Closed walks in a regular graph

Marsha Minchenko

School of Mathematical Sciences,Monash University

Research Group Meeting, 2009

Outline

Outline

1. Prelude
   - Introduction
   - A Motivating Set of Equivalences
2. Fugue
   - An Extension of These Equivalences
   - A Related Method
3. Descant
   - The Plan

Some Definitions

Graph, Spectra, Adjacency Matrix

- The `spectrum` of a graph with respect to its adjacency matrix consists of the eigenvalues of its adjacency matrix with their multiplicity.
- For this talk, let G be a simple graph with vertex set, $V(G)$ of size n.
- The `adjacency matrix`, $A = [a_{ij}]$, of G, is the $n \times n$ matrix defined as

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j \\ 0 & \text{otherwise} \end{cases}$$

Some Definitions.

Similar Matrices, Trace

- This matrix, A, is real and symmetric, thus:
  - A is similar to a diagonal matrix B with diagonal consisting of the eigenvalues of A.
- Similar matrices have the same trace, so:
  - the trace of A,

$$Tr(A) = Tr(B) = \sum \lambda_k$$

  where λ_k are the n eigenvalues of A.

Walks and Adjacency Matrices

Considering the adjacency algebra of G.

- So considering our entries of A,

$$a_{i,j} = 1 \text{ when we have } i \text{ adjacent to } j$$

- If we consider the matrix A^2 and look at one entry:

$$a^2_{i,j} = a_{i,1}a_{1,j} + a_{i,2}a_{2,j} + \ldots + a_{i,n}a_{n,j}$$

- We get that

$$a^2_{i,j} = \#\text{ walks of length 2 from } i \text{ to } j$$

- And if you carry on in this way, and consider one entry of A^r:

$$a^r_{i,j} = \#\text{ walks of length } r \text{ from } i \text{ to } j$$

Walks and Adjacency Matrices

Considering the adjacency algebra of G.

- So considering our entries of A,

$$a_{i,j} = 1 \text{ when we have } i \text{ adjacent to } j$$

- If we consider the matrix A^2 and look at one entry:

$$a^2_{i,j} = a_{i,1}a_{1,j} + a_{i,2}a_{2,j} + \dots + a_{i,n}a_{n,j}$$

- We get that

$$a^2_{i,j} = \# \text{ walks of length 2 from } i \text{ to } j$$

- And if you carry on in this way, and consider one entry of A^r:

$$a^r_{i,j} = \# \text{ walks of length } r \text{ from } i \text{ to } j$$

Closed Walks and Adjacency Matrices.

The trace acting on the adjacency algebra of G.

What about the diagonal?

- The entries along the diagonal in A^r give the number of walks of length r from a given vertex to itself
- $Tr(A^r)$ gives the total number of closed walks of length r in G.
- Considering our diagonal matrix B:

$$Tr(A^r) = Tr(B^r) = \sum_{k=1}^{n} \lambda_k^r$$

Outline

1. Prelude
   - Introduction
   - A Motivating Set of Equivalences
2. Fugue
   - An Extension of These Equivalences
   - A Related Method
3. Descant
   - The Plan

Closed Walks and Adjacency Matrices.

The trace acting on the adjacency algebra of G.

It can be shown that for n as before, e edges, and t triangles or 3-cycles,

$$\sum_{k=1}^{n} \lambda_k^1 = Tr(A^1) = 0$$

$$\sum_{k=1}^{n} \lambda_k^2 = Tr(A^2) = 2e$$

$$\sum_{k=1}^{n} \lambda_k^3 = Tr(A^3) = 6t$$

Or simply given the spectrum of G

Closed Walks and Adjacency Matrices.

The trace acting on the adjacency algebra of G.

It can be shown that for n as before, e edges, and t triangles or 3-cycles,

$$\sum_{k=1}^{n} \lambda_k^1 = Tr(A^1) = 0$$

$$\sum_{k=1}^{n} \lambda_k^2 = Tr(A^2) = 2e$$

$$\sum_{k=1}^{n} \lambda_k^3 = Tr(A^3) = 6t$$

Or simply given the spectrum of G

Closed Walks and Adjacency Matrices.

The trace acting on the adjacency algebra of G.

It can be shown that for n as before, e edges, and t triangles or 3-cycles,

$$\sum_{k=1}^{n} \lambda_k^1 = Tr(A^1) = 0$$

$$\sum_{k=1}^{n} \lambda_k^2 = Tr(A^2) = 2e$$

$$\sum_{k=1}^{n} \lambda_k^3 = Tr(A^3) = 6t$$

Or simply given the spectrum of G

Can these results be extended for higher powers of A?

- $K_{1,4}$ and $K_1 \cup C_4$ have the same same spectrum:

$$\{-2^1, 0^3, 2^1\}$$

  but they don't have the same number of 4-cycles
- We need to look further than the sole contribution of n-cycles to the number of closed walks of length n in G.

Can these results be extended for higher powers of A?

- $K_{1,4}$ and $K_1 \cup C_4$ have the same same spectrum:

$$\{-2^1, 0^3, 2^1\}$$

  but they don't have the same number of 4-cycles
- We need to look further than the sole contribution of n-cycles to the number of closed walks of length n in G.

Outline

Closed Walks For Higher Powers Of A.

When G is 4-regular bipartite.

Has any other work been done to extend these results?

- A paper by Stevanovic et al., stated that for 4-regular bipartite graphs; where n is again the number of vertices, q the number of 4-cycles, and h the number of 6-cycles,

$$Tr(A^0) = n$$
$$Tr(A^2) = 4n$$
$$Tr(A^4) = 28n + 8q$$
$$Tr(A^6) = 232n + 144q + 12h$$

$$Tr(A^8) \geq 2092n + 2024q + 288h$$

Closed Walks For Higher Powers of A

Walking in the corresponding tree

These results are based on an equivalence established between the number of closed walks in k-regular graphs and infinite k-regular trees.

Counting Closed Walks in the Corresponding Tree

Recursion

- We can look at walks in trees recursively
  - Let $w_k(d, l)$ denote the number of walks of length l between the vertices at a distance d in an infinite k-regular tree.
  - $w_k(d, l) = w_k(d - 1, l - 1) + (k - 1)w_k(d + 1, l - 1)$
- The authors do not find a closed form except when $d = 0$

$$w_k(0, l) = \frac{2k - 2}{k - 2 + k\sqrt{1 - 4kx}}$$

Counting Closed Walks in the Corresponding Tree
Conceptually

- What closed walks of G correspond with walks where $d = 0$ in our tree?
- Which don't?

Summary Of This Extension By Stevanovic et al.

The authors managed to

- find a recursive formula to count the number of closed walks of length l containing the cycle C in a k-regular graph
- let $k = 4$ and find the number of closed walks for $l \leq 6$ of bipartite graphs in terms of n and the number of various cycles
- find a bound on walks of length 8:

$$Tr(A^8) \geq 2092n + 2024q + 288h$$

with note that they need to account for not only 8-cycles but also subgraphs like two 4-cycles sharing a common vertex.

Outline

Revisiting $w_k(0, l)$

Curiously, the same closed form for generating closed walks in an infinite rooted *nearly-regular* tree is derived in a soon to be pubished paper by an **AMS 2009 medal** winning author, Wanless.

- Let T_r count closed rooted walks in an infinite tree with root, degree r, and every other vertex, degree $k + 1$.

$$T_r = \frac{2k}{2k - r + r\sqrt{1 - 4(k)x}}$$

- Resulting is a polynomial in x with the coefficient of x^l corresponding to the number of walks of length $2l$

Counting Certain Closed Walks

This generating function, T_r is used to count closed walks in a graph G which are specifically:

- `totally-reducible`: back-tracks itself completely
- and `not tree-like`: contains a cycle at some intermediate step of the back-tracking process

The author recognizes that all of the *desired* closed walks contain a particular kind of walk about a cycle.

Closed Walks That Extend A Given Walk

The generating function, T_r is used to craft a generating function that takes a certain walk around a cycle of length $2l$ that induces a certain subgraph in G and

- adds totally-reducible bits
- moves the start/end point of the walk

Summary Of This Method by Wanless

The author managed to:

- Obtain a generating function for all totally-reducible walks about a given closed walk
- Express the number of totally-reducible not tree-like walks of length $2l$ as polynomials in n, k, and the number of certain subgraphs of the $(k+1)$-regular graph G
- Confirm some known results for $l \leq 5$ and publish results for $l \leq 8$
- Confirm $l \leq 6$ and publish $l \leq 10$ in the bipartite case

Summary Of This Method by Wanless

Examples, where ϵ_l denotes the number of totally-reducible not tree-like walks of length $2l$:

$$\begin{aligned}
\epsilon_4 &= 48kC_3 + 8C_4 \\
\epsilon_5 &= 270k^2C_3 + 80kC_4 + 10C_5 - 40\theta_{2,2,1} \\
\epsilon_6 &= (1320k^3 - 6)C_3 + 528k^2C_4 + 120kC_5 + 12C_6 + 192K_4 \\
&\quad - (480k + 12)\theta_{2,2,1} - 48(\theta_{3,2,1} + \theta_{2,2,2} + C_{3\cdot 3})
\end{aligned}$$

Outline

The Given Results For Regular Graphs

The number of closed walks with n as before, e edges, and t 3-cycles,

$$\sum_{k=1}^{n} \lambda_k^1 = 0$$

$$\sum_{k=1}^{n} \lambda_k^2 = 2e$$

$$\sum_{k=1}^{n} \lambda_k^3 = 6t$$

The Given Results For 4-Regular Bipartite Graphs

The number of closed walks with n as before, q 4-cycles, and h 6-cycles,

$$\sum_{k=1}^{n} \lambda_k^0 = n$$

$$\sum_{k=1}^{n} \lambda_k^2 = 4n$$

$$\sum_{k=1}^{n} \lambda_k^4 = 28n + 8q$$

$$\sum_{k=1}^{n} \lambda_k^6 = 232n + 144q + 12h$$

My Plan

- I plan to extend these results for k-regular graphs and to consider k-regular bipartite graphs for general k
- The number of closed walks will be given by sets of polynomials, a polynomial for each length of walk
- These will be polynomials on n, k, and the number of certain subgraphs of the original graph
- A series of equations will be formed from the two ways of counting closed walks on regular graphs of length l: the trace of the l-th power of the adjacency matrix of the graph and the above polynomials

Further Possibilities

- These equations have unknowns on n, k, the number of various subgraphs of the graph, and the eigenvalues of the graph
- Thus given different knowns, the possibilities for the unknowns could be determined
- For example: Stevanovic et al. used the equations determined here to
  - refine their list of feasible spectra of 4-regular bipartite integral graphs
  - extend the list of known 4-regular integral graphs

Further Possibilities

In this way, using known properties of regular graph spectrum for families of graphs

- on certain numbers of vertices
- or of certain subgraph configurations
- or with certain properties;

I hope to obtain results relating these graphs to their algebraic properties

FINE

For Further Reading I

D. M. Cvetkovic, P. Rowlinson, S. Simic.
Eigenspaces of Graphs.
Cambridge University Press, 1997.

C. D. Godsil, G. Royle.
Algebraic Graph Theory.
Springer-Verlag, 2001.

Dragan Stevanovic and Nair M.M. de Abreu and Maria A.A. de Freitas and Renata Del-Vecchio.
Walks and regular integral graphs.
Linear Algebra and its Applications, 423(1):119–135, 2007.

I. M. Wanless.
Counting matchings and tree-like walks in regular graphs.
2009.