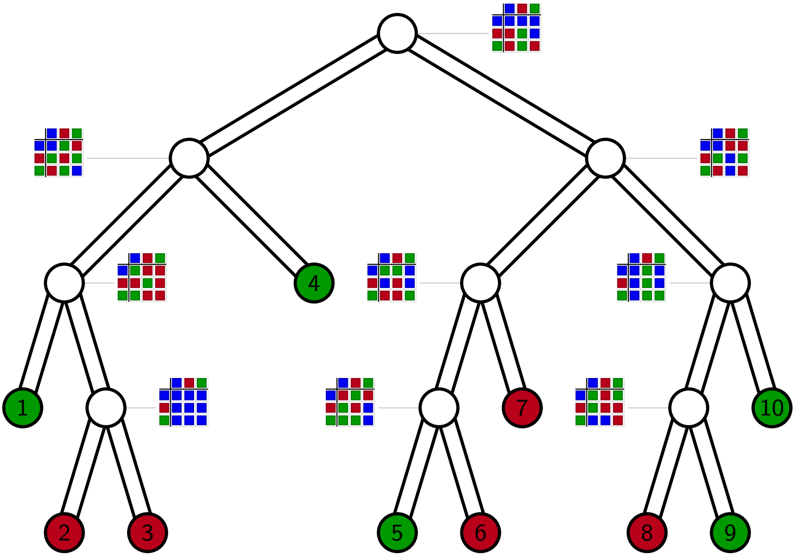

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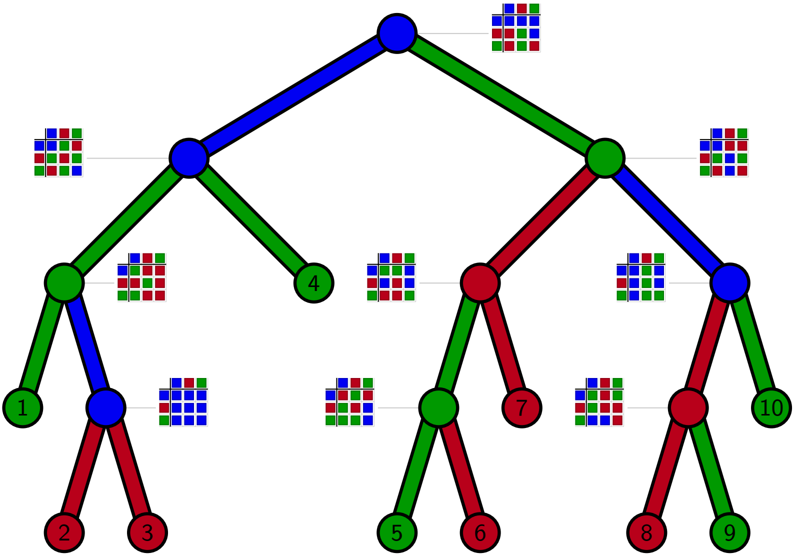

Automatic set-systems

Given an automaton with two distinguished colours that encode subsets, there is a corresponding set-system on the leaf-set of any tree.

What families of set-systems arise in this way?

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What families of set-systems arise in this way?

Definition

Let $\mathcal{M}$ be family of set-systems.

Assume there is an automaton, A, and for every $(E, \mathcal{I}) \in \mathcal{M}$, there is a tree, T_M, with leaf-set E, such that A accepts the subsets in $\mathcal{I}$ and rejects the subsets not in $\mathcal{I}$.

Then we say that $\mathcal{M}$ is automatic.

What families of set-systems are automatic?

Characterising automatic set-systems

Theorem

Let $\mathcal{M}$ be a family of set-systems. Then $\mathcal{M}$ is automatic if and only if it has bounded decomposition-width.

Decomposition-width

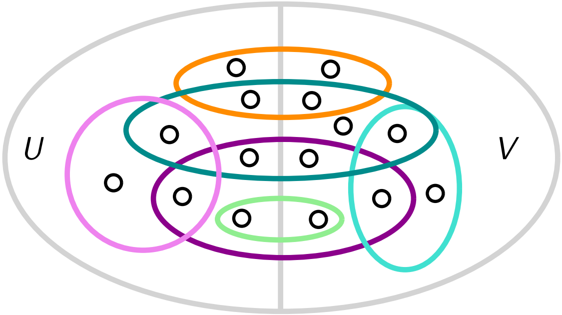
Definition

Let $(E, \mathcal{I})$ be a set-system, and let (U, V) be a partition of E.

We define $\sim_U$, an equivalence relation on subsets of U.

Subsets $X, X' \subseteq U$ satisfy $X \sim_U X'$ if, whenever Z is a subset of V, both $X \cup Z$ and $X' \cup Z$ are in $\mathcal{I}$, or neither are.

Decomposition-width



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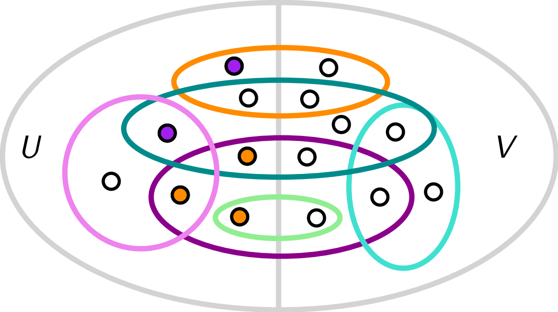

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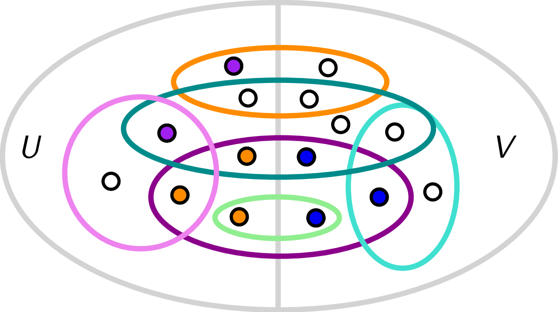

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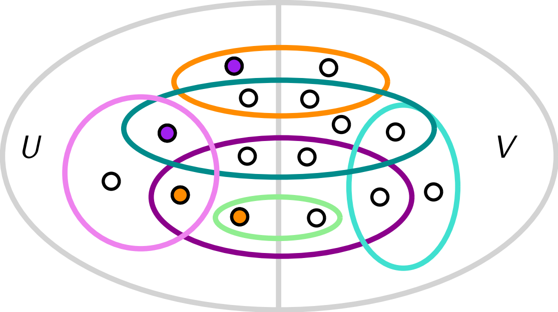

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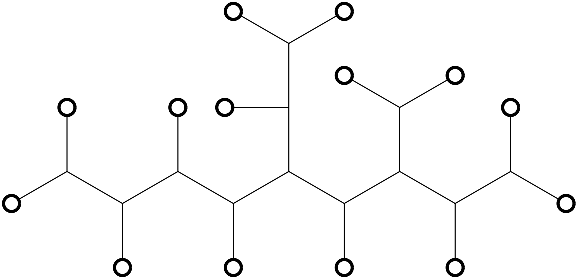
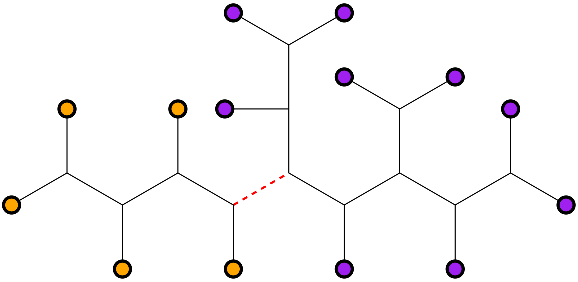


A decomposition of a set-system $(E, \mathcal{I})$ is a bijection between E and the leaves of a tree where every non-leaf vertex has degree three.

Decomposition-width



A decomposition of a set-system $(E, \mathcal{I})$ is a bijection between E and the leaves of a tree where every non-leaf vertex has degree three.

A displayed set is any set corresponding to the leaves in a connected component created by deleting an edge of the tree.

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Definition

If $\mathcal{M}$ has bounded decomposition-width, then there is an integer, K, and for every $M = (E, \mathcal{I}) \in \mathcal{M}$, we have a decomposition of M such that whenever U is a displayed set, then $\sim_U$ has at most K equivalence classes.

An equivalent notion of decomposition-width was discussed by Král and Strozecki.

Tree automata and Hliněný's Theorem

If $\mathcal{M}$ is an automatic family of set-systems, then there is an automaton, A, and for every $(E, \mathcal{I}) \in \mathcal{M}$, there is a tree, T_M, with leaf-set E, such that A accepts the subsets in $\mathcal{I}$ and rejects the subsets not in $\mathcal{I}$.

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We can use a bootstrapping procedure to build an automaton that will quickly test any given MS_0 sentence for set-systems in $\mathcal{M}$.

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We can use a bootstrapping procedure to build an automaton that will quickly test any given MS_0 sentence for set-systems in $\mathcal{M}$.

This explains why tree automata are at the heart of Hliněný's Theorem.

The challenge is constructing T_M, given $M \in \mathcal{M}$.

Our conclusions

We have reproved Courcelle's Theorem and Hliněný's Theorem.

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We have done this for the following classes.

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- Bicircular matroids
- Lattice-path matroids
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We have some negative results: if $\mathbb{F}$ is an infinite field, then Hliněný's Theorem cannot extend to the class of $\mathbb{F}$-representable matroids. Nor can it extend to the class of gain-graphic matroids over an infinite group (assuming $\mathsf{P} \neq \mathsf{NP}$).