Asymptotic enumeration of correlation-immune functions

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Correlation-immune functions

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The function is correlation-immune of order k if the function value is uncorrelated with any k of the arguments.

Suppose the fraction λ of all 2^n argument values give a function value 1. Then correlation-immune means that if any arbitrary k of the arguments are fixed to arbitrary values, the same fraction λ of the remaining 2^{n-k} argument values give a function value 1.

The weight of the function is $\lambda 2^n$ — the number of argument lists that give function value 1.

Example (Sloane): $n = 12$, $k = 3$, $\lambda = 24/2^{12}$, weight $= 24 = 2^k 3$.

The rows of the table give the argument lists for which the function value is 1.

0	1	1	1	1	1	1	1	1	1	1	1
0	0	1	0	1	1	1	0	0	0	1	0
0	0	0	1	0	1	1	1	0	0	0	1
0	1	0	0	1	0	1	1	1	0	0	0
0	0	1	0	0	1	0	1	1	1	0	0
0	0	0	1	0	0	1	0	1	1	1	0
0	0	0	0	1	0	0	1	0	1	1	1
0	1	0	0	0	1	0	0	1	0	1	1
0	1	1	0	0	0	1	0	0	1	0	1
0	1	1	1	0	0	0	1	0	0	1	0
0	0	1	1	1	0	0	0	1	0	0	1
0	1	0	1	1	1	0	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	1	1	1	0	1
1	1	1	0	1	0	0	0	1	1	1	0
1	0	1	1	0	1	0	0	0	1	1	1
1	1	0	1	1	0	1	0	0	0	1	1
1	1	1	0	1	1	0	1	0	0	0	1
1	1	1	1	0	1	1	0	1	0	0	0
1	0	1	1	1	0	1	1	0	1	0	0
1	0	0	1	1	1	0	1	1	0	1	0
1	0	0	0	1	1	1	0	1	1	0	1
1	1	0	0	0	1	1	1	0	1	1	0
1	0	1	0	0	0	1	1	1	0	1	1

This is an orthogonal array of 2 levels, 12 variables, 24 runs and strength 3.

Our task

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Define $N(n, k, q)$ to be the number of n-variable correlation-immune functions of order k and weight $2^k q$.

Also define $N(n, k) = \sum_q N(n, k, q)$.

We seek the asymptotic values of $N(n, k, q)$ and $N(n, k)$ as $n \to \infty$, with k and q being some functions of n.

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Define

$$M = \sum_{i=0}^{k} \binom{n}{i} \quad \text{and} \quad Q = \sum_{i=1}^{k} i \binom{n}{i}.$$

Theorem (Denisov, 1992)
If $k \ge 1$ is a constant integer, then as $n \to \infty$,

$$N(n, k) \sim 2^{2^n + Q - k} (2^{n-1} \pi)^{-(M-1)/2}. \quad \square$$

Denisov's method (translated)

For $S \subseteq \{1, 2, \ldots, n\}$, let β_S be the number of rows $(\beta_1, \beta_2, \ldots, \beta_n)$ of the matrix such that $\beta_i = 1$ for $i \in S$. Also $\beta_\emptyset = 2^k q$ (i.e., all the rows).

Then the matrix is that of a correlation-immune function of weight $2^k q$ iff

$$\beta_S = 2^{k-|S|} q \quad \text{for} \quad |S| \leq k.$$

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Consider

$$\prod_{\beta \in \{0,1\}^n} \left(1 + \prod_{|S| \leq k} x_S^{\prod_{i \in S} \beta_i}\right),$$

where $\{x_S \mid S \subseteq \{1, 2, \ldots, n\}\}$ are indeterminates.

Then $N(n, k, q)$ is the coefficient of the monomial

$$\prod_{|S| \leq k} x_S^{2^{k-|S|} q}.$$

Denisov extracts $N(n, k)$ by Fourier inversion.

The inversion integral is concentrated at two equivalent places, where it is approximately gaussian.

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This is unfortunate, since the 1992 result is correct and the 2000 result is incorrect!

Alternative approach

For a boolean function $g(x_1, \ldots, x_n)$, the Walsh transform of g is the real-valued function $\hat{g}$ over $\{0,1\}^n$ defined by

$$\hat{g}(w_1, \ldots, w_n) = \sum_{(x_1,\ldots,x_n)\in\{0,1\}^n} g(x_1, \ldots, x_n)(-1)^{w_1x_1+\cdots+w_nx_n}.$$

It is known that g is correlation-immune of order k iff $\hat{g}(w_1, \ldots, w_n) = 0$ whenever the number of 1s in $w_1, \ldots, w_n$ is between 1 and k.

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Put $R = \lambda/(1-\lambda)$. Define

$$F(\boldsymbol{x}) = \prod_{\alpha\in\{\pm1\}^n} \Big(1 + R \prod_{|S|\le k} x_S^{\alpha_S}\Big),$$

where

$$\alpha_S = \prod_{i\in S} \alpha_i.$$

Theorem: $N(n,k,q)$ is the constant term of $(Rx_\emptyset)^{-2^kq}F(\boldsymbol{x})$.

Apply the Cauchy coefficient formula, using unit circles as contours, and change variables as $x_S = e^{i\theta_S}$ for each S.

Then

$$N(n,k,q) = \frac{(1+R)^{2^n}}{(2\pi)^M R^{2^k q}}\, I(n,k,q),$$

where

$$I(n,k,q) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} G(\boldsymbol{\theta})\, d\boldsymbol{\theta},$$

$$G(\boldsymbol{\theta}) = e^{-i2^k q \theta_\emptyset} \prod_{\alpha \in \{\pm 1\}^n} \frac{1 + R e^{i f_\alpha(\boldsymbol{\theta})}}{1+R},$$

$$f_\alpha(\boldsymbol{\theta}) = \sum_{|S| \le k} \alpha_S \theta_S.$$

Here $\boldsymbol{\theta}$ is a vector of the variables θ_S, $|S| \le k$, in arbitrary order.

Analysis of the domain of integration

The integrand

$$G(\boldsymbol{\theta}) = e^{-i2^k q\theta_\emptyset} \prod_{\alpha\in\{\pm1\}^n} \frac{1 + Re^{if_\alpha}}{1 + R}$$

has greatest absolute value 1 when

$$f_\alpha = f_\alpha(\boldsymbol{\theta}) = \sum_{|S|\le k} \alpha_S \theta_S$$

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Define the difference operator

$$\delta_j f_{(\alpha_1,\ldots,\alpha_j,\ldots,\alpha_n)} = f_{(\alpha_1,\ldots,\alpha_j,\ldots,\alpha_n)} - f_{(\alpha_1,\ldots,-\alpha_j,\ldots,\alpha_n)}.$$

and in general $\delta_S = \prod_{j\in S} \delta_j$.

If each f_α is a multiple of 2π, then so are all the differences. Now we compute

$$\delta_S f_\alpha = 2^{|S|} \sum_{T\supseteq S} \alpha_T \theta_T.$$

and apply this with decreasing $|S|$.

Conclusion:

$|G(\boldsymbol{\theta})| = 1$ iff there are integers j_S such that

$$\sum_{T \supseteq S} \theta_T = 2^{-|S|+1} j_S \pi$$

for every $S \subseteq \{1, 2, \ldots, n\}$ with $|S| \leq k$.

There are 2^Q such critical points, where $Q = \sum_{i=1}^{k} i \binom{n}{i}$.

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There are 2^Q such critical points, where $Q = \sum_{i=1}^{k} i \binom{n}{i}$.

Define the critical region $\mathcal{R}$ to be the set of points $\boldsymbol{\theta}$ such that, for some critical point $\hat{\boldsymbol{\theta}}$

$$|\theta_S - \hat{\theta}_S| \leq \Delta (2n)^{-|S|}$$

for each S, where $\Delta = 2^{-n/2+k+3} \lambda^{-1/2} n^{k+1/2} M^{1/2}$.

These 2^Q cuboids are disjoint and equivalent.

The integrand outside the critical region

If $\boldsymbol{\theta}$ is not in the critical region,

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Proof:

(1) There is some S such that, outside the critical region, $\delta_S f_\alpha$ is at least $(2 - e^{1/2})\Delta n^{-|S|}$ from any multiple of 2π for all α.

(2) Divide the 2^n vectors α into $2^{n-|S|}$ classes of size $2^{|S|}$, where two vectors are in the same class iff they agree outside S.

(3) For each class,

$$\prod_\alpha \left| \frac{1 + Re^{if_\alpha}}{1 + R} \right| \leq \exp(-stuff).$$

(4) That does it.

The integrand inside the critical region

Since the 2^Q components of the critical region are all equivalent, consider the component containing the origin.

If $\boldsymbol{\theta}$ is in the critical region near the origin,

$$G(\boldsymbol{\theta}) = \exp\Big(-\tfrac{1}{2}\lambda(1-\lambda)2^n \sum_{|S|\le k} \theta_S^2 + O(\lambda 2^n \Delta^3)\Big).$$

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Proof:

Use Taylor expansion.

The linear term vanishes thanks to the choice of R.

Conclusion

Theorem: If $\omega\big(2^{5k}n^{6k+3}M^3\big) \le q \le 2^{n-k} - \omega\big(2^{5k}n^{6k+3}M^3\big)$, then

$$N(n,k,q) \sim \frac{2^{Q-(n+1)M/2}}{\pi^{M/2}\big(\lambda^\lambda(1-\lambda)^{(1-\lambda)}\big)^{2^n+M/2}}.$$

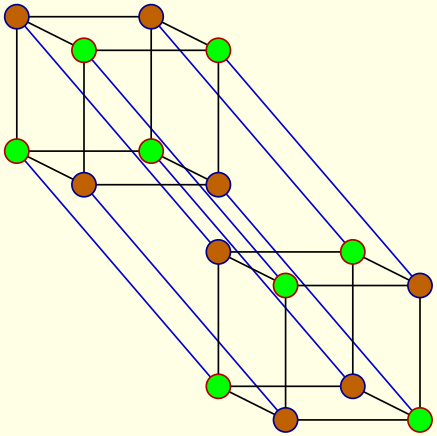
This allows *some* values of q if $k \le cn/\log n$ (compared to constant k for Denisov).

In that case:

$$N(n,k) \sim 2^{2^n+Q-k}(2^{n-1}\pi)^{-(M-1)/2}.$$

More on the case $k = 1$

A balanced colouring of a hypercube is a colouring with two colours such that the center of mass is at the center of the hypercube.



0 0 0 0
1 0 0 0
0 1 1 0
1 1 1 0
1 0 0 1
0 1 0 1
0 0 1 1
1 1 1 1

This corresponds to colouring according to the value of $f(x_1, x_2, \ldots, x_n)$ for a correlation-immune function of order 1.

Palmer, Read and Robinson did an exact enumeration (1992) but it seems unsuitable for asymptotics.

Naive estimate ($2q$ uses of first colour):

Choose, uniformly at random, a set of $2q$ distinct elements of $\{0,1\}^n$. The event of any particular column having exactly q 0s and q 1s has probability

$$\binom{2^{n-1}}{q}^2 \Big/ \binom{2^n}{2q}.$$

Therefore, if these n events are close to being independent,

$$N(n,1,q) \sim \binom{2^n}{2q} \left(\frac{\binom{2^{n-1}}{q}^2}{\binom{2^n}{2q}} \right)^n .$$

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For small q, actually $q = o(2^{n/2})$, we can estimate $N(n,1,q)$ probabilistically: make each column randomly with q zeros and q ones. Then use Bonferroni to show that the rows are distinct with probability $1 - o(1)$.

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Conclusion: The naive estimate is correct for all q.

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```
-1  1  1  1  1 -1 -1 -1
 1 -1  1  1 -1  1 -1 -1
 1  1 -1  1 -1 -1  1 -1
 1  1  1 -1 -1 -1 -1  1
 1 -1 -1 -1 -1  1  1  1
-1  1 -1 -1  1 -1  1  1
-1 -1  1 -1  1  1 -1  1
-1 -1 -1  1  1  1  1 -1
```

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-1  1 -1 -1  1 -1  1  1
-1 -1  1 -1  1  1 -1  1
-1 -1 -1  1  1  1  1 -1
```

Hadamard conjecture: A Hadamard matrix exists iff $n = 2$ or n is a multiple of 4.

Multiply rows by -1 as needed so the first column is 1 then delete the first column. Then change -1 into 0. The result is a correlation-immune function of $n-1$ variables, order 2, and weight n.

Are there any??
De Launey and Levin used similar methods to show that at least $n^{1/12-\epsilon}$ rows of a Hadamard rectangle always exist.