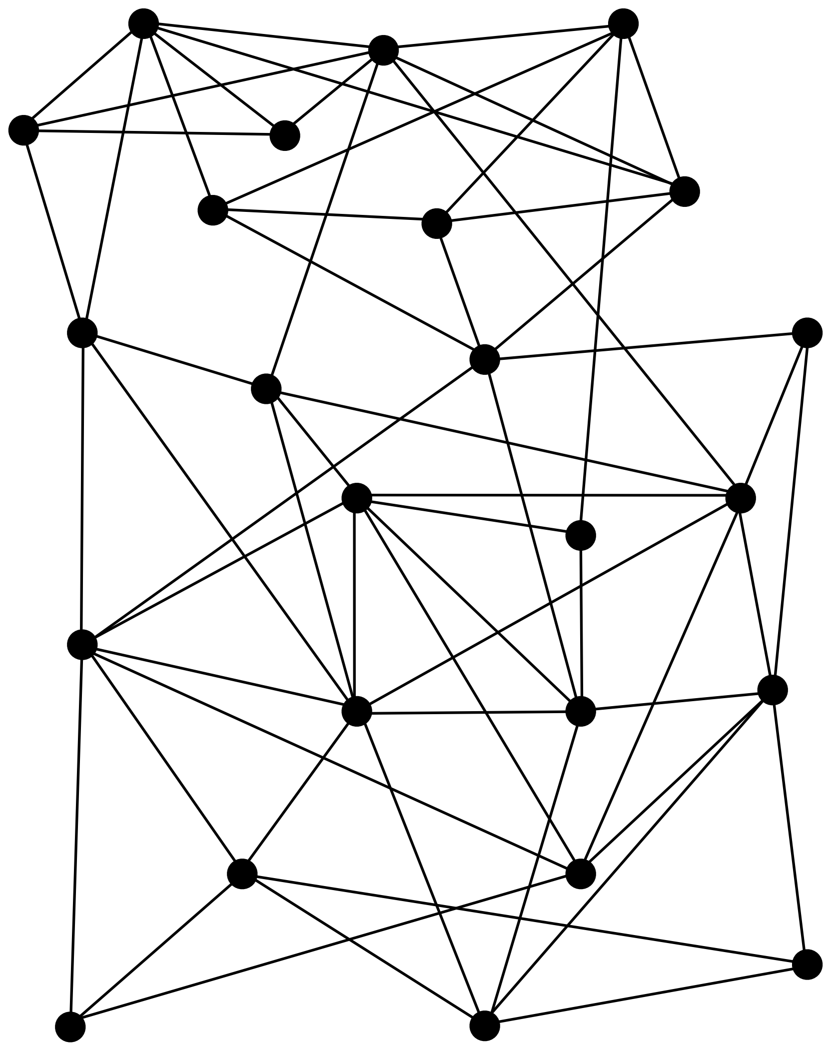
Spanning universality in random graphs

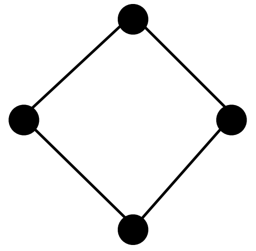
Rajko Nenadov
Monash University

(joint work with Asaf Ferber)

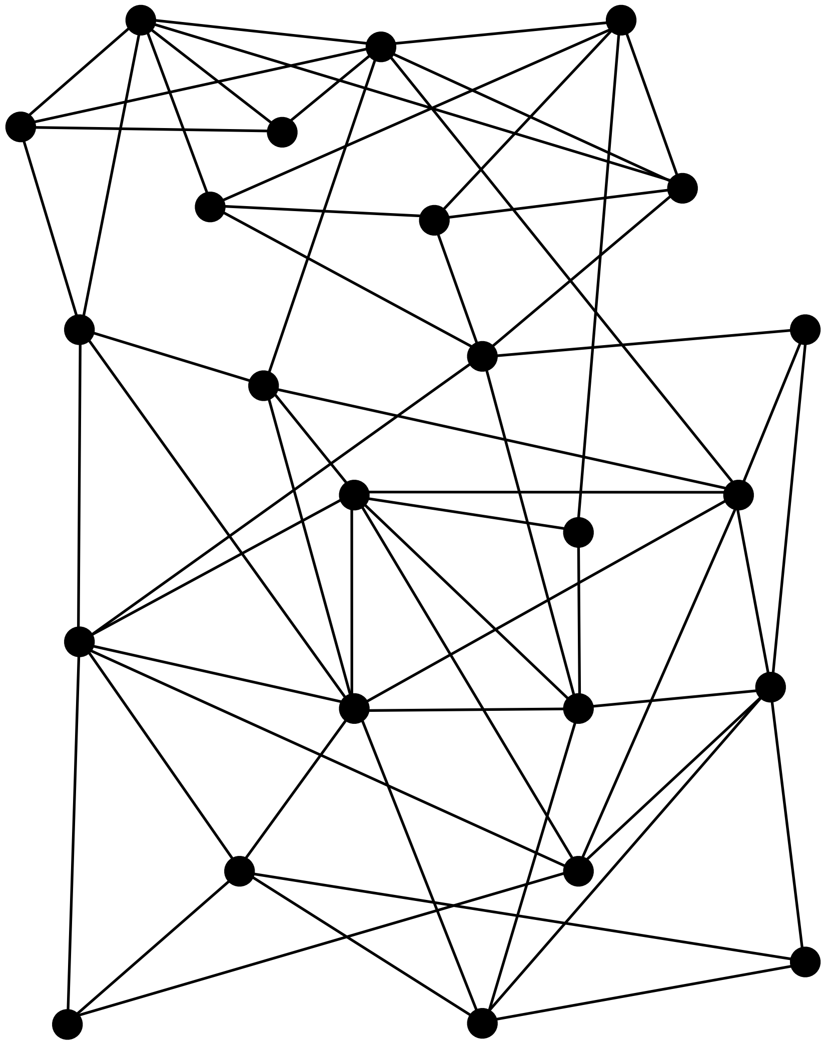


G

n vertices



H



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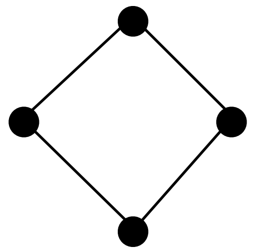

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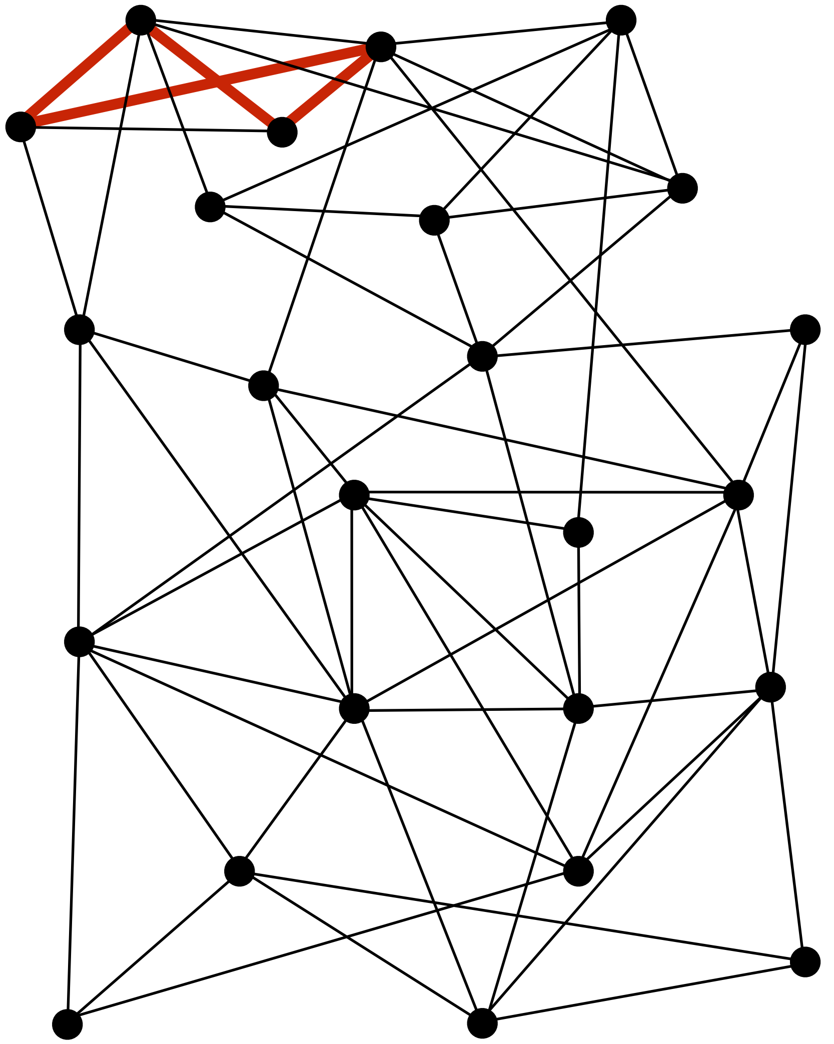

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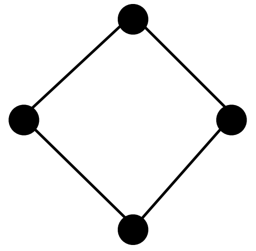

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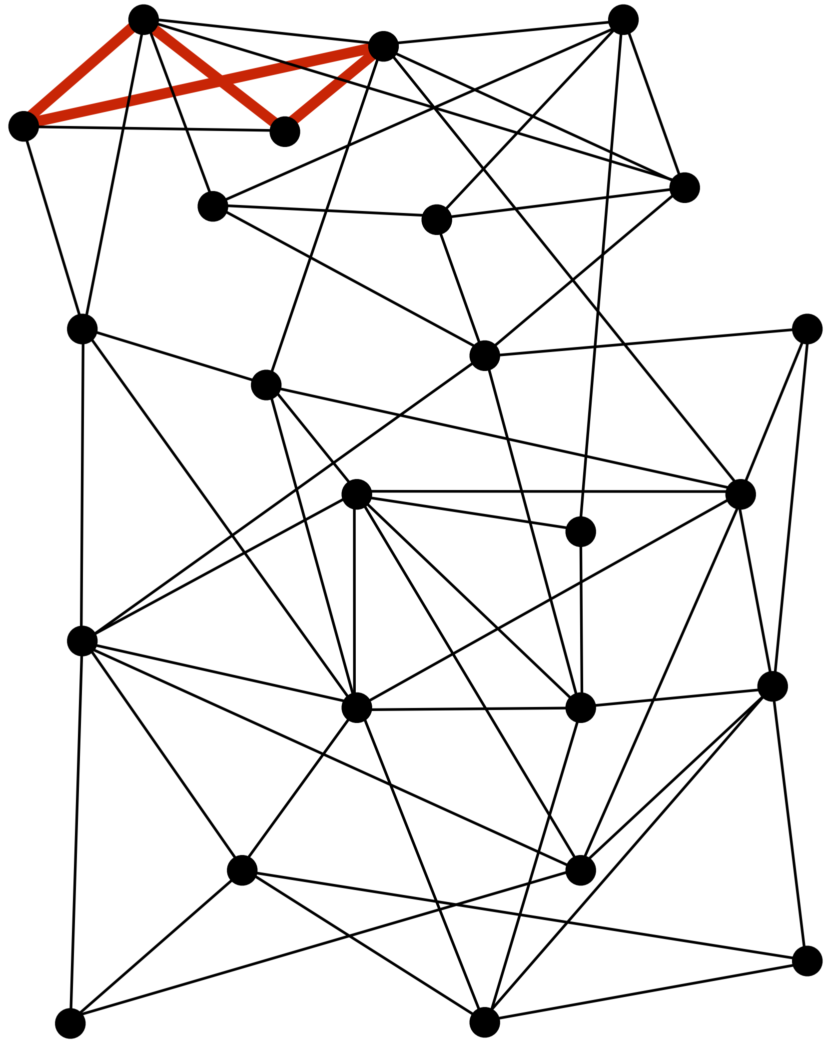

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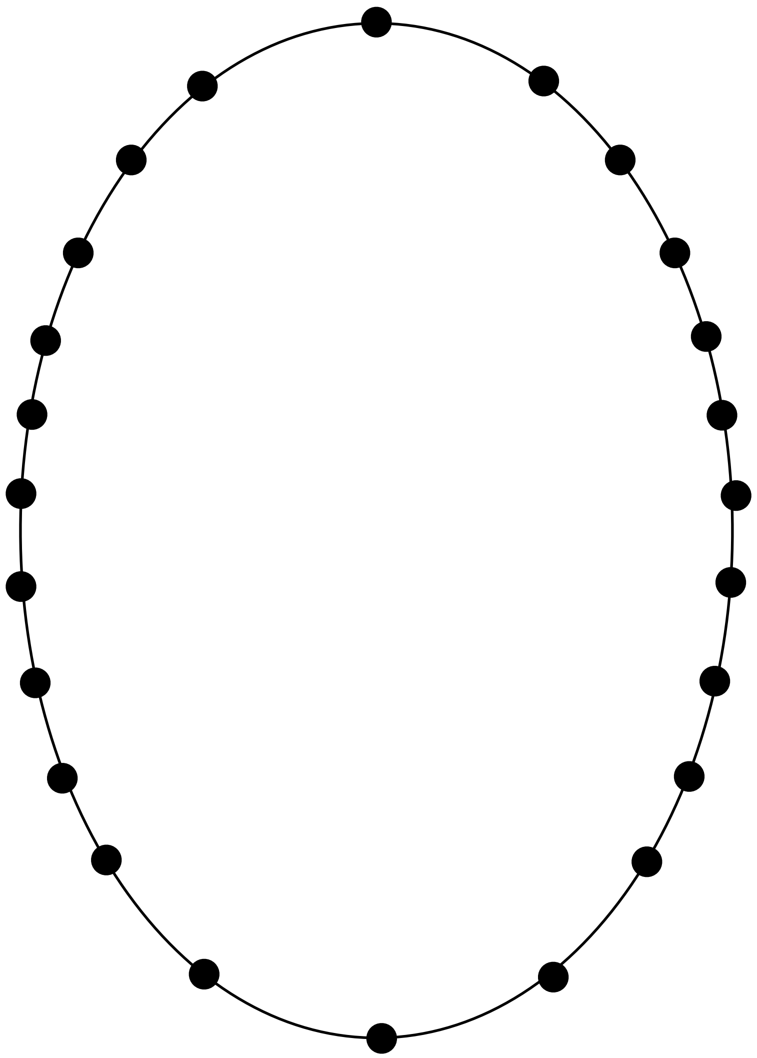
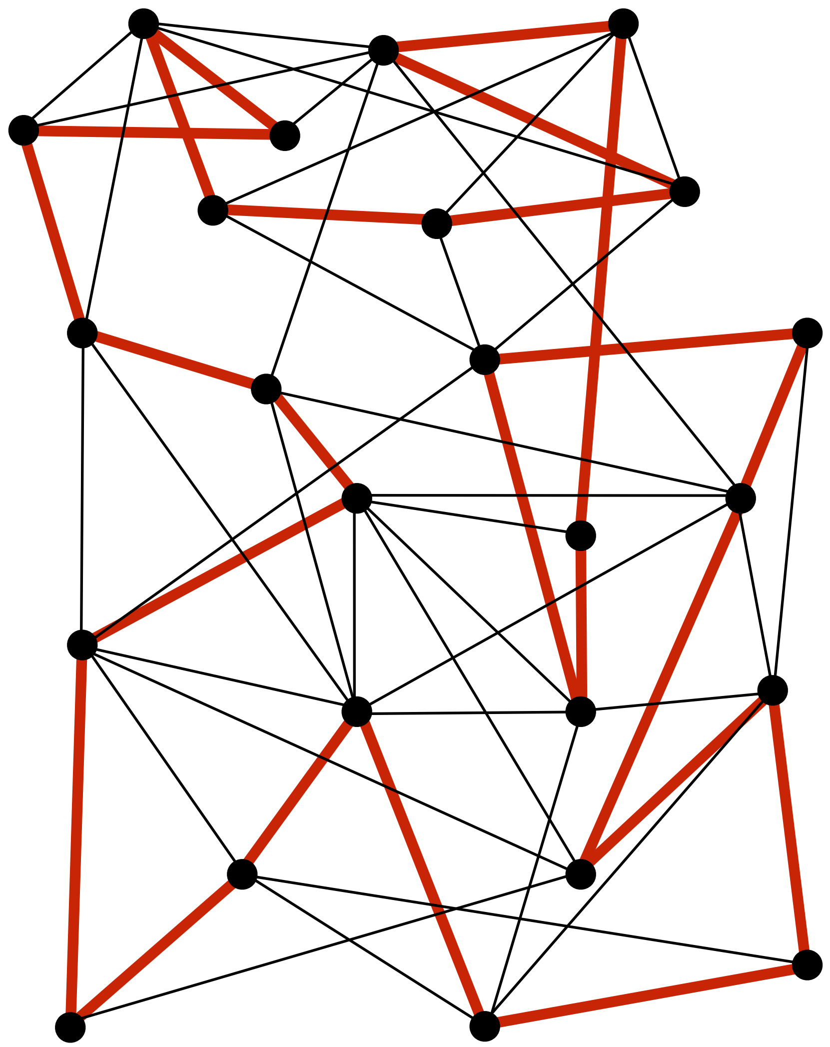
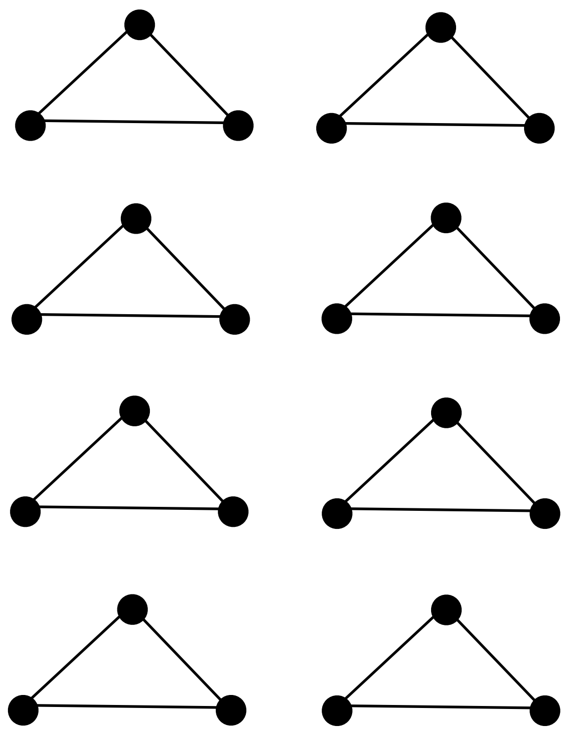
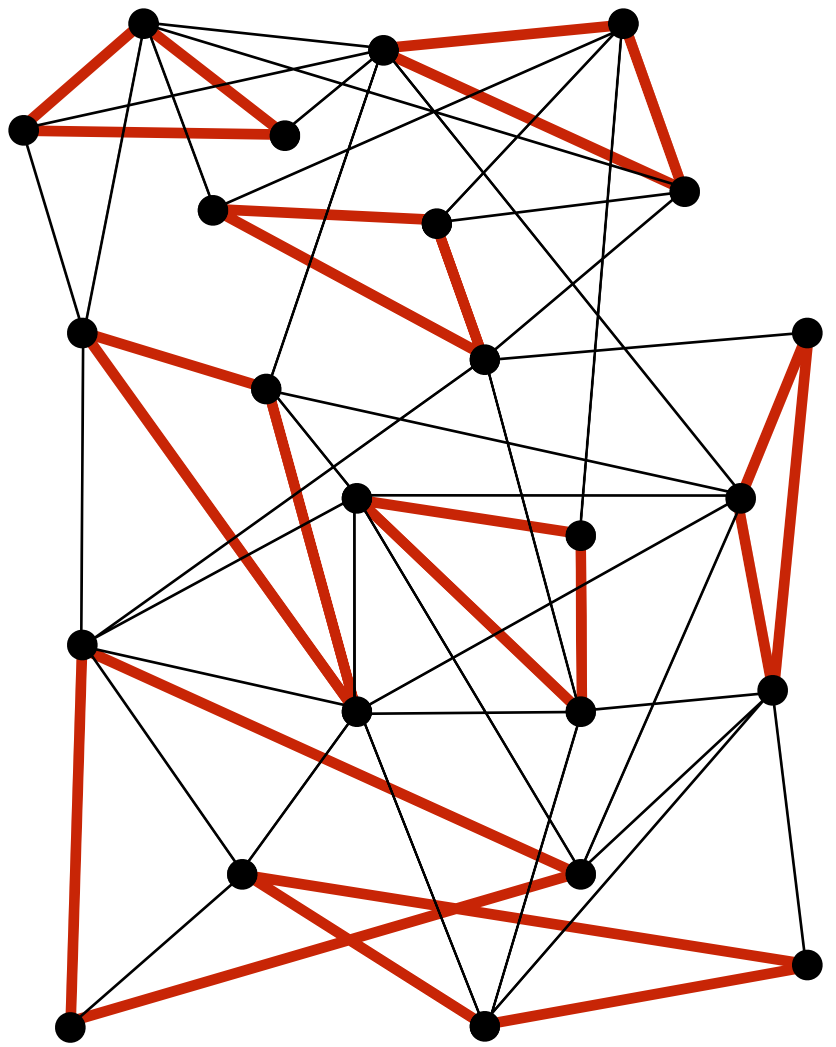
H is a **small** graph





H is **spanning**





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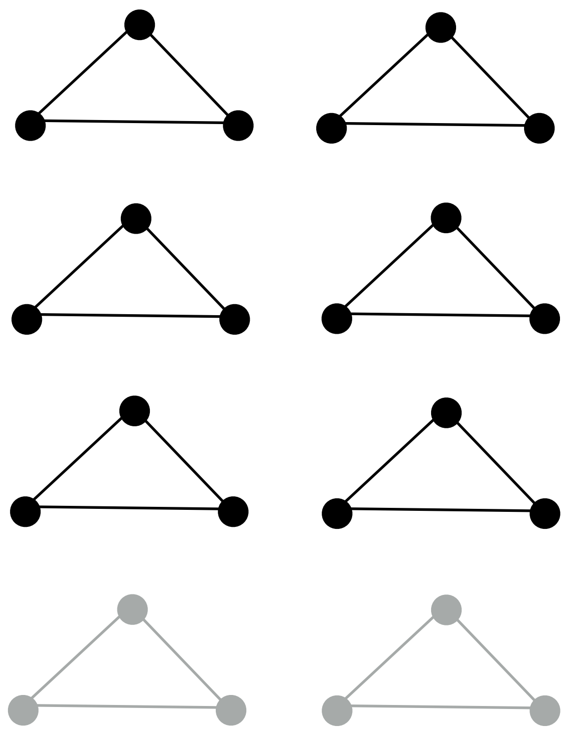
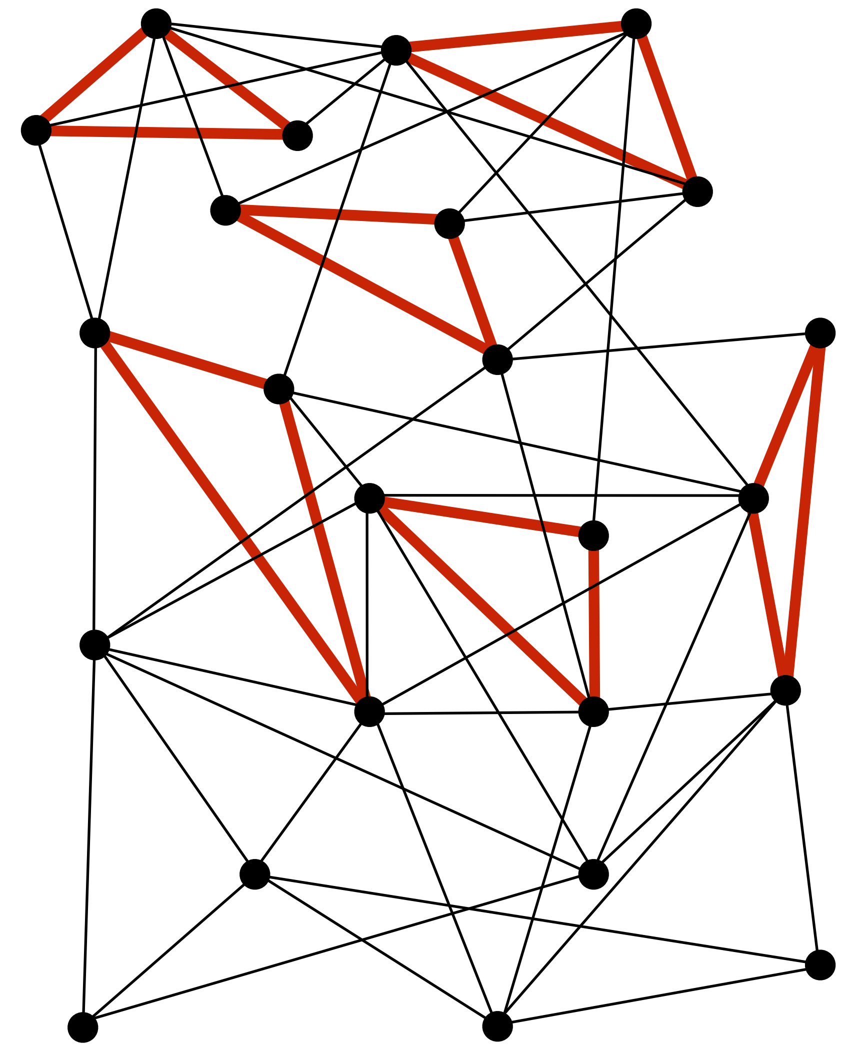


H

(1-ε)**n** vertices



G

n vertices

H is almost-spanning

Universality

Given a family of graphs $\mathcal{H}$, we say that a graph **G** is

$\mathcal{H}$**-universal**

if it contains every graph **H** from $\mathcal{H}$ as a subgraph

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In this talk – $\mathcal{H}_{n,D}$ is a family of all graphs with

- **n** vertices
- **maximum degree** at most **D**

Universality in random graphs

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Q. Given **D**, for which **p** do we have

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not the same!!!

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  - $p > n^{-(1 - 1/D)\,2/(D + 1)}$

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- Johannson, Kahn, Vu ('08)
  — K_{D+1}-factor, $p > n^{-2/(D+1)}$
- Ferber, Luh, Nguyen ('17)
  — almost-spanning, $p > n^{-2/(D+1)}$

Results - universality

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Further related results

- Kohayakawa, Rödl, Schacht, Szemerédi ('11)
  — **size-Ramsey number** of bounded-degree graphs = $\mathbf{O(n^{2-1/D})}$
- Allen, Böttcher, Hàn, Kohayakawa, Person ('17+)
  — **blow-up lemma** for random graphs **[**$p > n^{-1/D}$**]**
- Allen, Böttcher, Ehrenmüller, Taraz ('17+)
  — **Bandwidth Theorem** for random graphs **[**$p > n^{-1/D}$**]**

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  Conlon, **N.** ('17+)* : $O(n^{2-1/D-\varepsilon})$

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<u>Believed to be the truth:</u>

$$p > n^{-2/(D+1)}$$

(needed for K_{D+1}-factor)

$p > n^{-1/D}$

For $p > n^{-1/D}$ **every subset** of at most **D vertices** has many **common neighbours**

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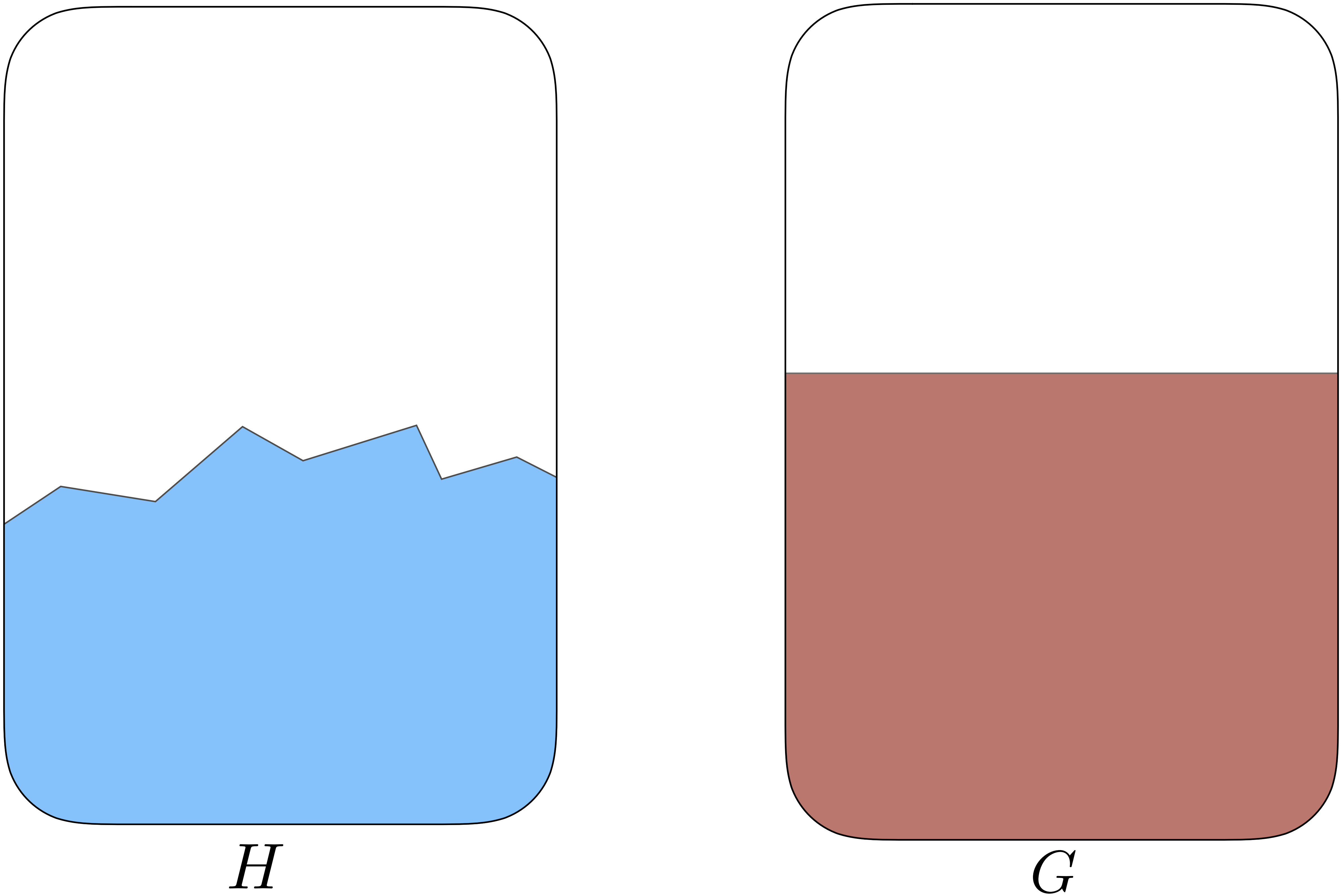

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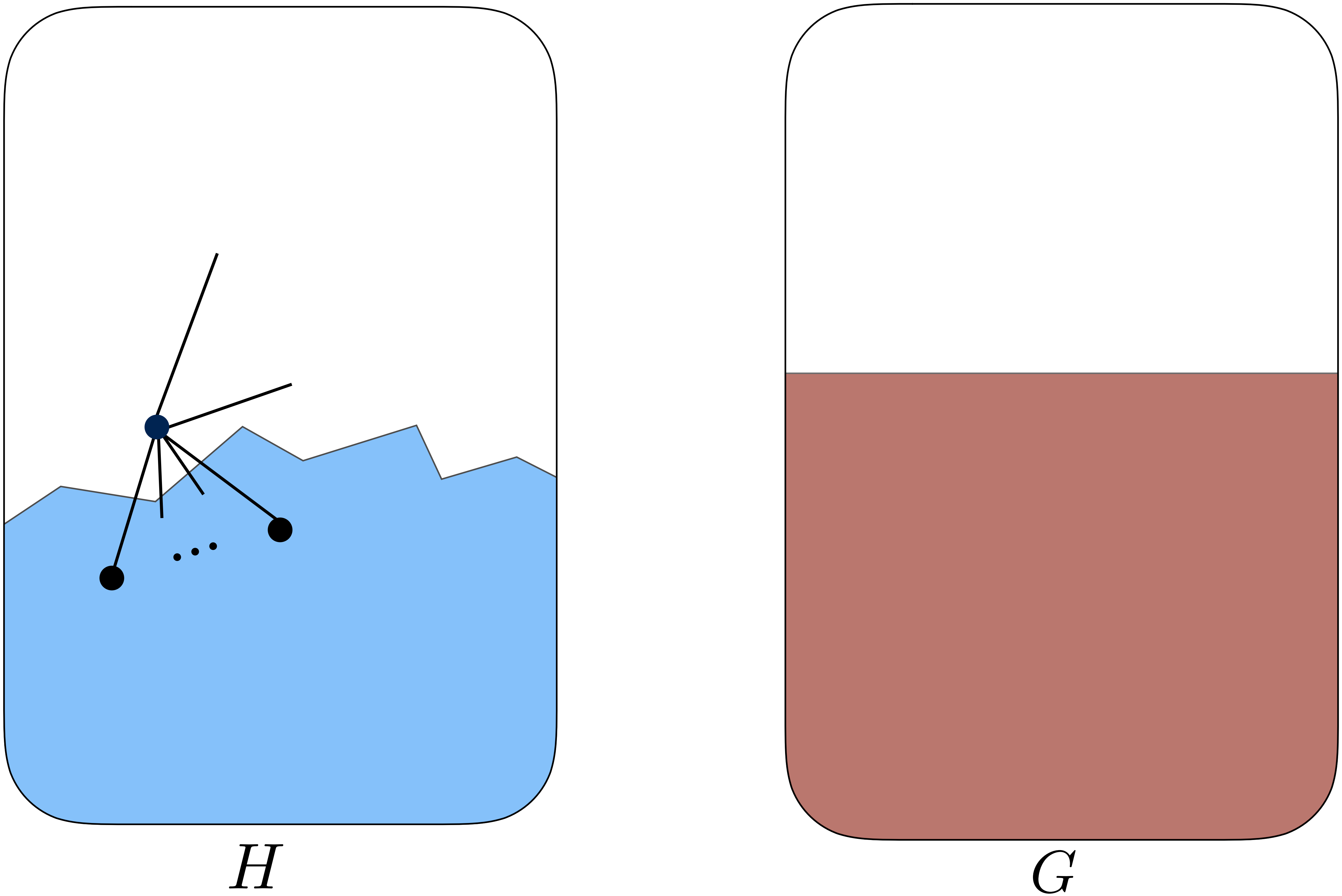

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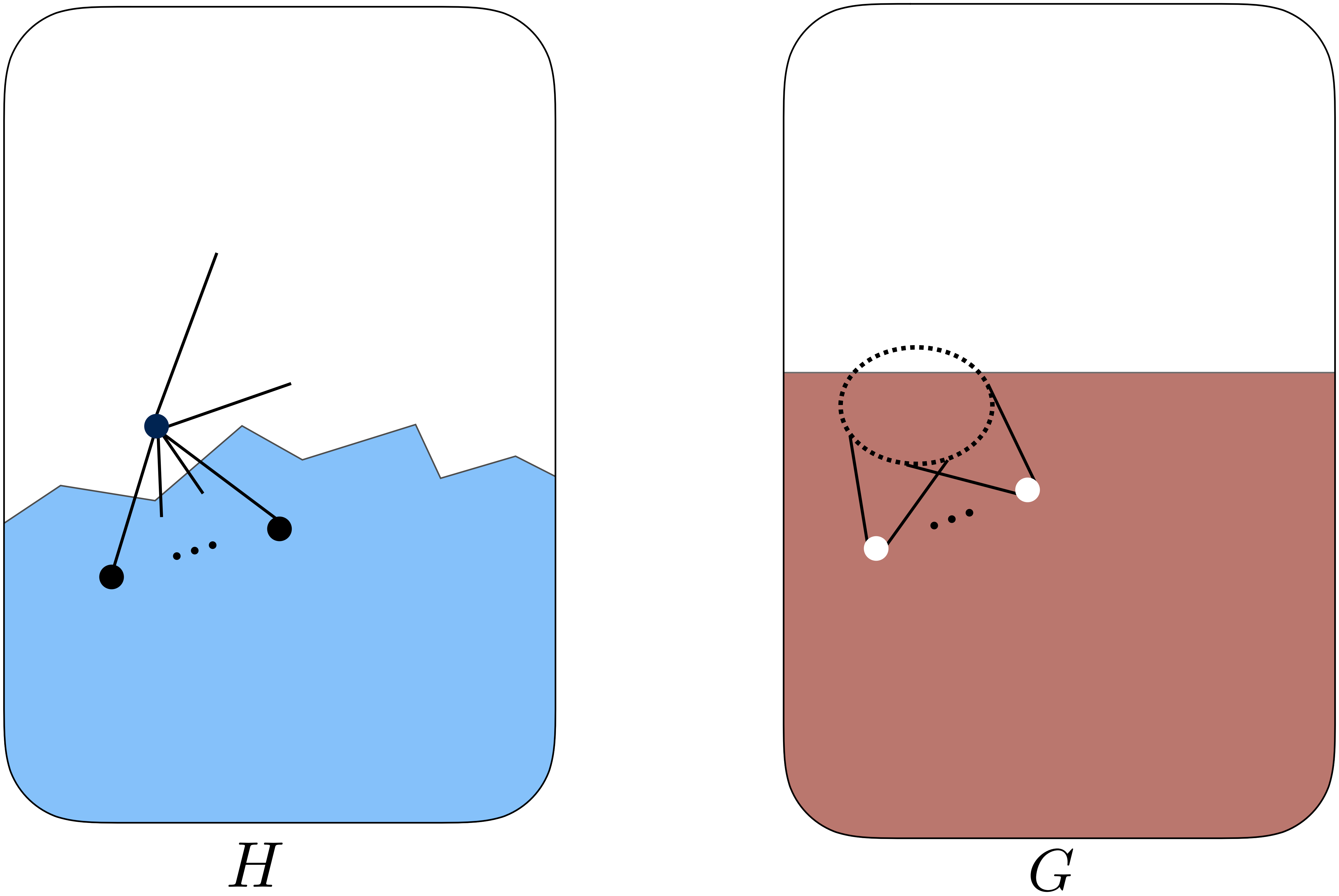

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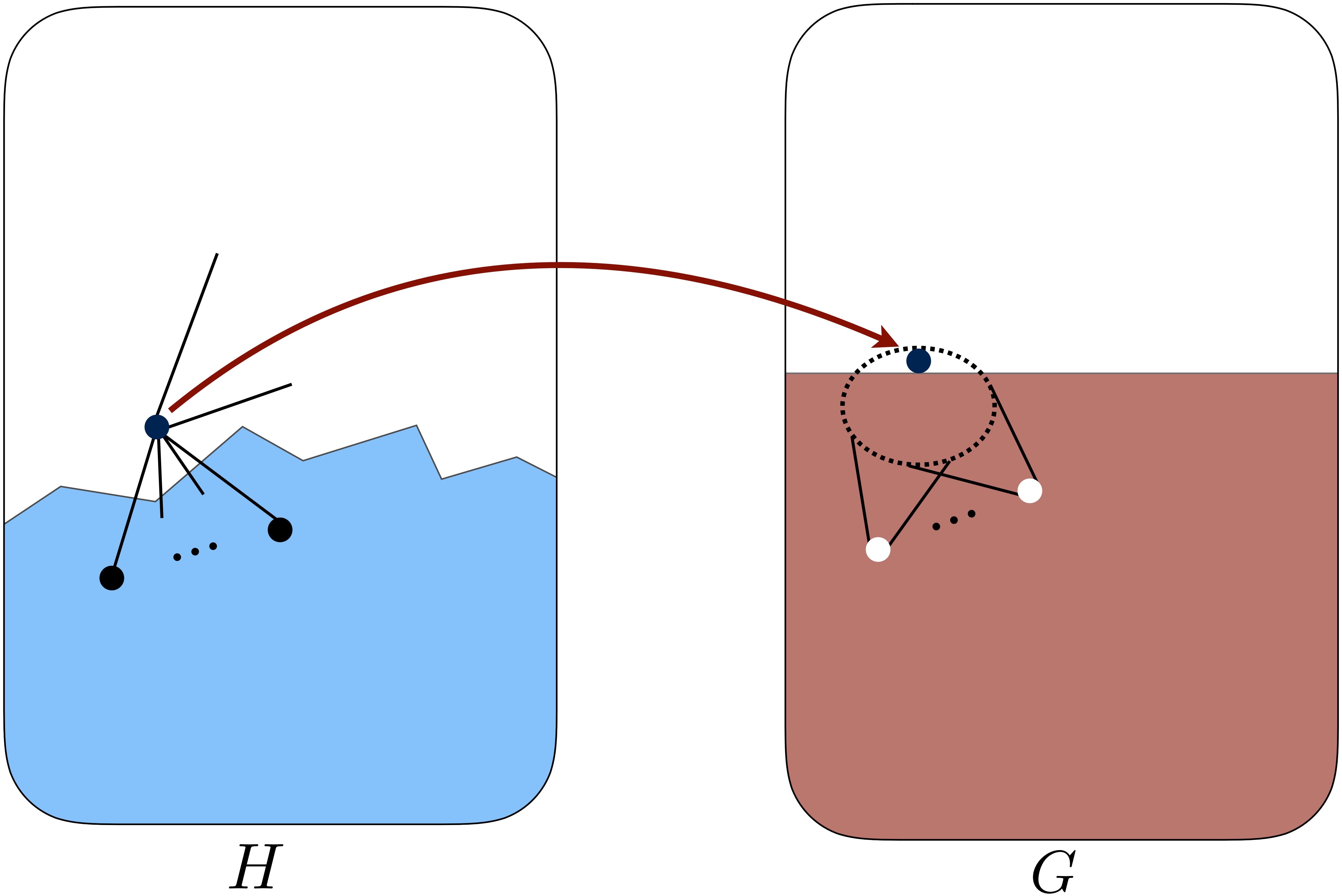

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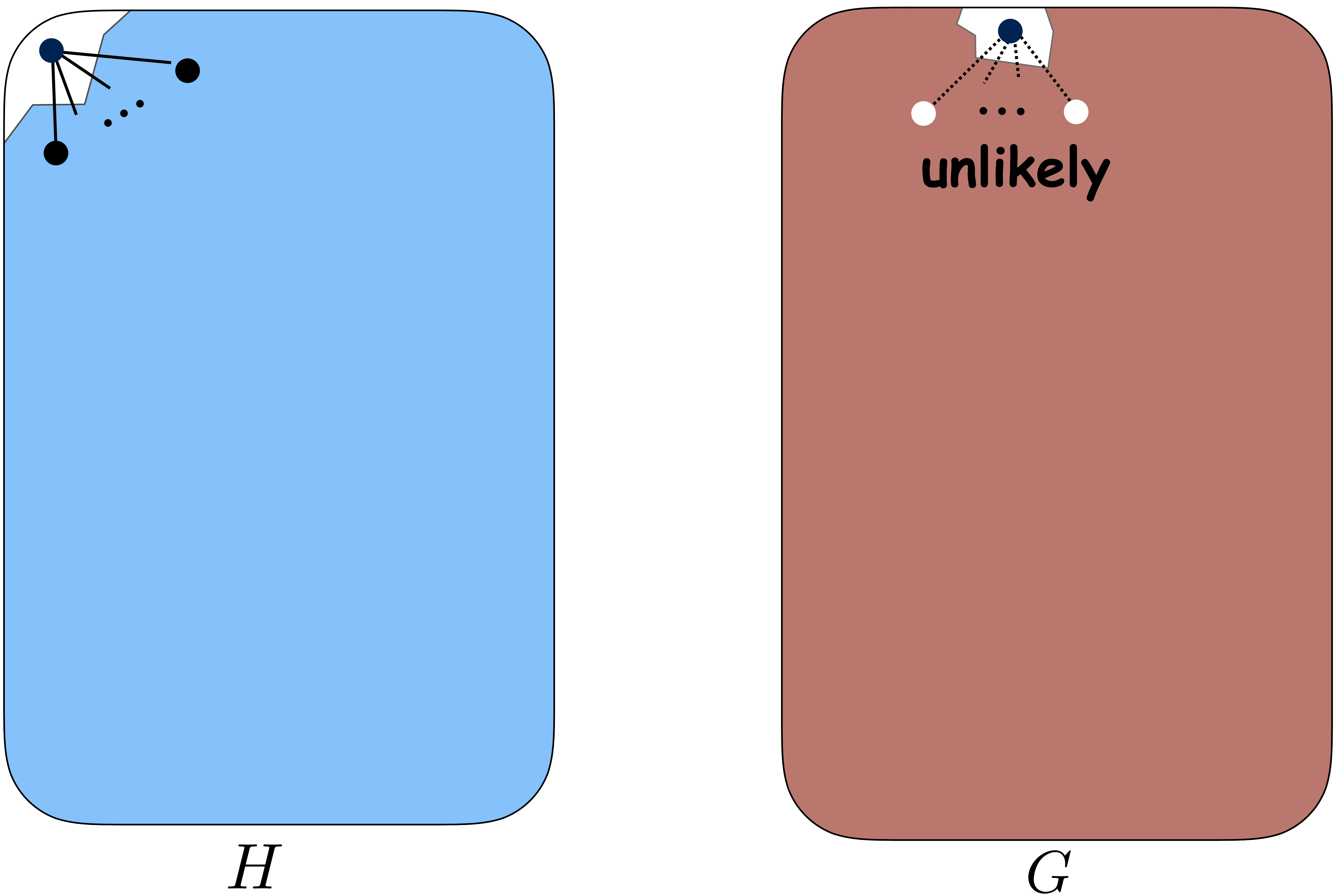

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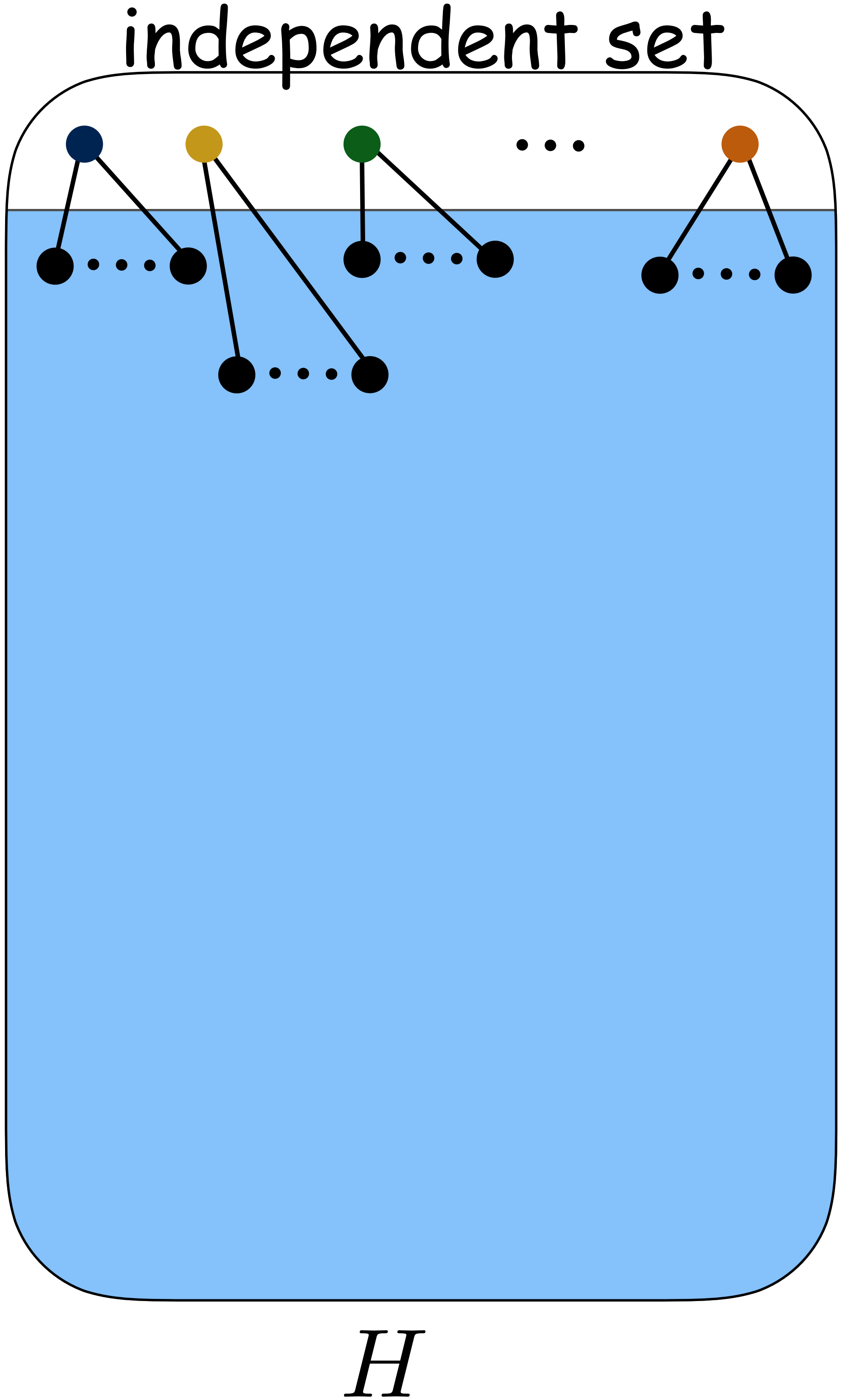

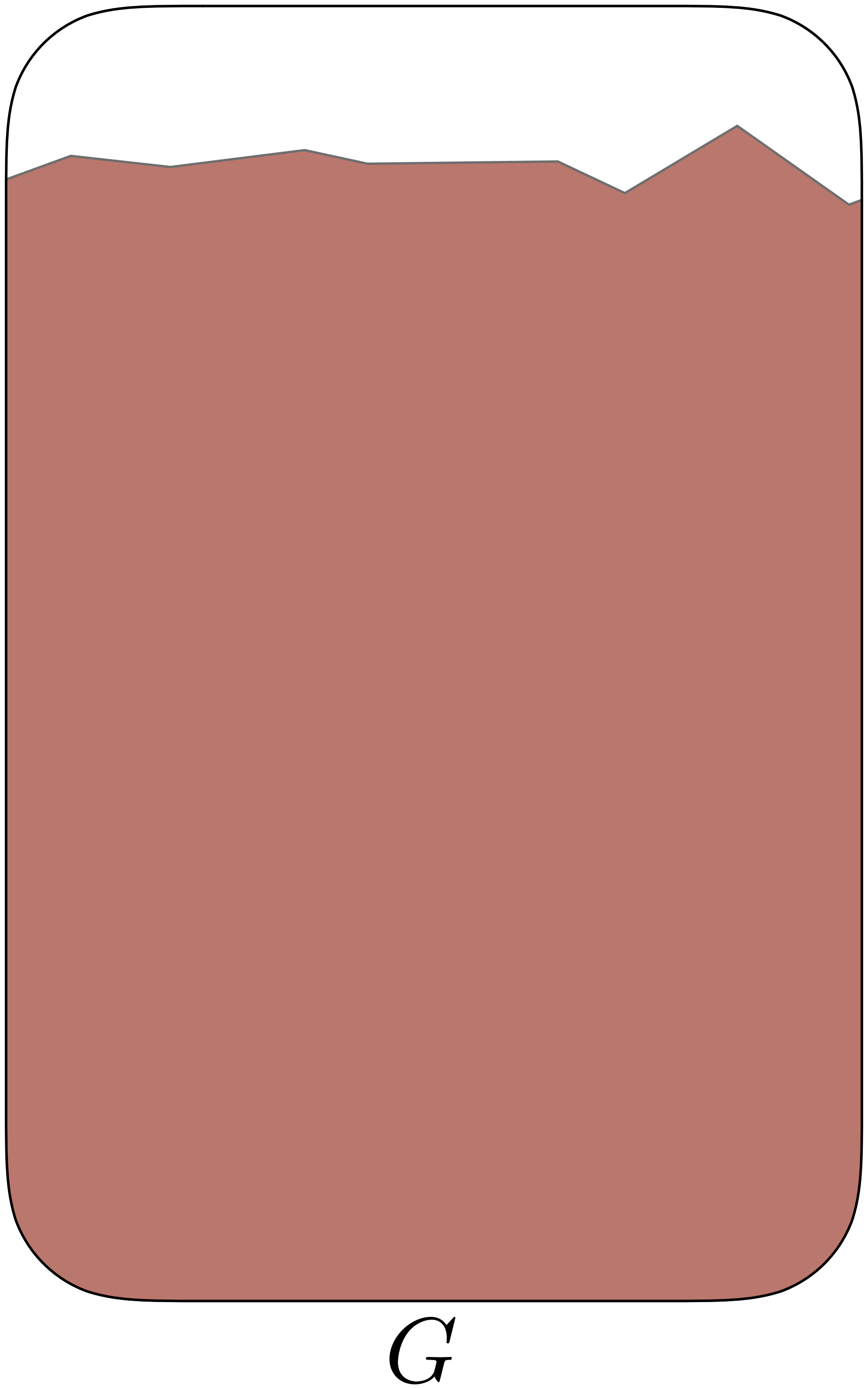

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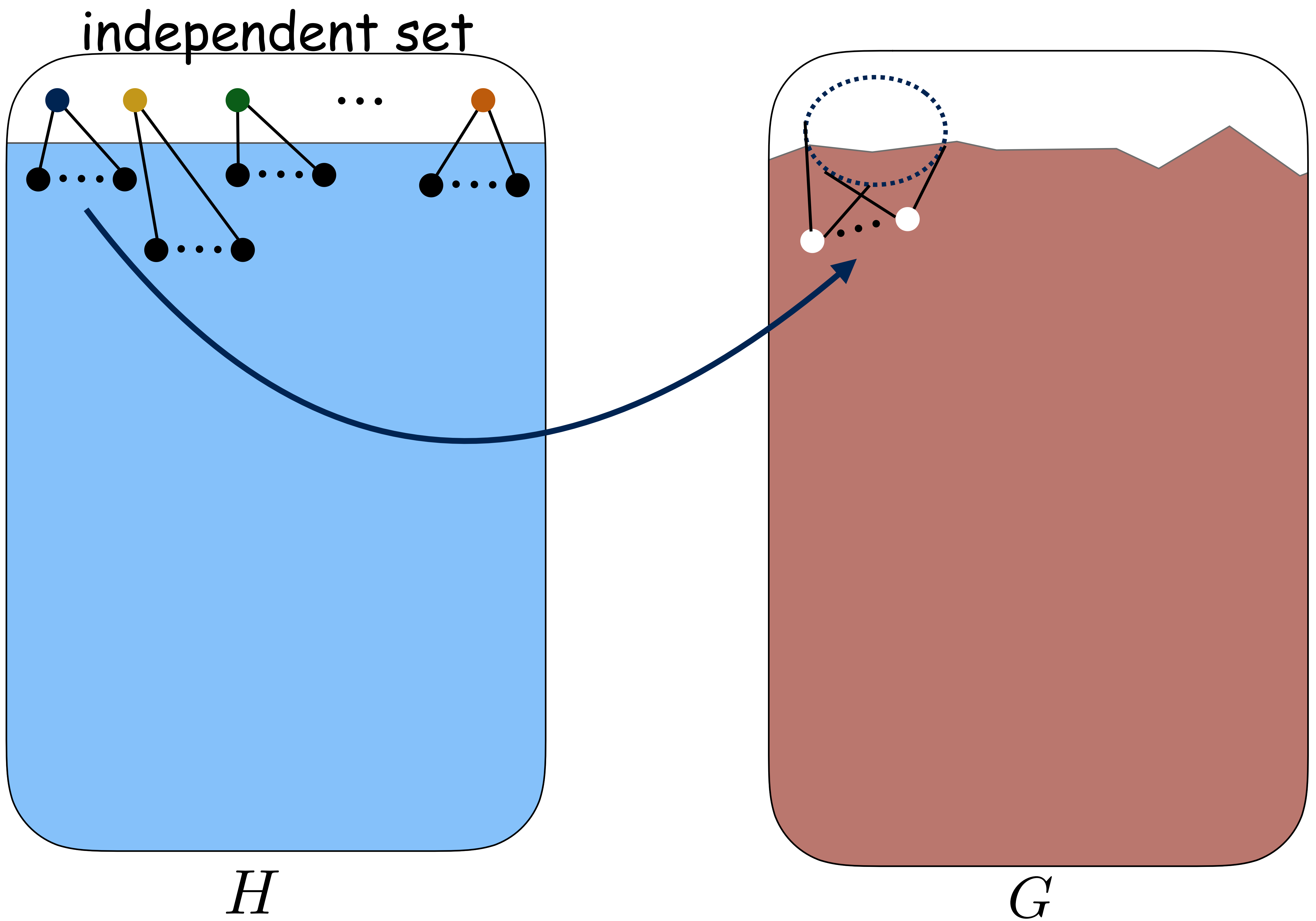

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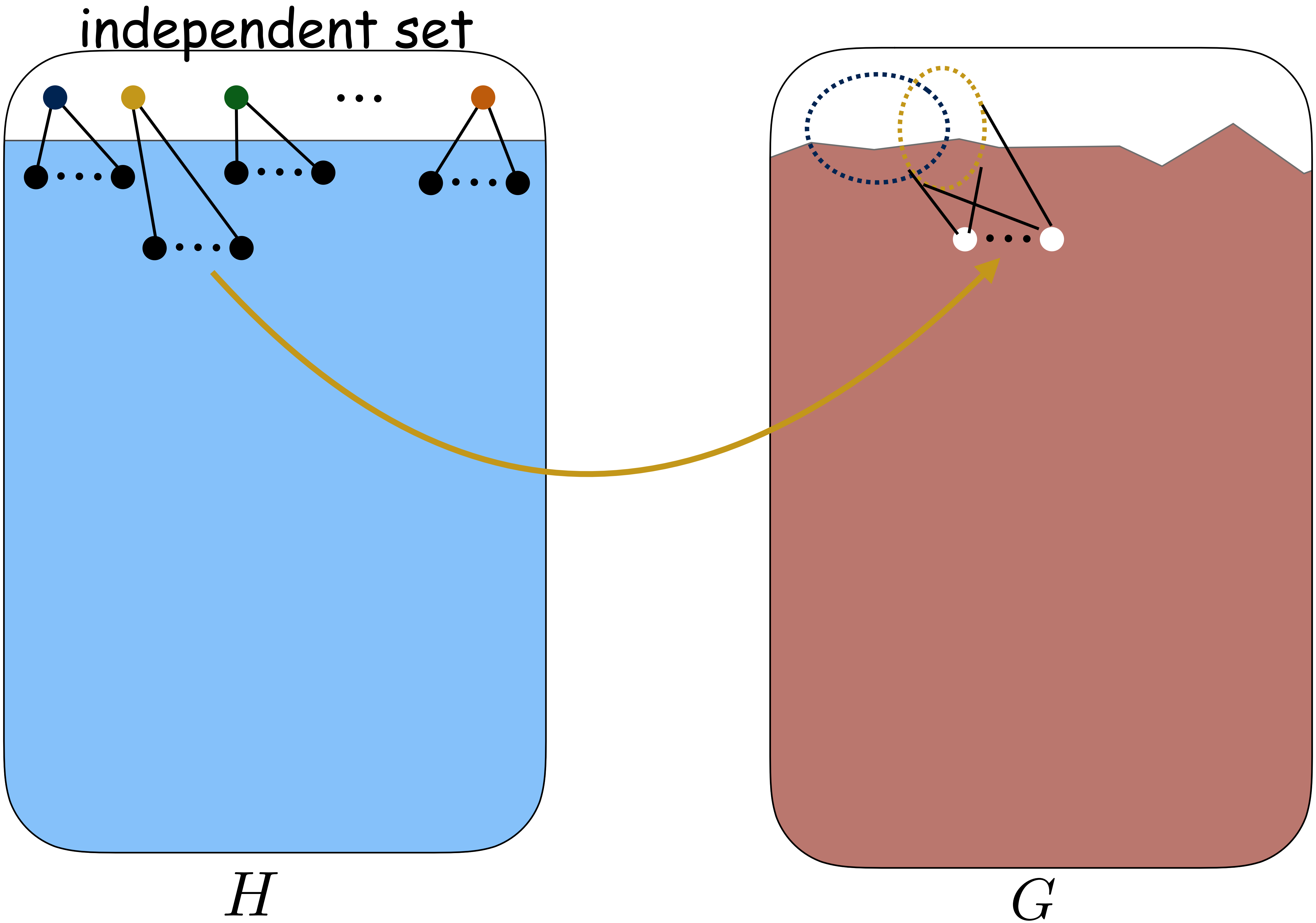

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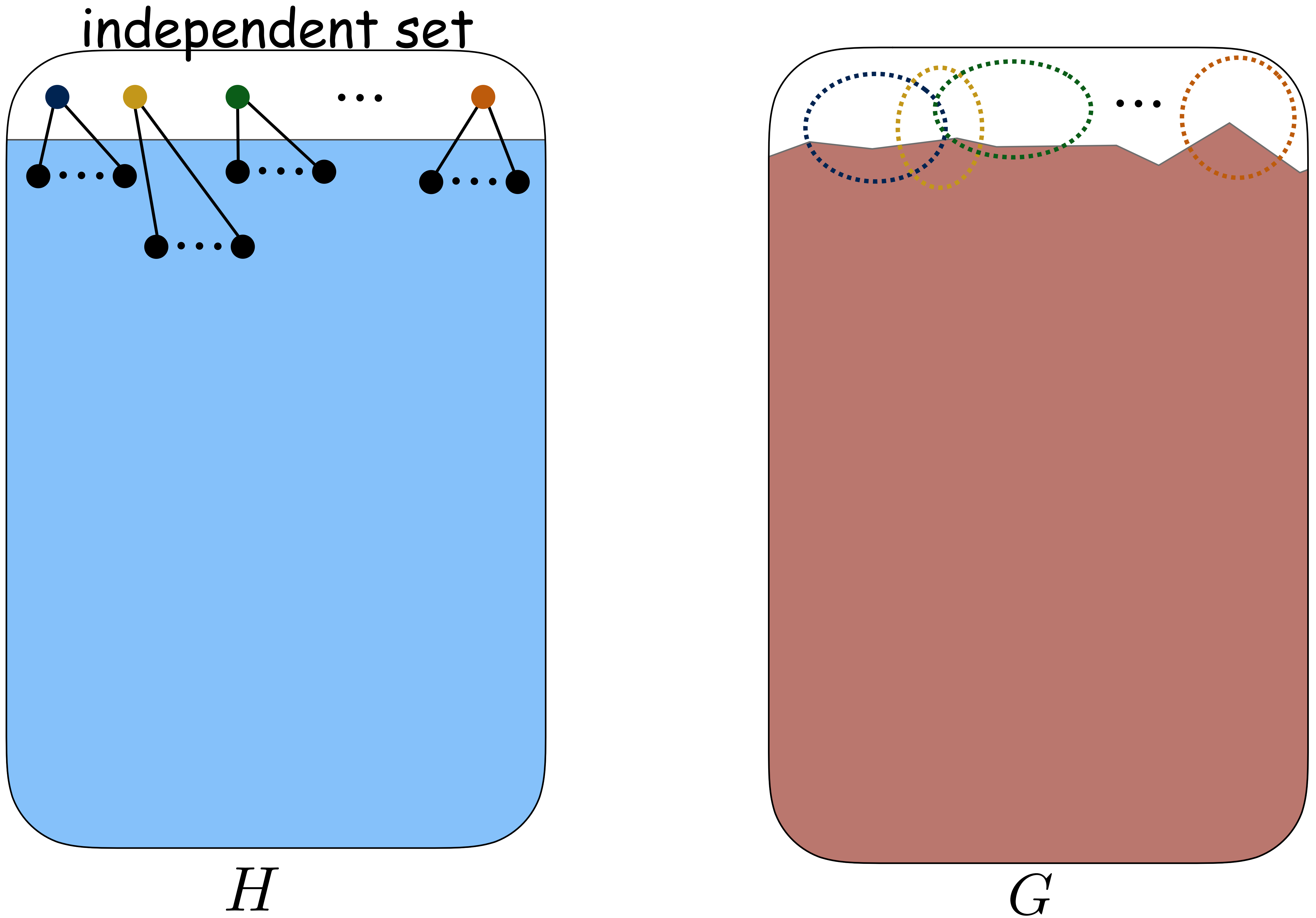

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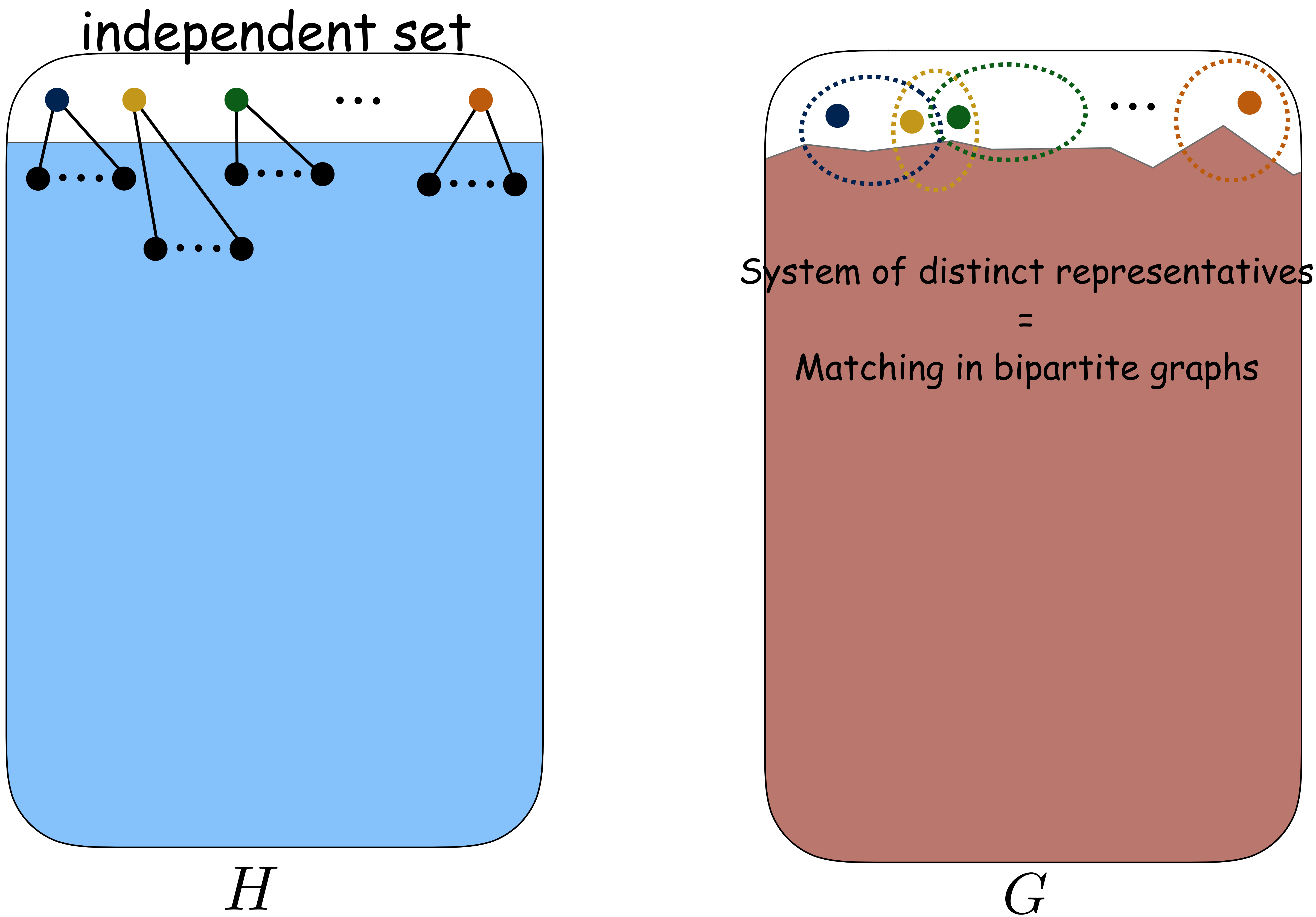

$$n^{-\varepsilon - 1/D} < p < n^{-1/D}$$

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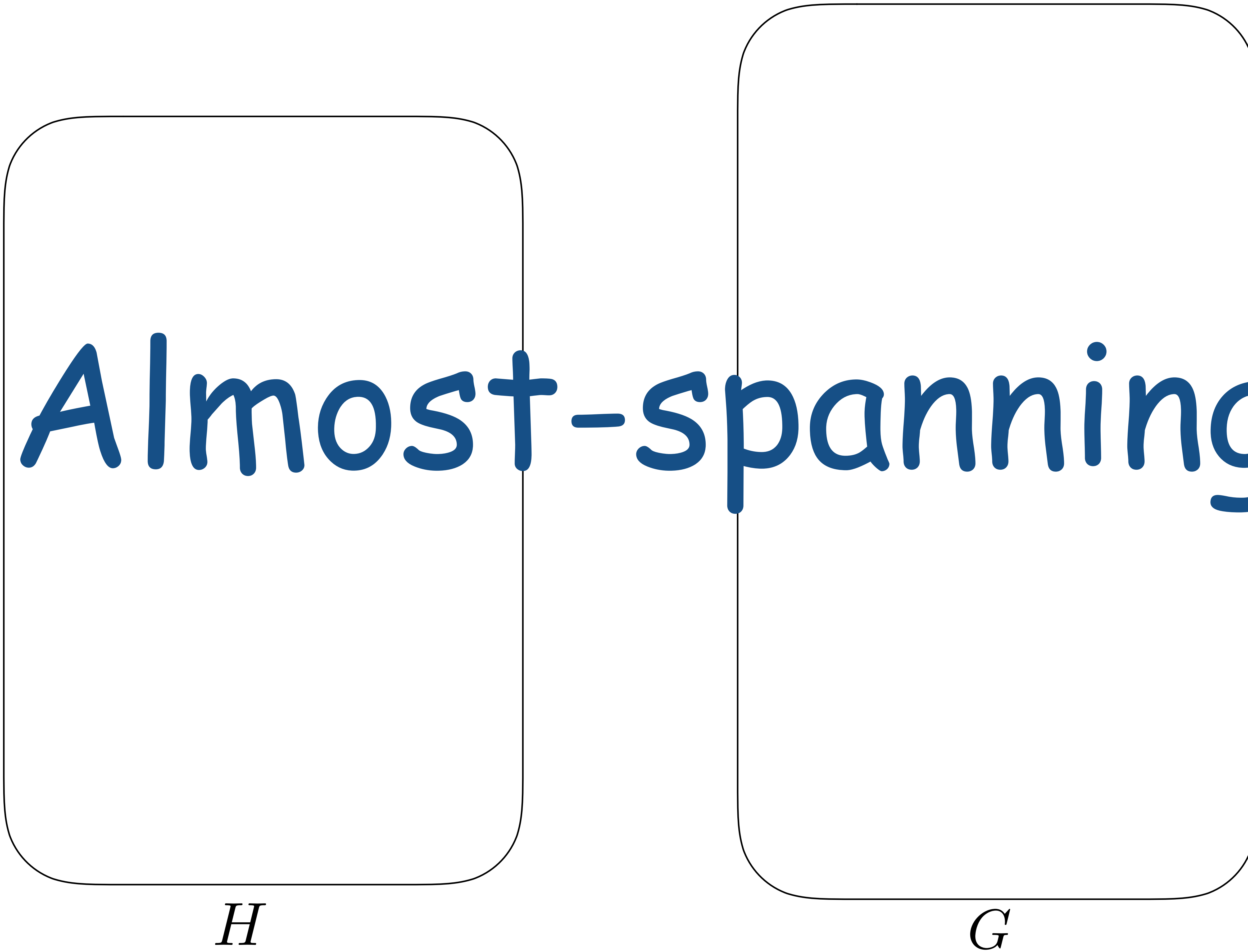
Almost-spanning



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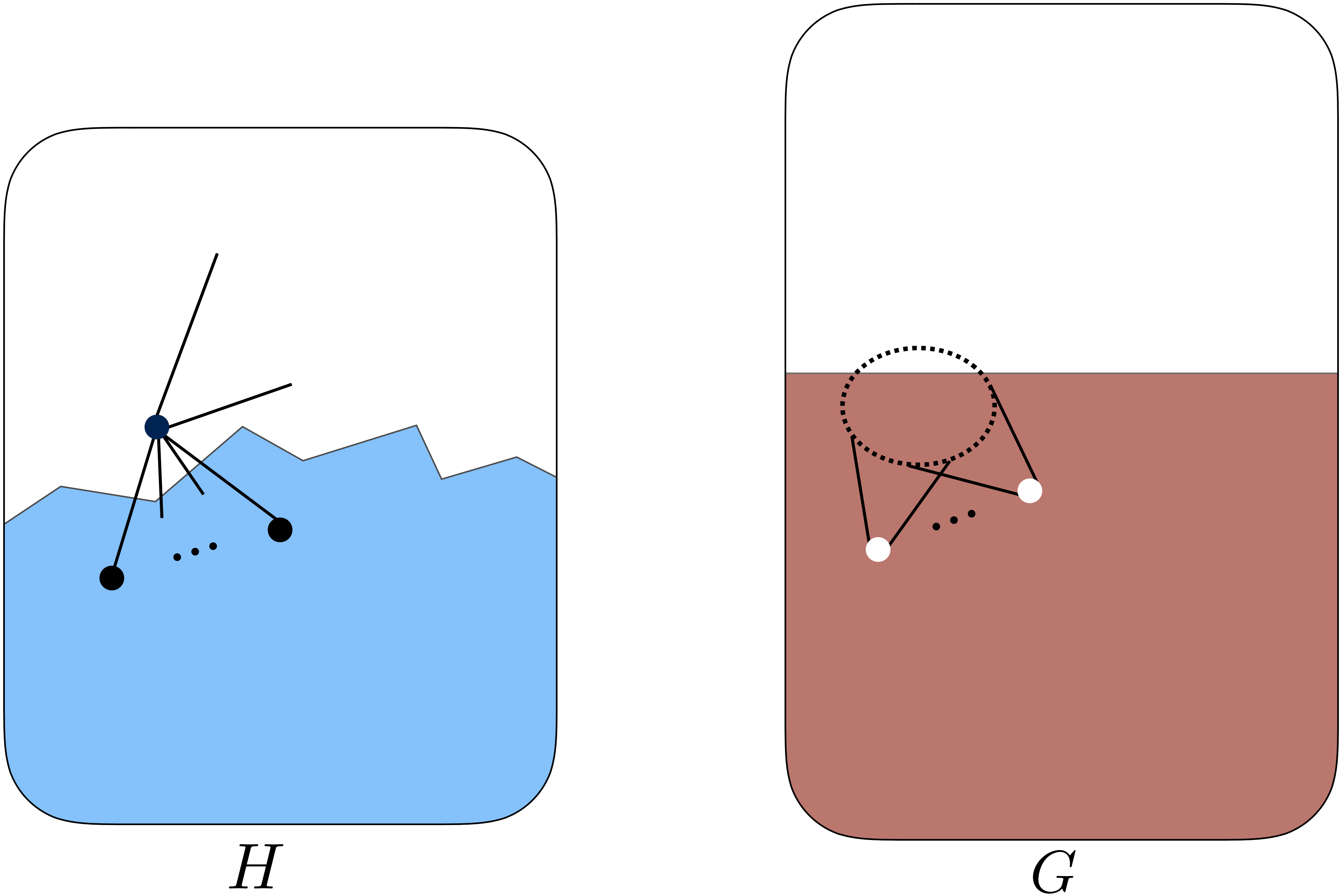

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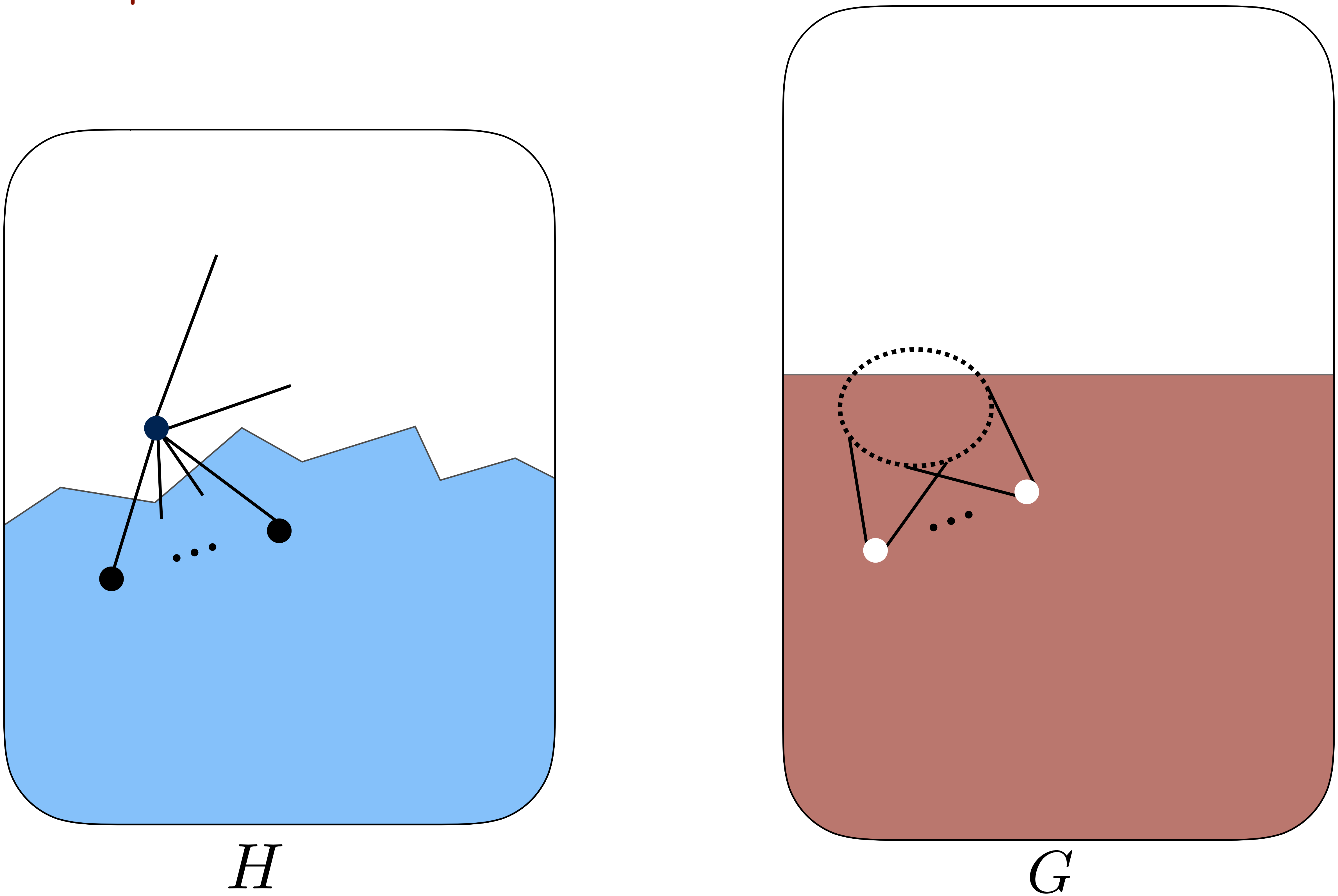

For ~~$p > n^{-1/D}$~~ **not!** **every subset** of at most **D vertices** has many **common neighbours**

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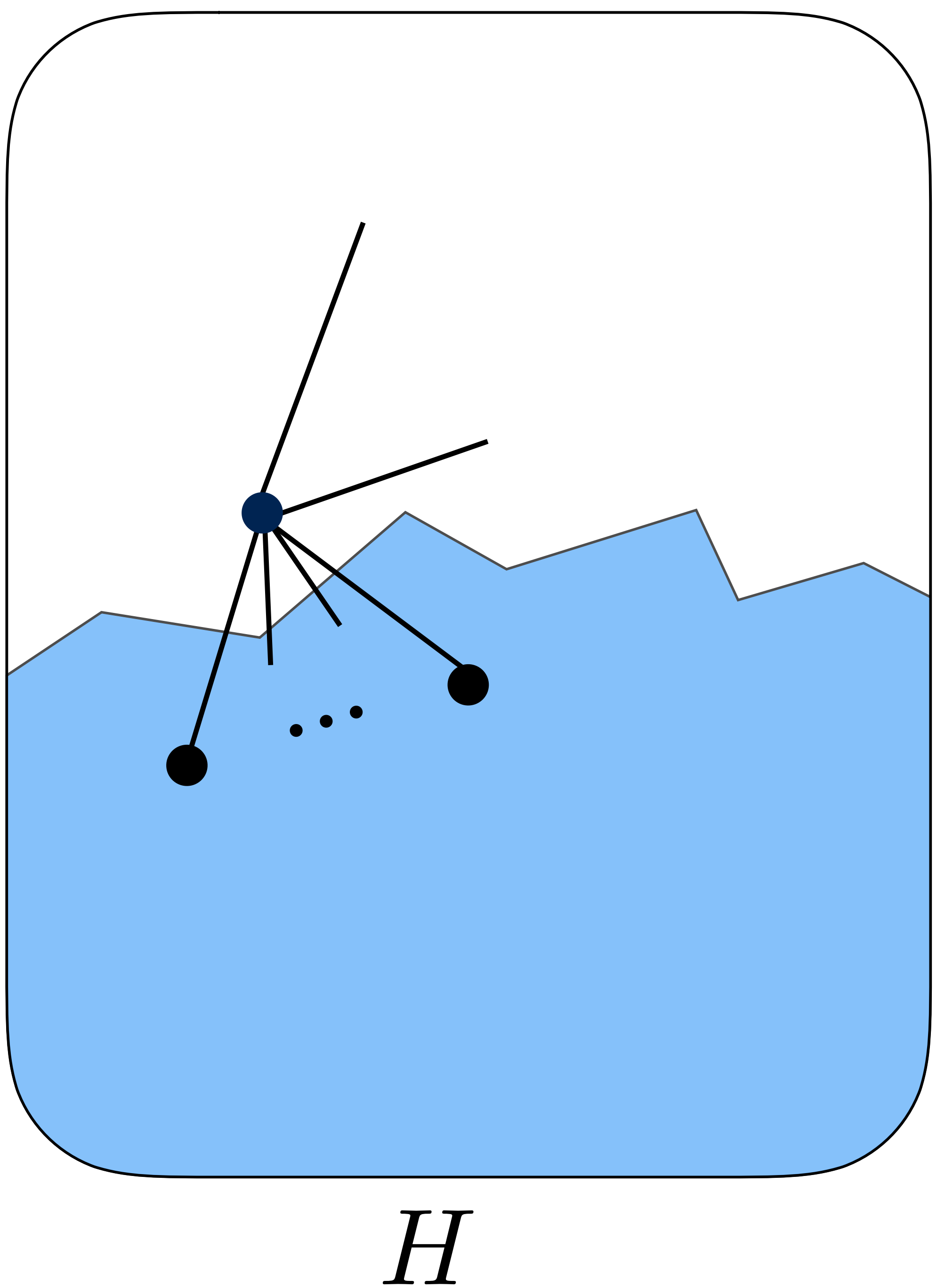
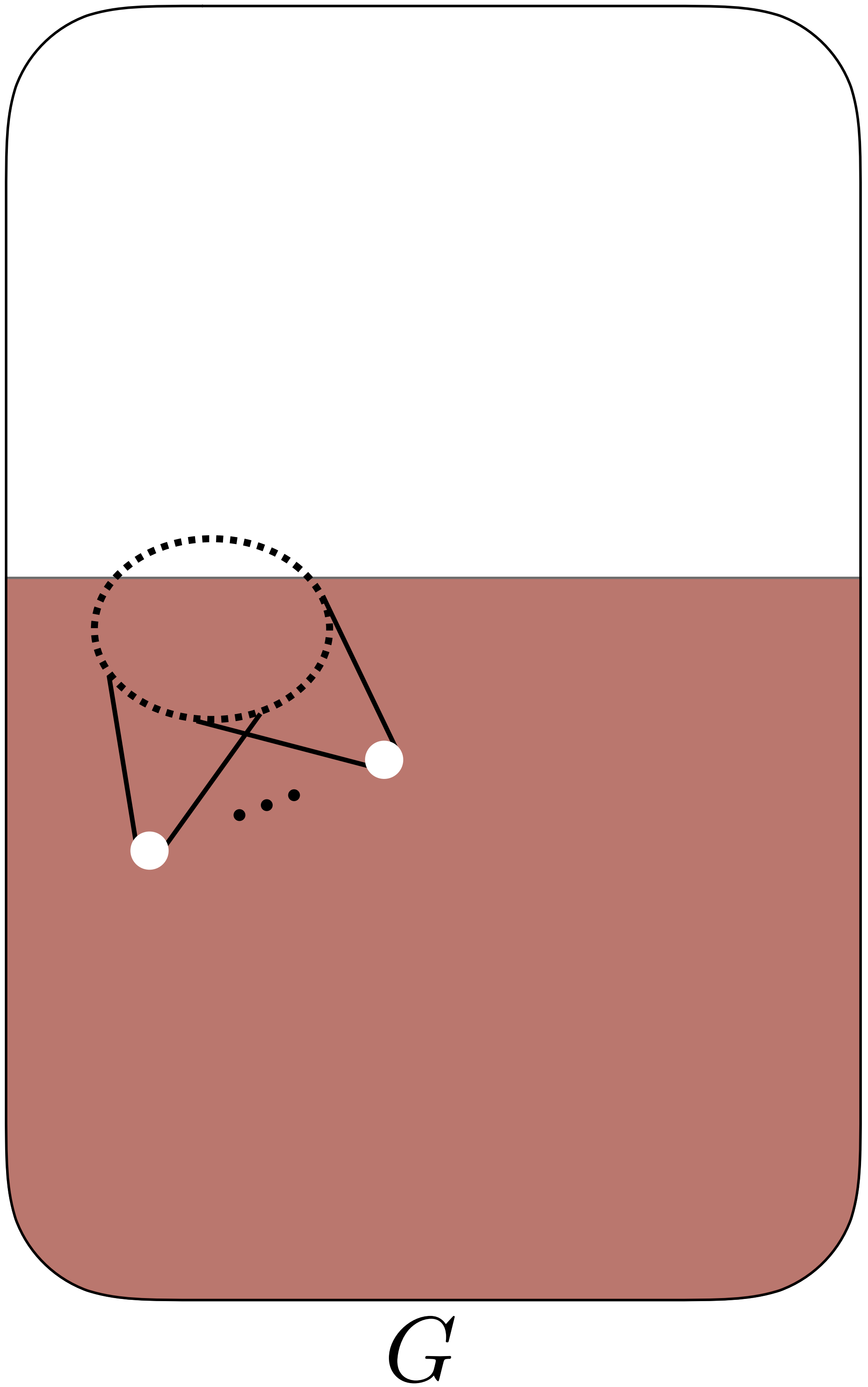
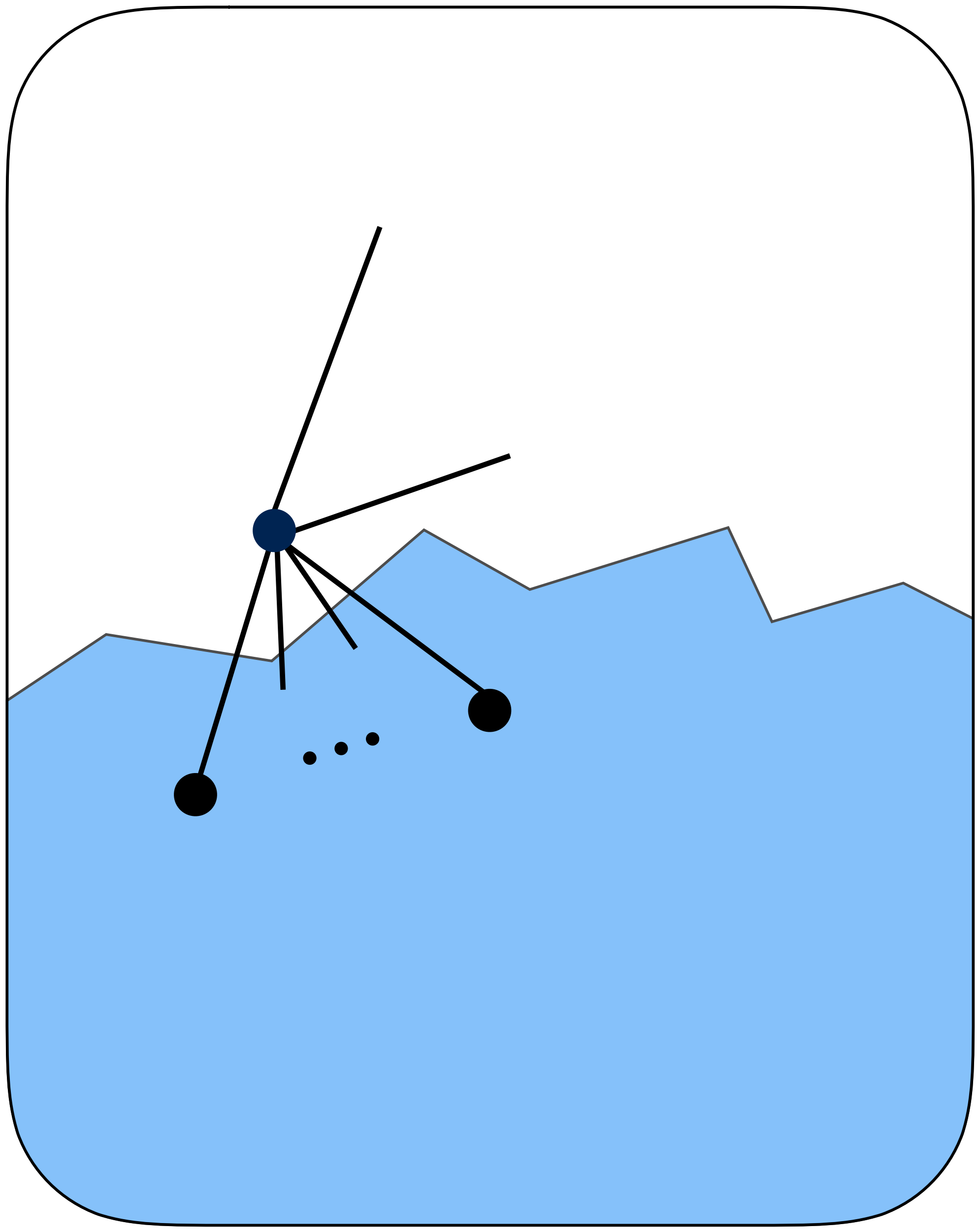




For ~~$p > n^{-1/D}$~~ **every subset** of at most **D-1** **vertices** has many **common neighbours**

$n^{-\varepsilon - 1/D} < p < n^{-1/D}$



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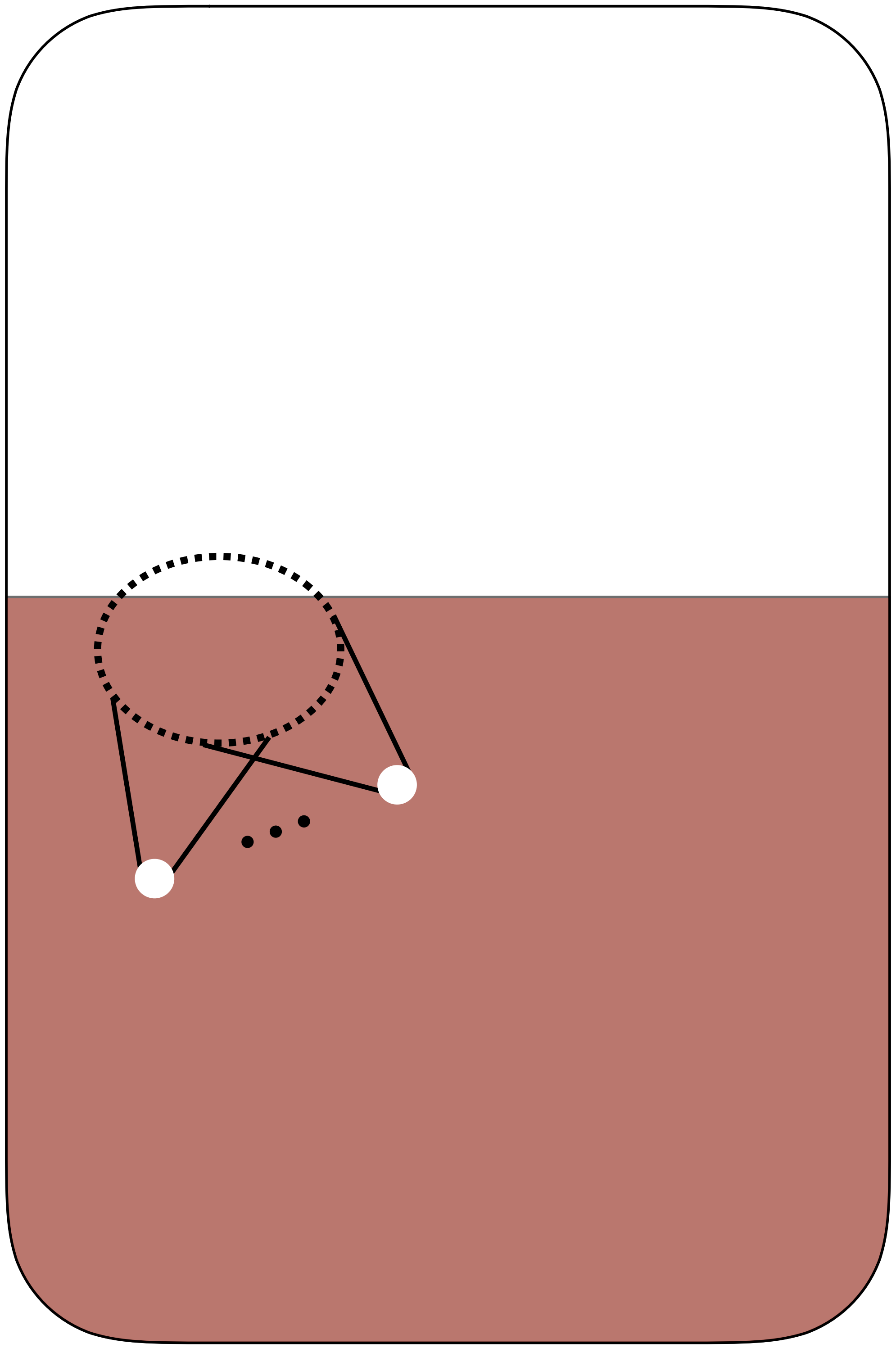

G

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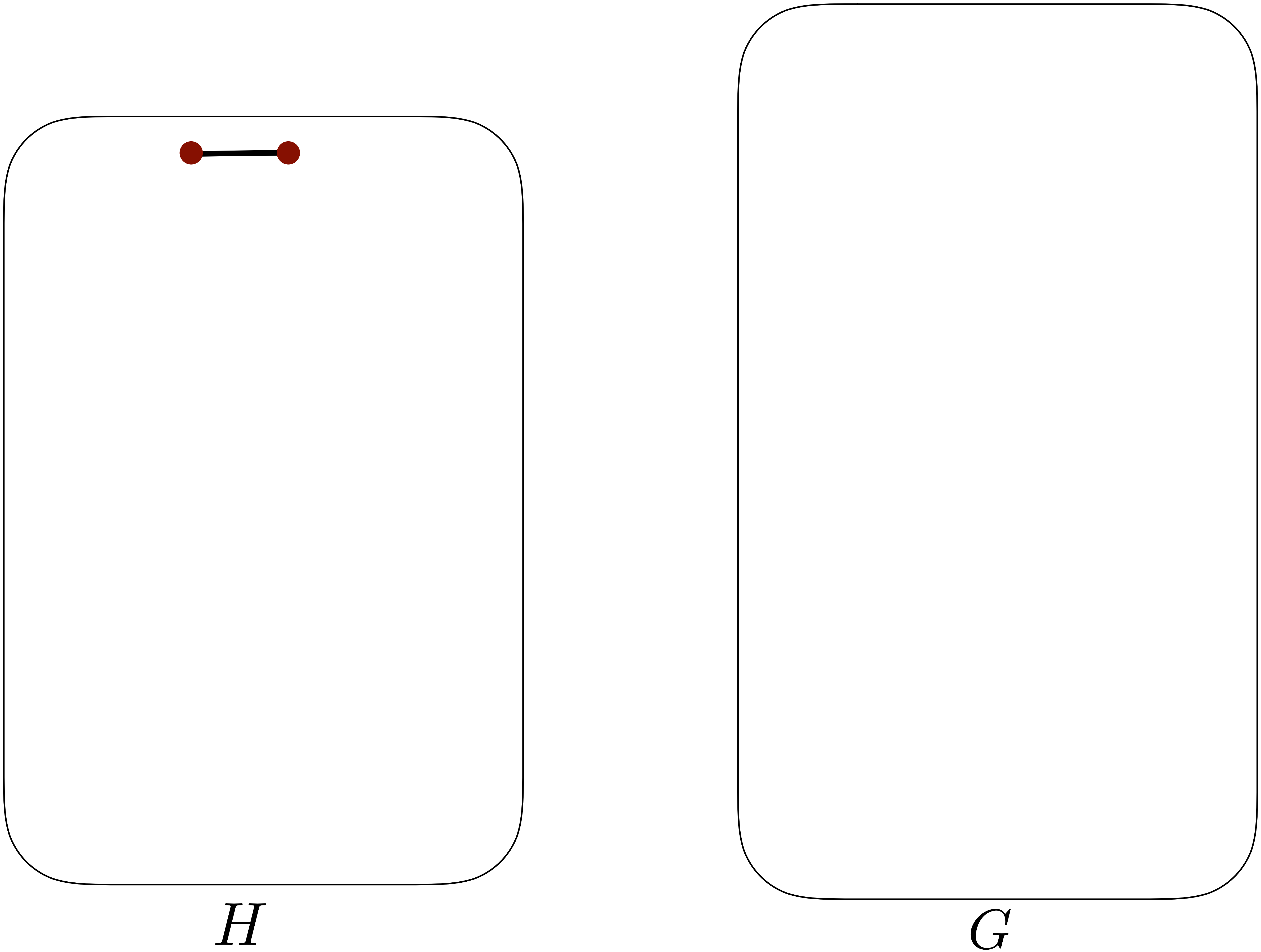

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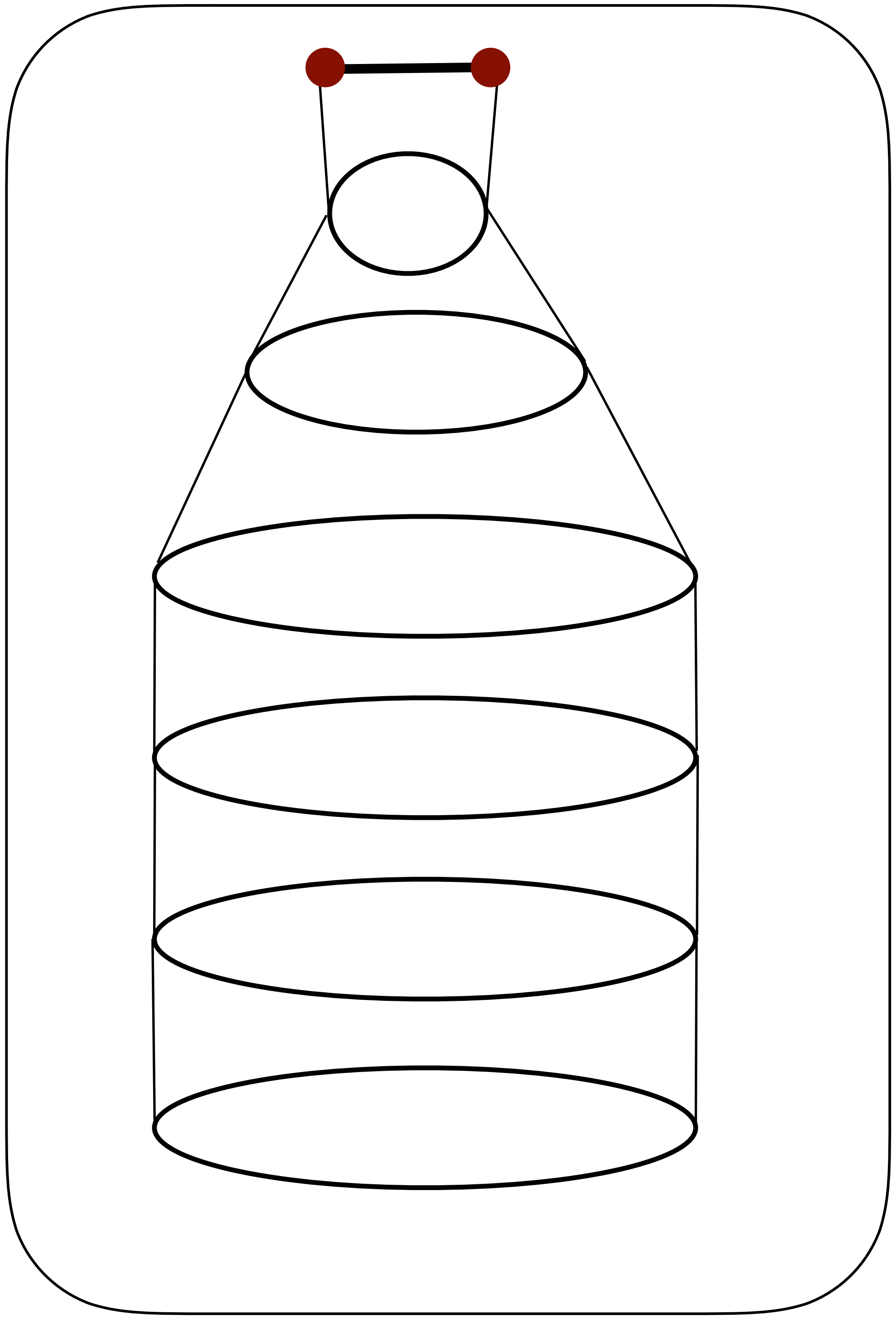

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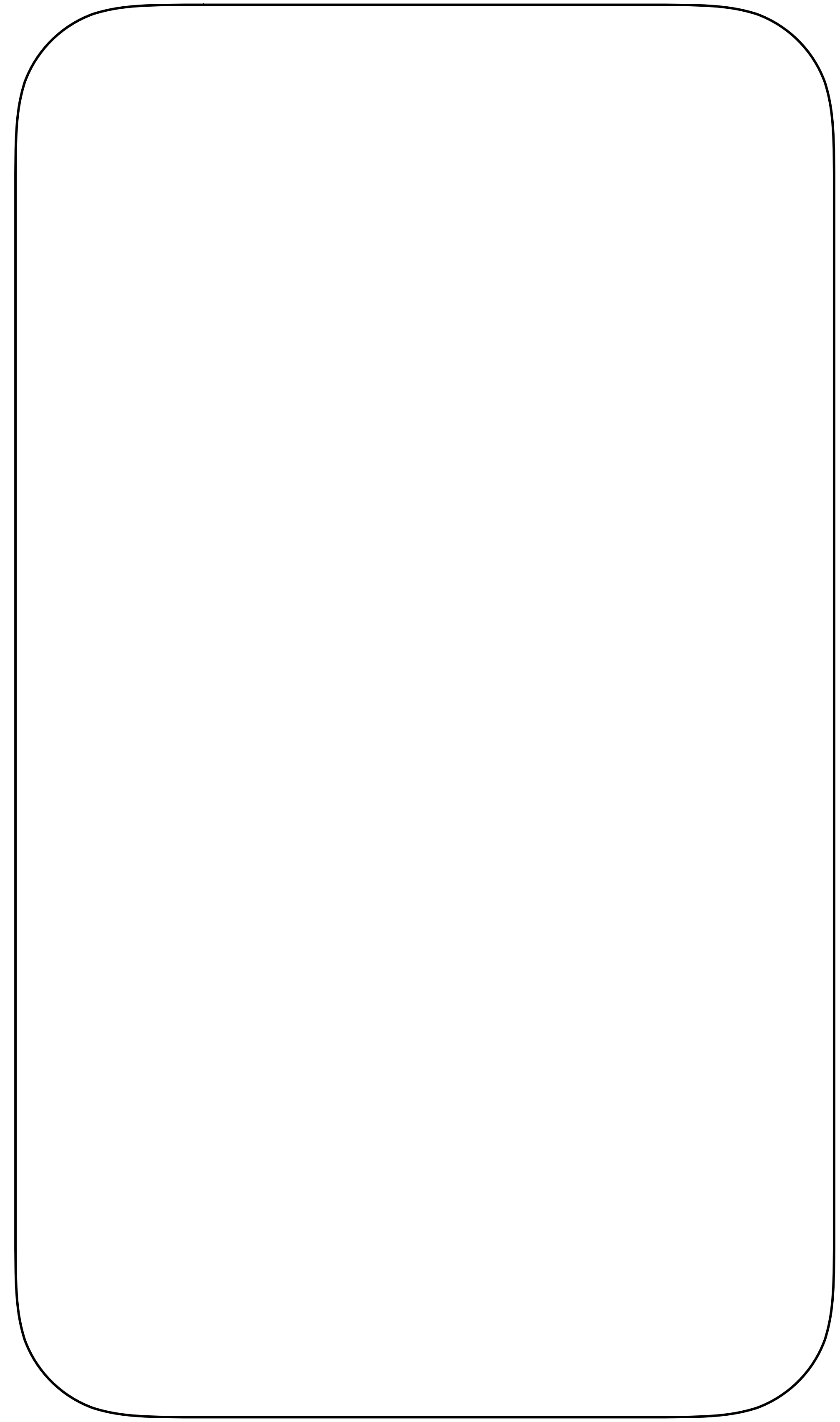
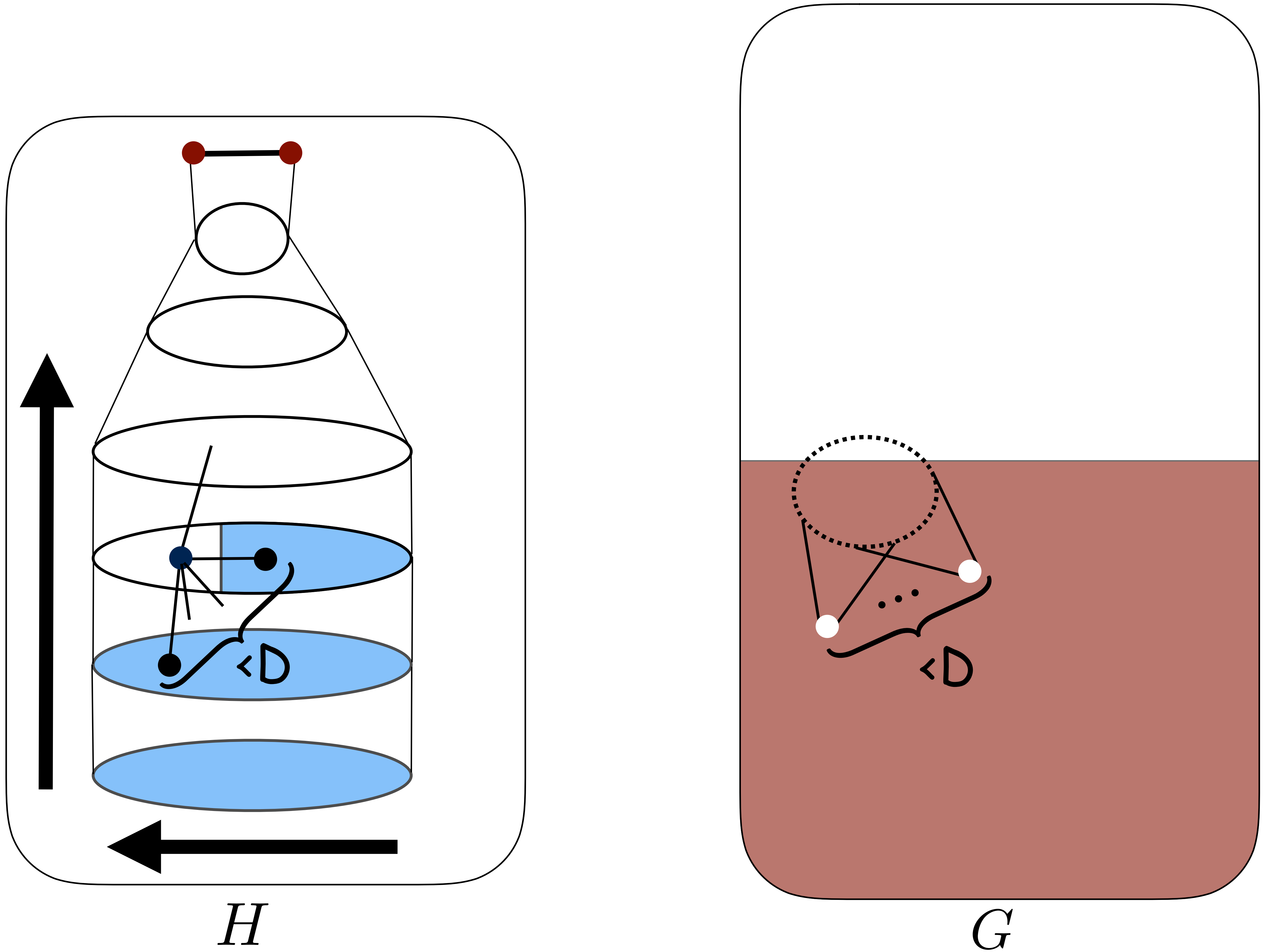


G

$$n^{-\varepsilon - 1/D} < p < n^{-1/D}$$



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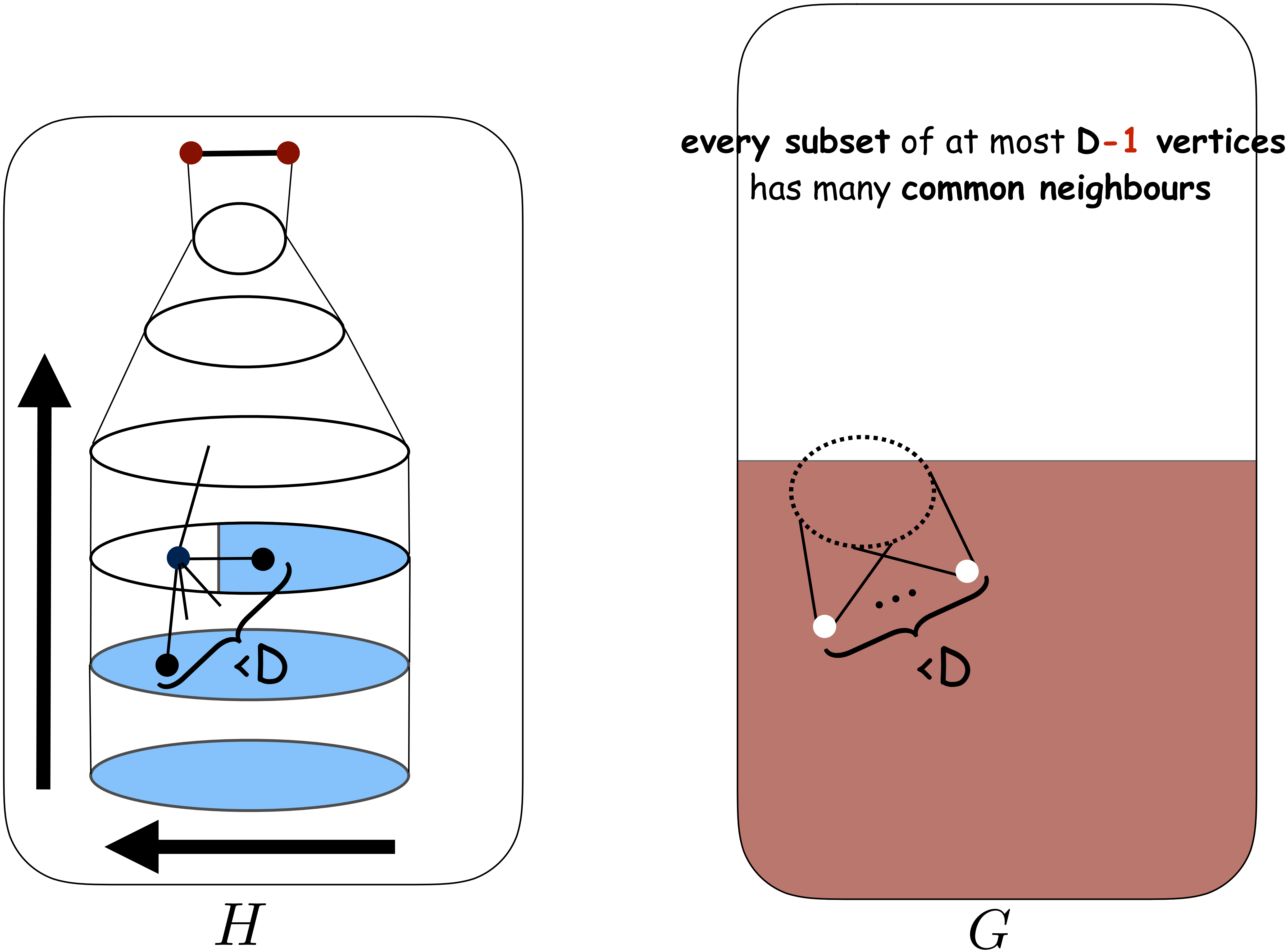

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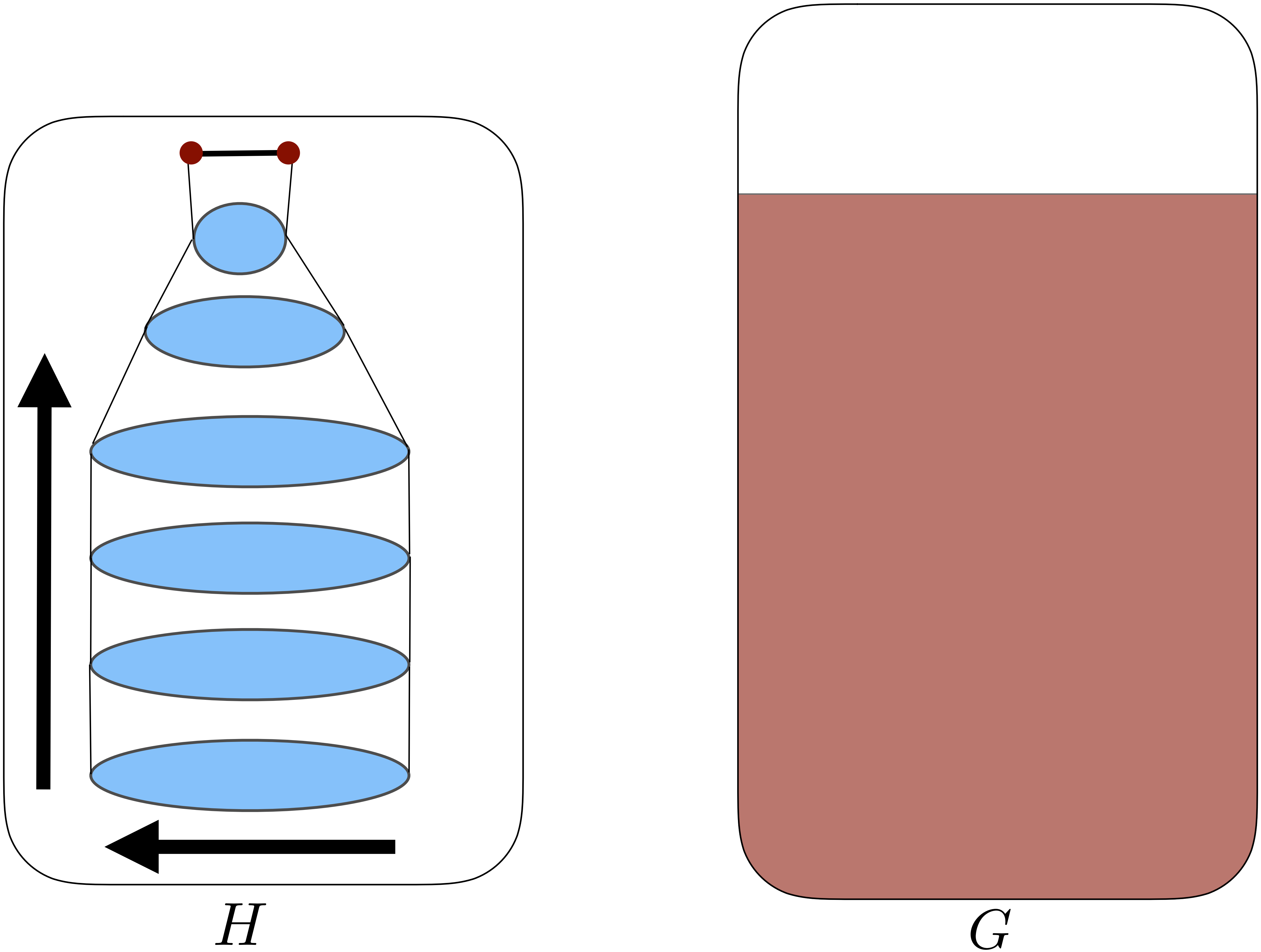
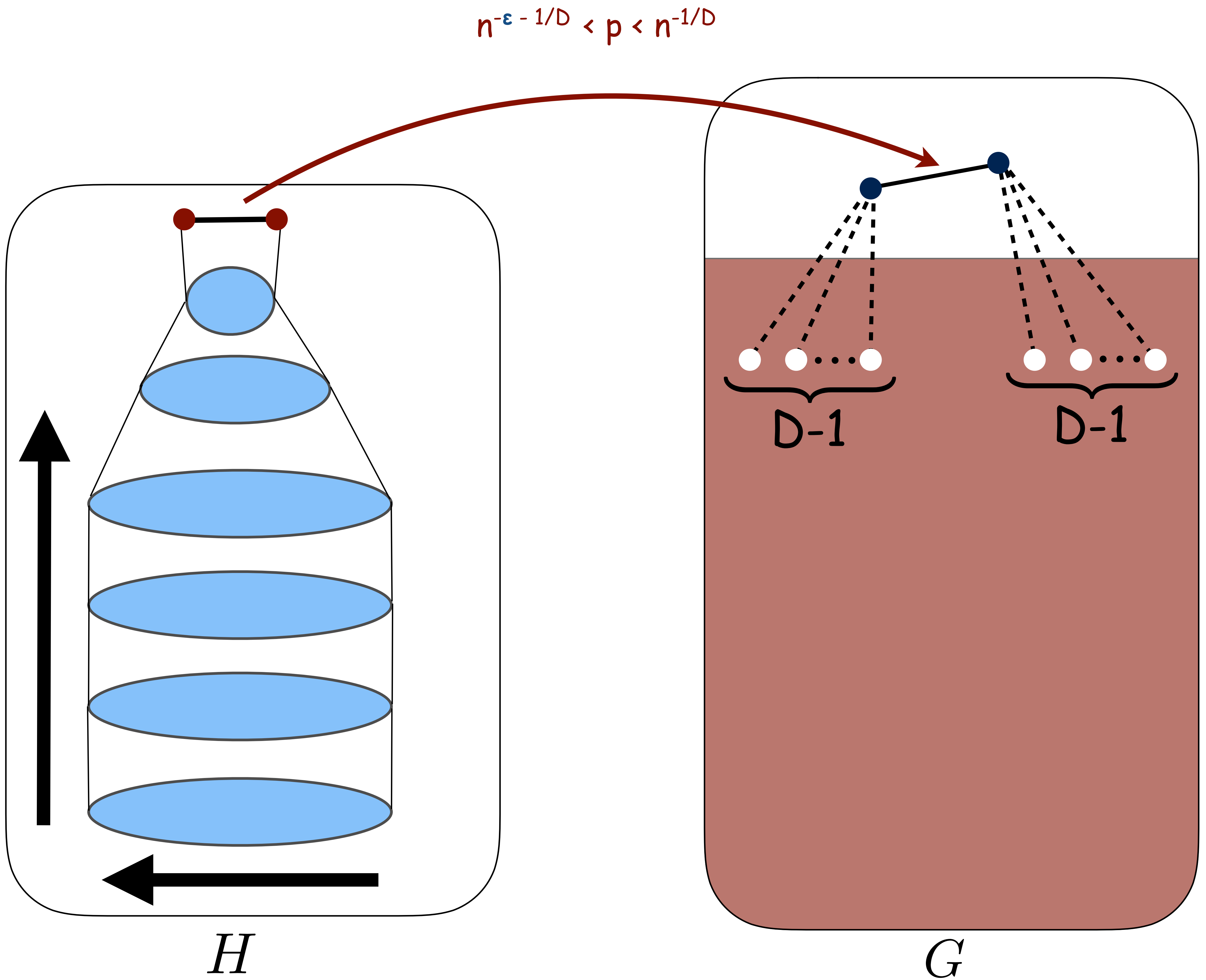
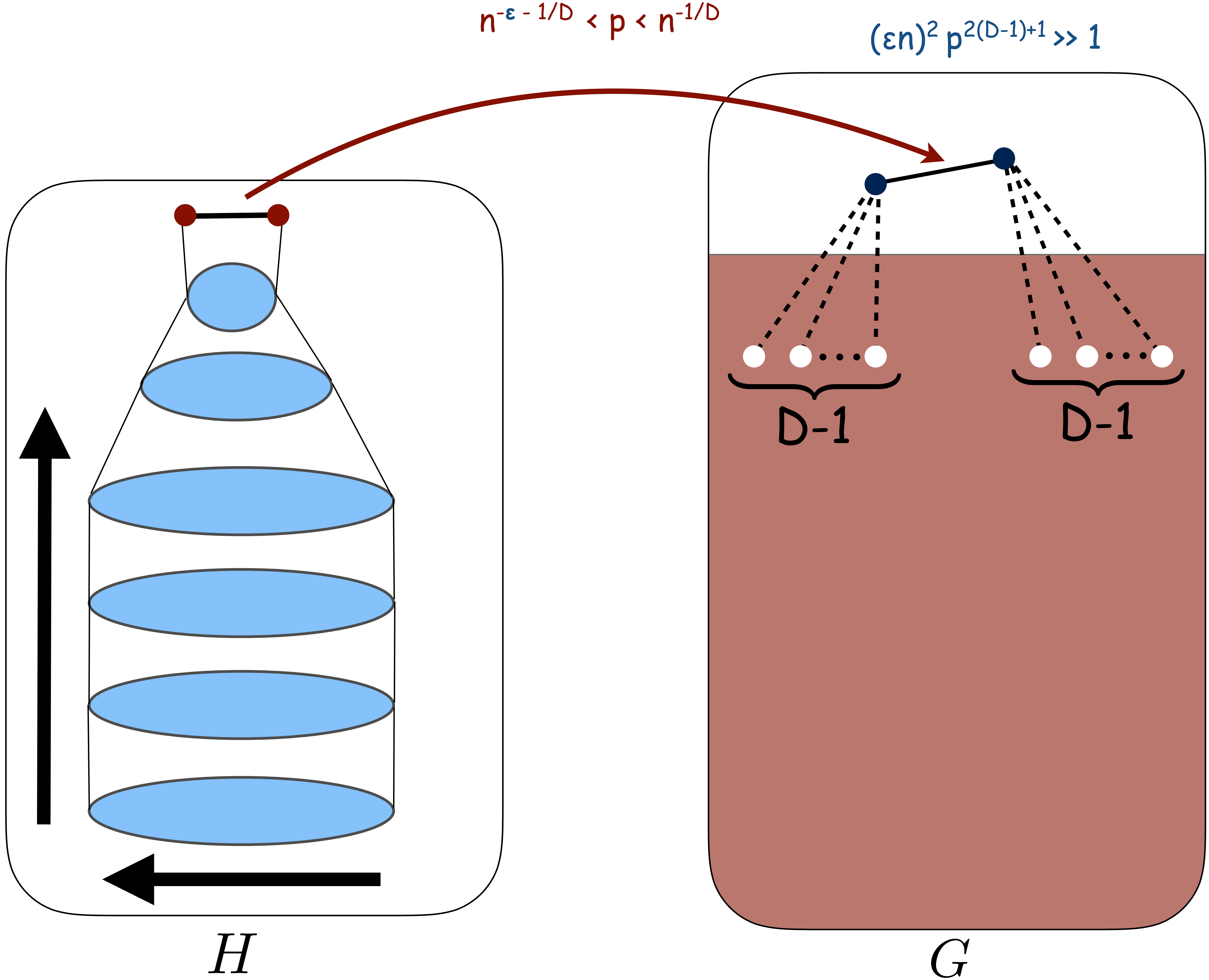
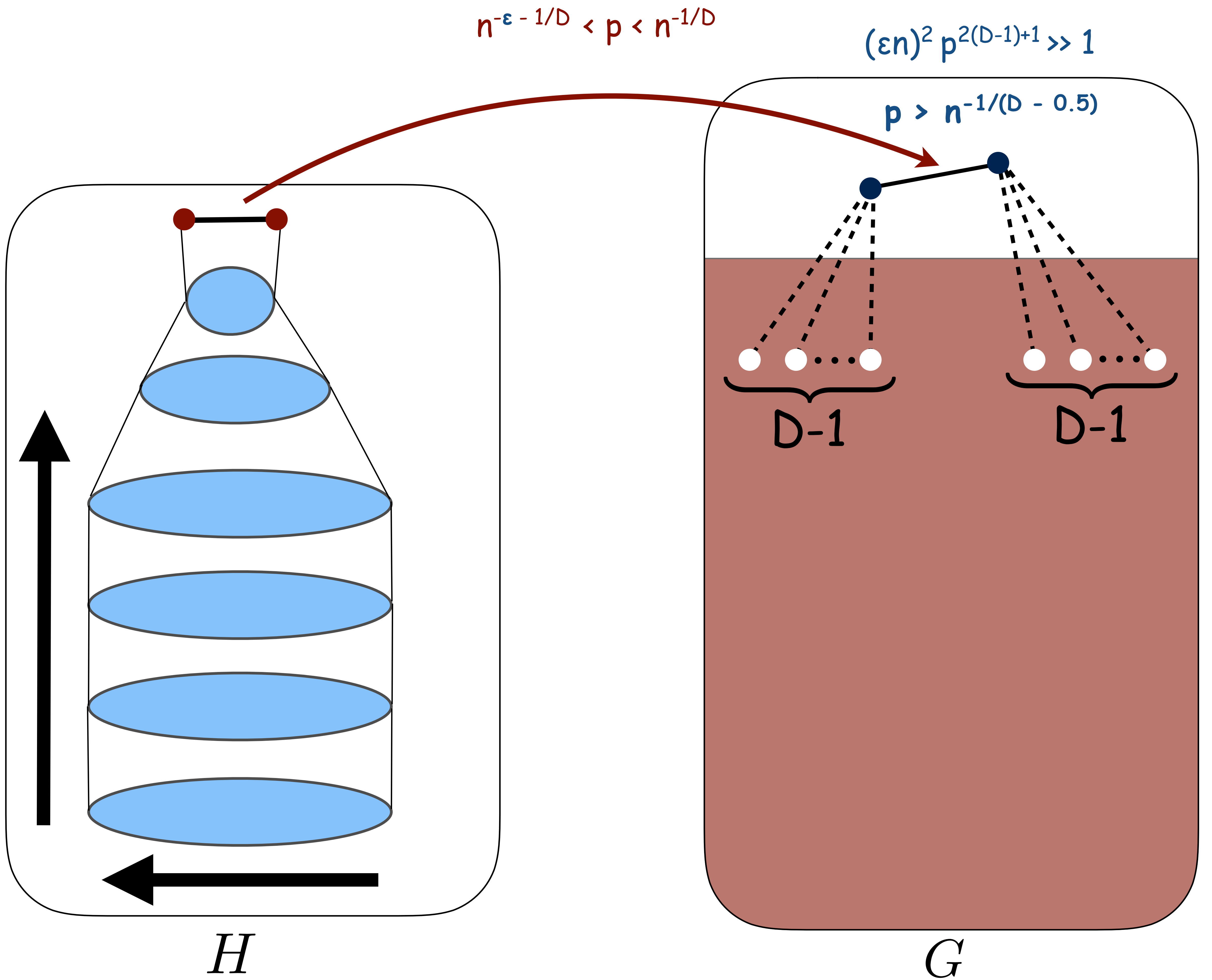




n^{-ε - 1/D} < p < n^{-1/D}
D-1
D-1
H
G



$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
$(\varepsilon n)^2 p^{2(D-1)+1} >> 1$
D-1
D-1
H
G


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
$(\varepsilon n)^2 p^{2(D-1)+1} >> 1$
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D-1
D-1
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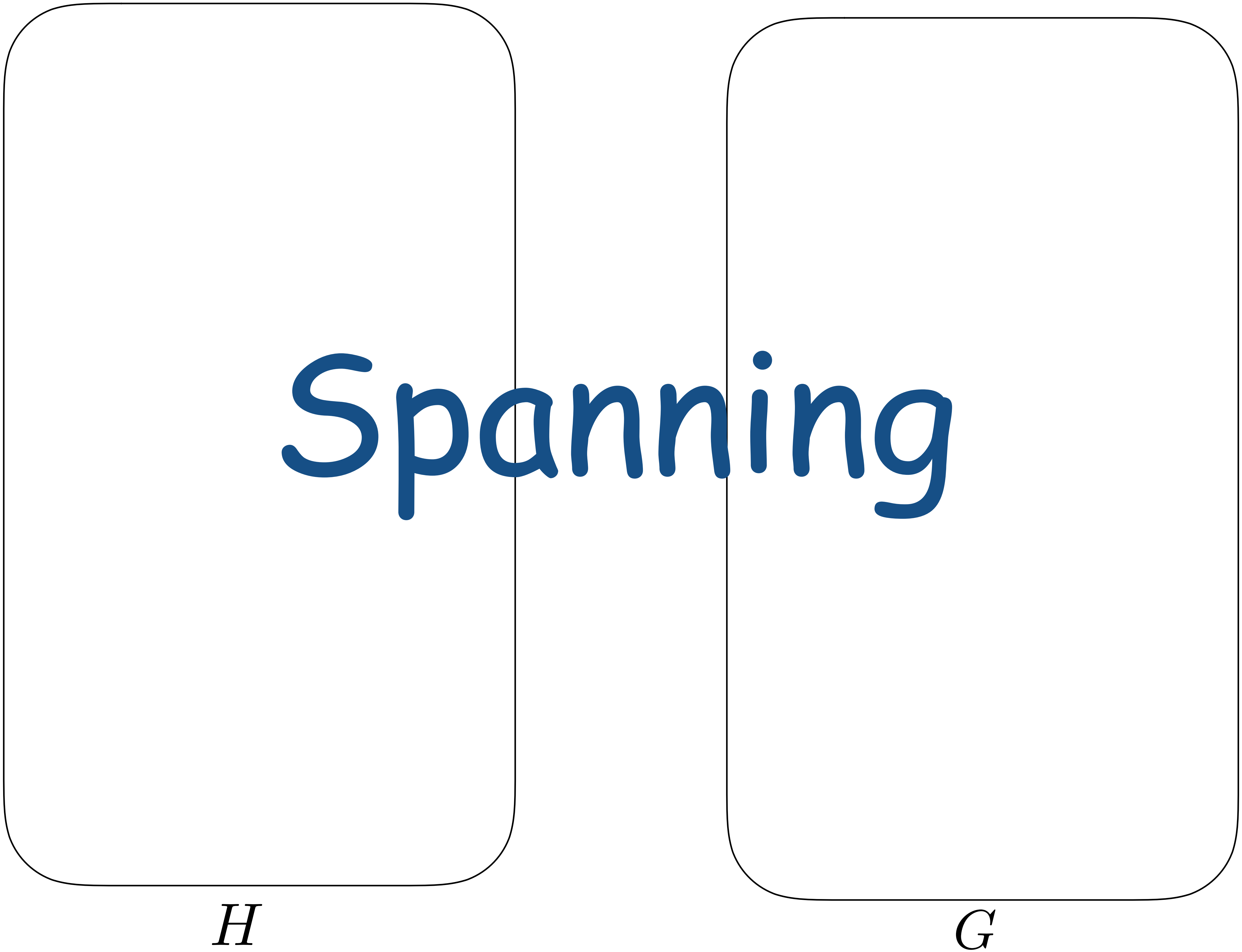

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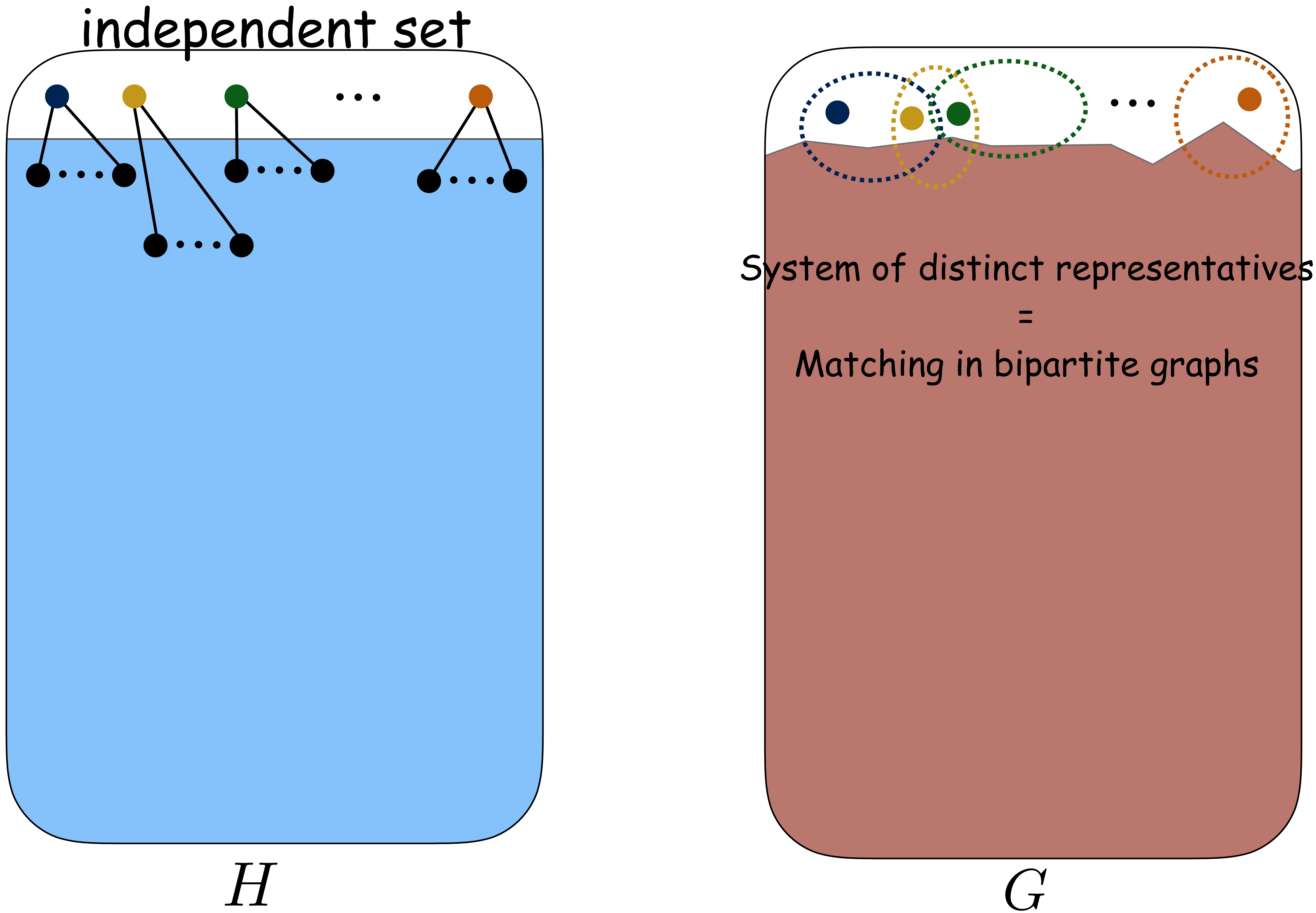

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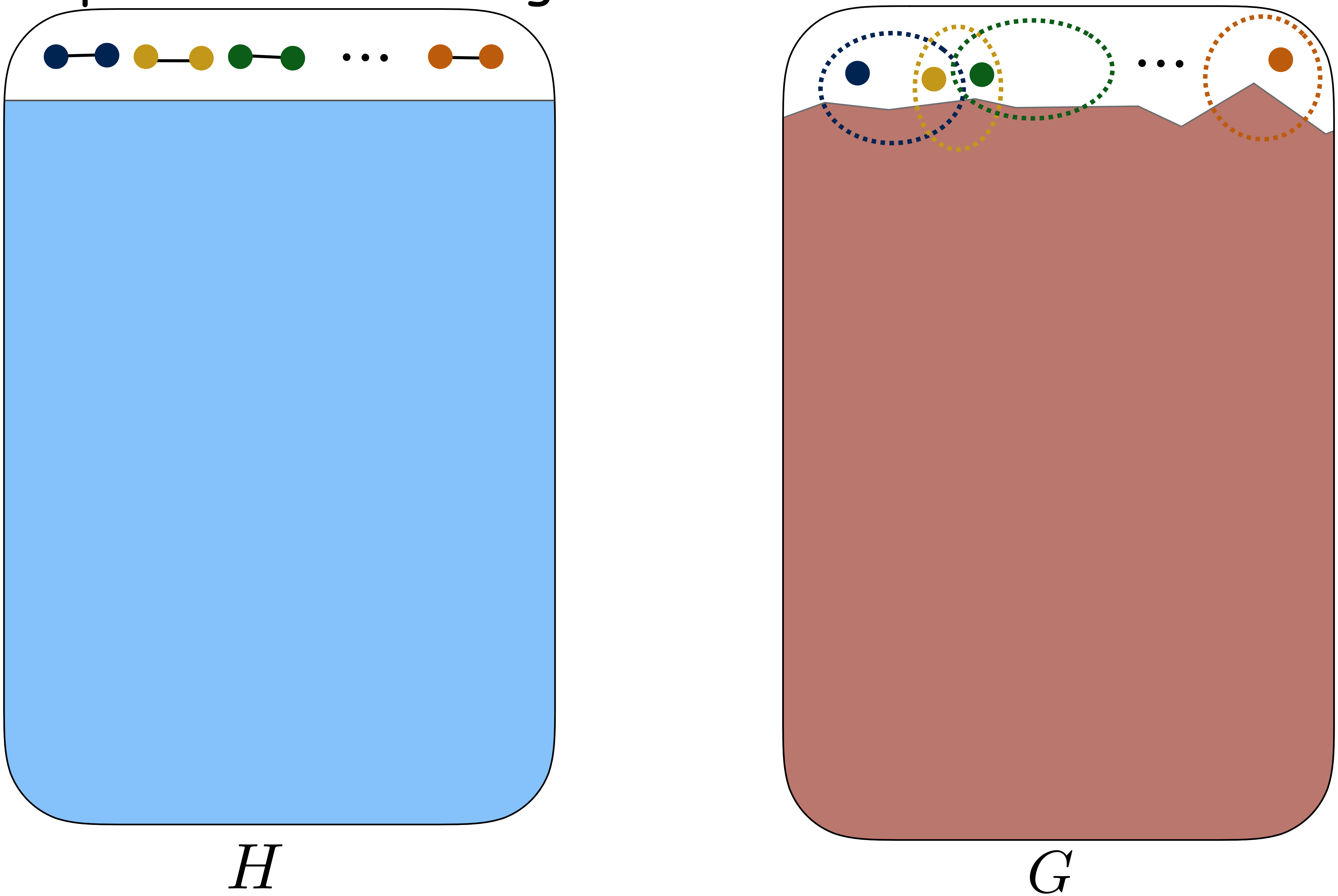
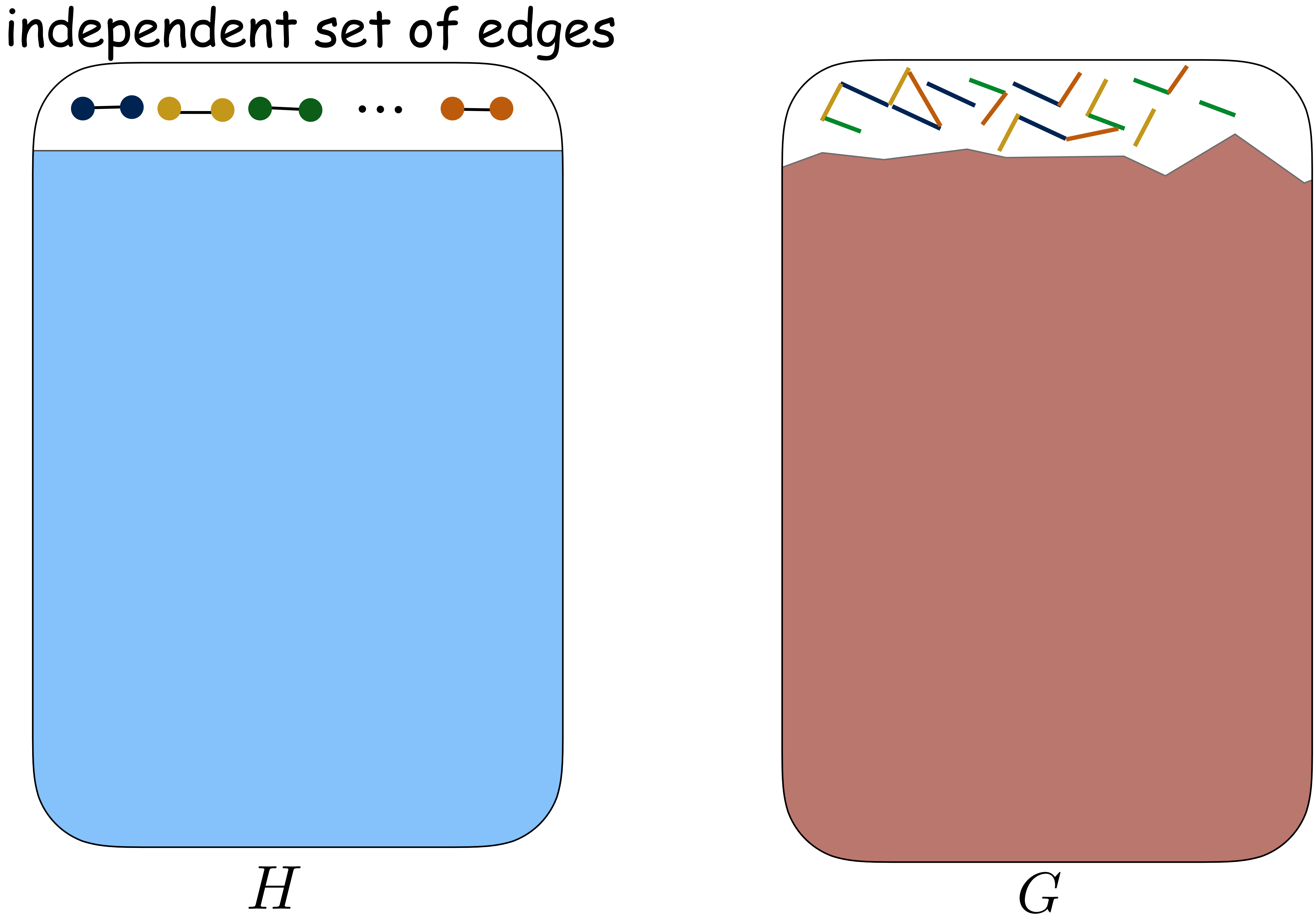
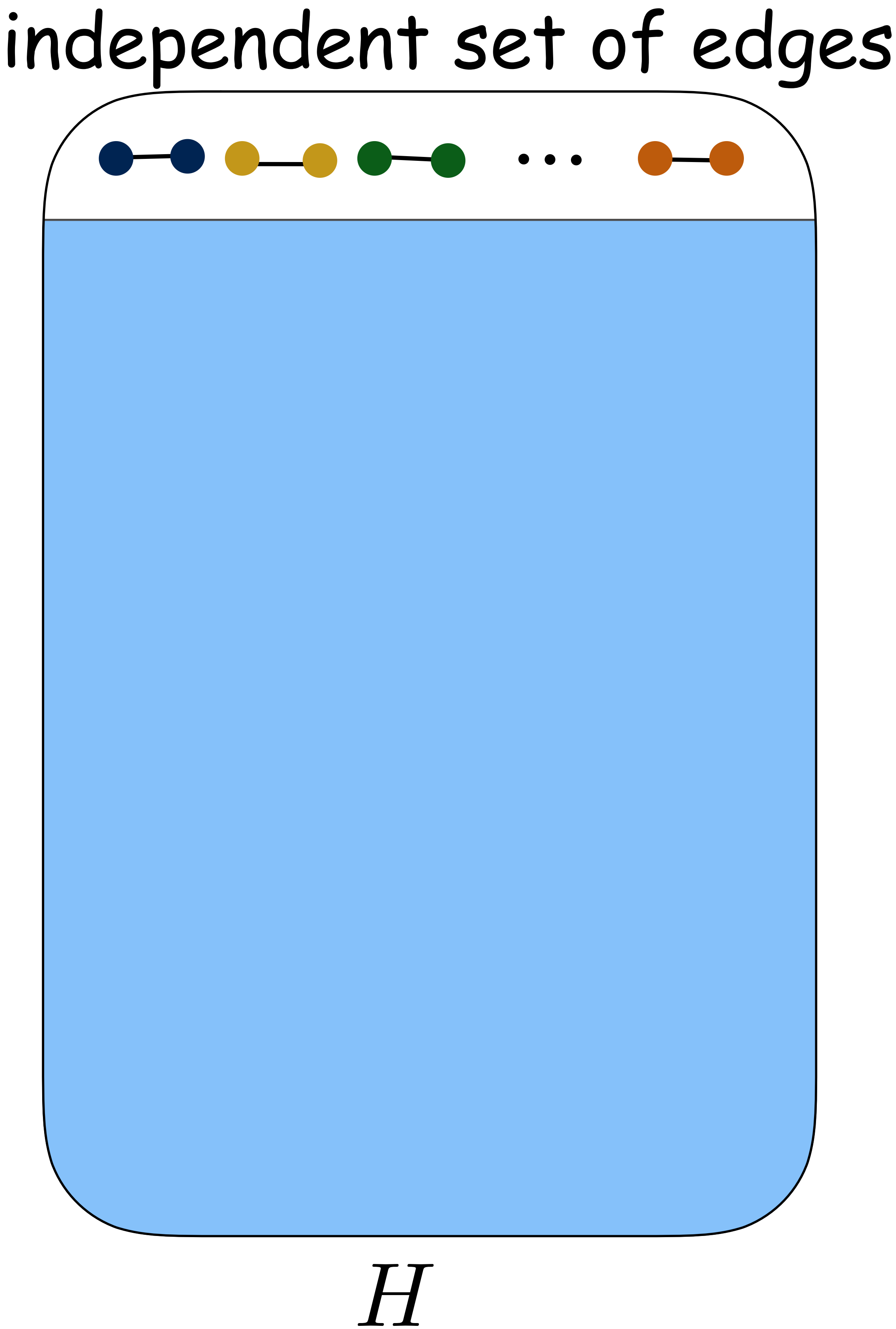
independent set of edges

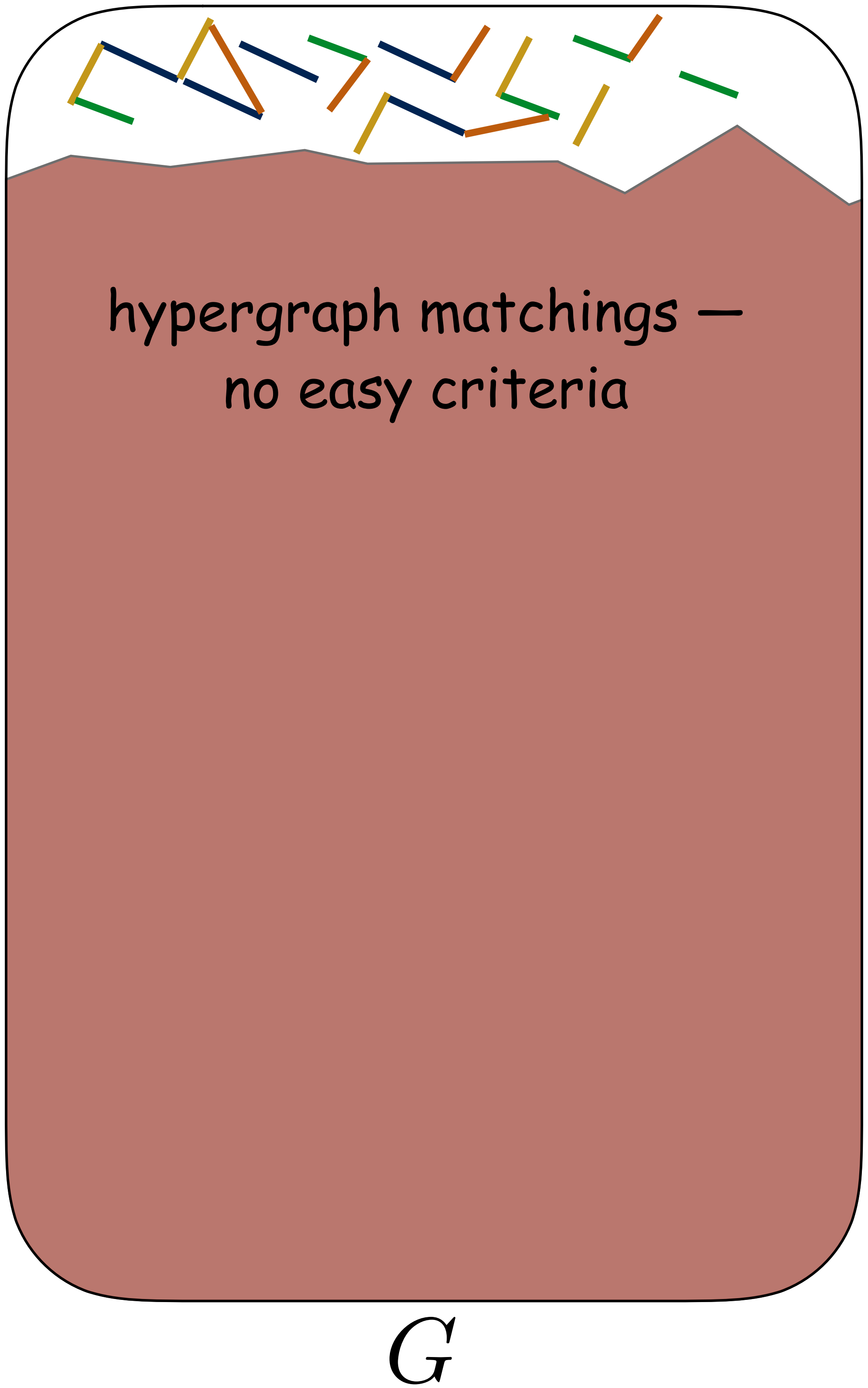


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$

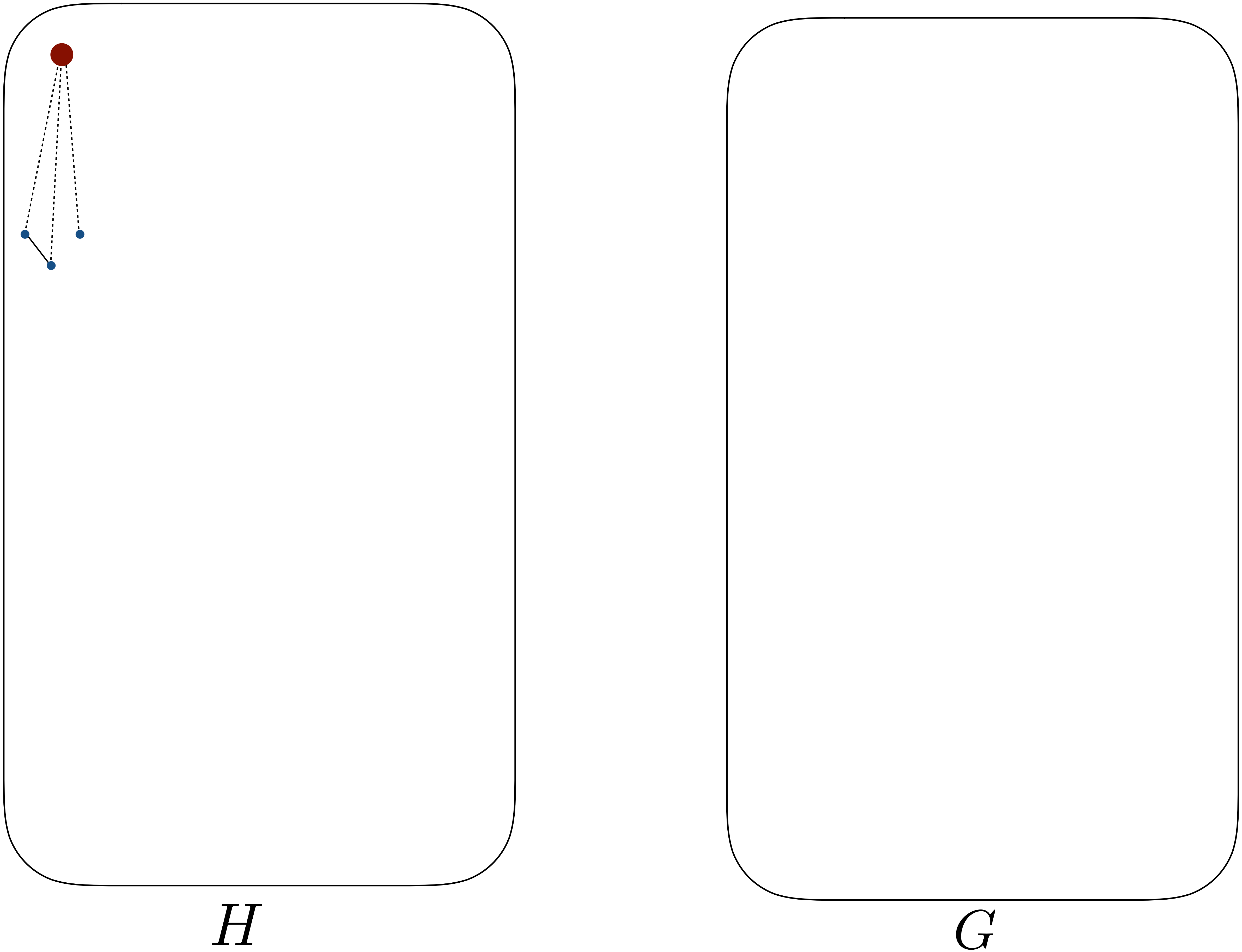


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$

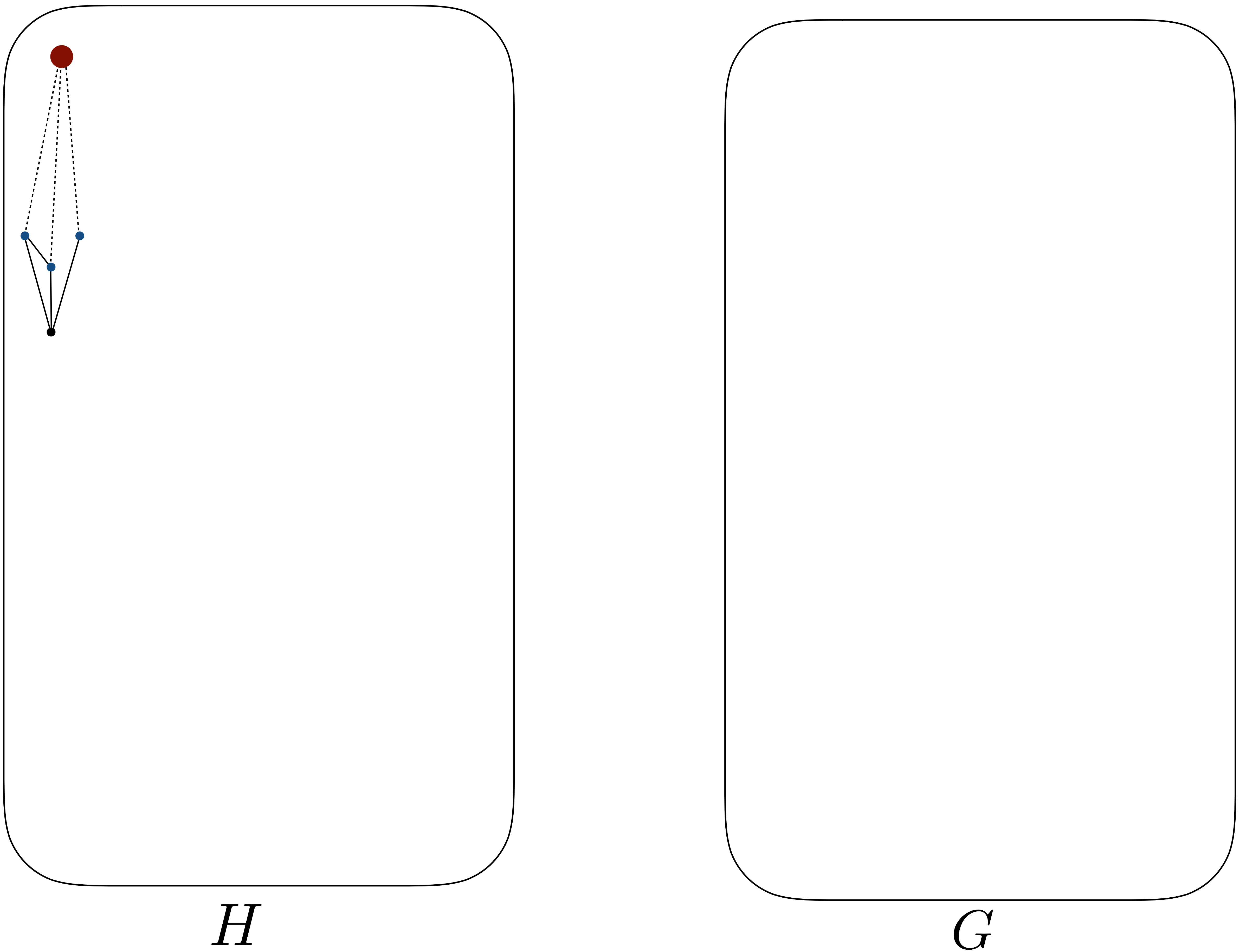


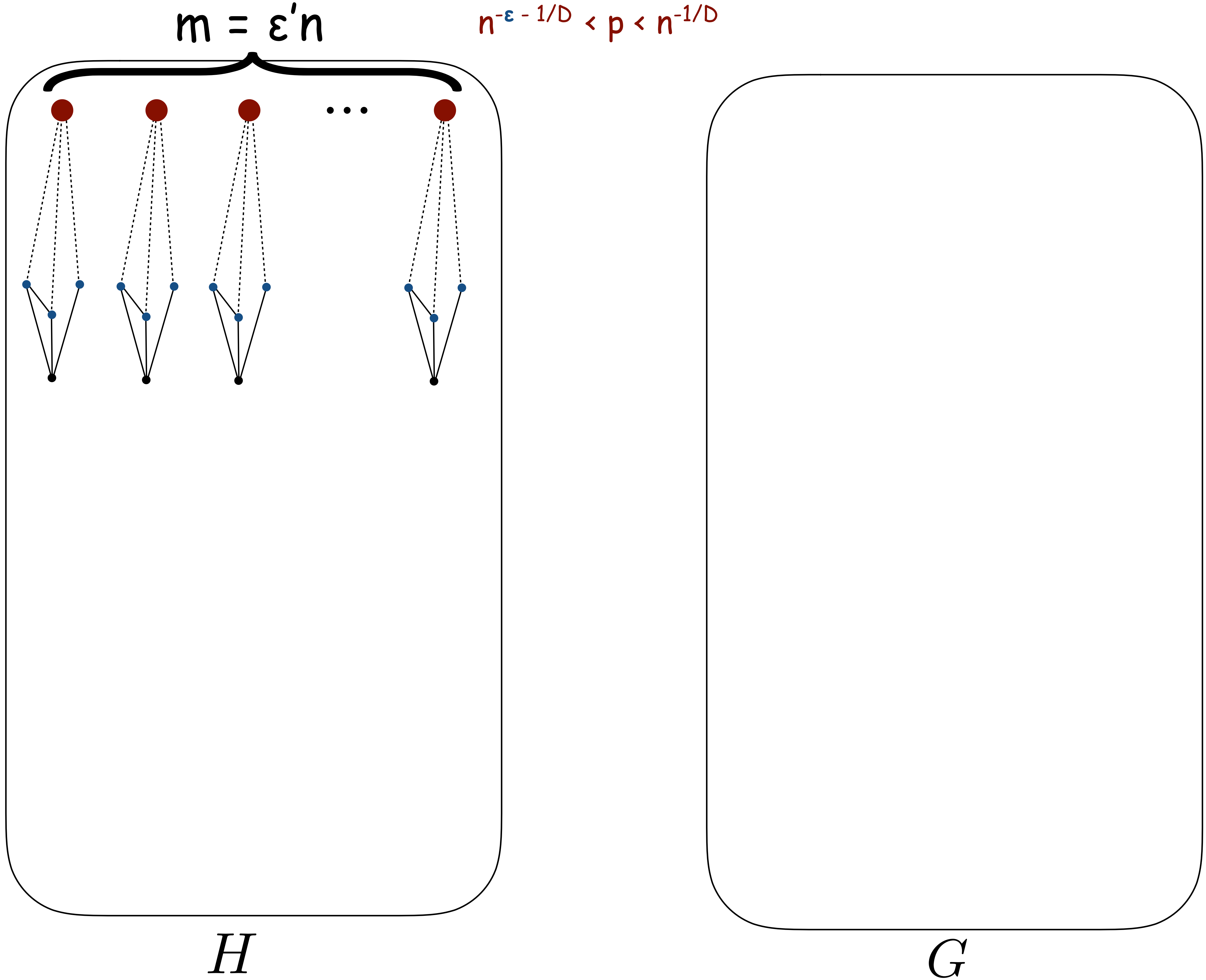


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$

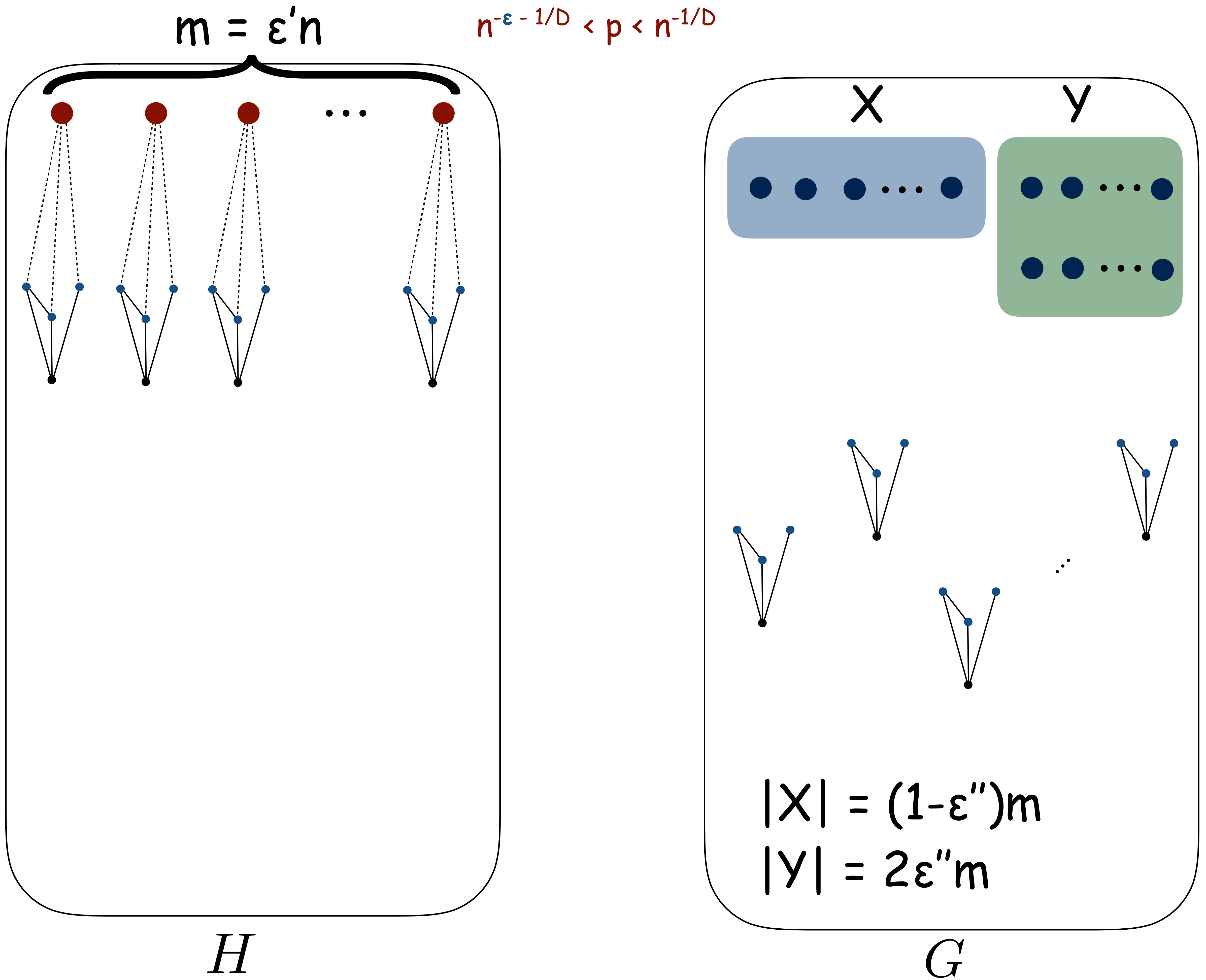


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$

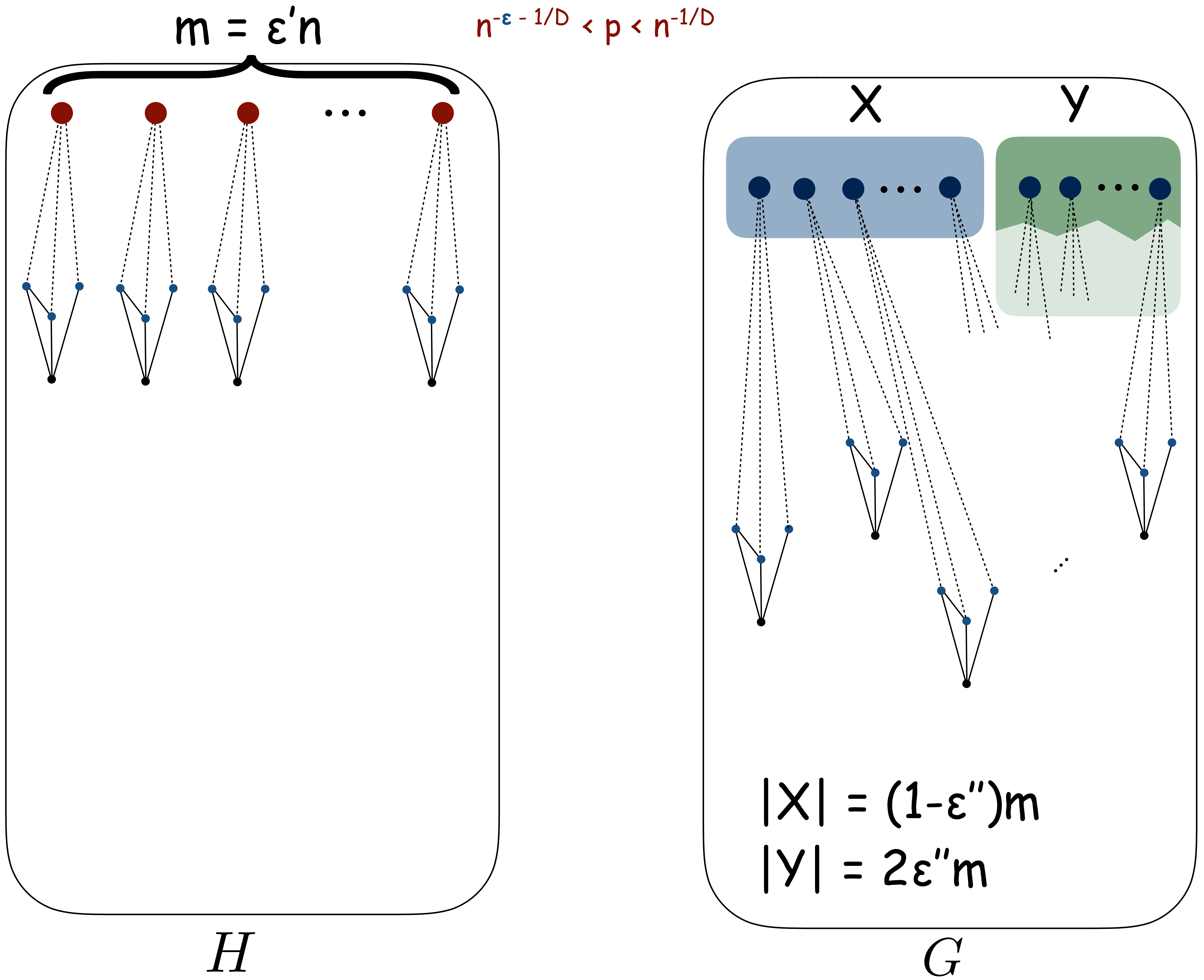




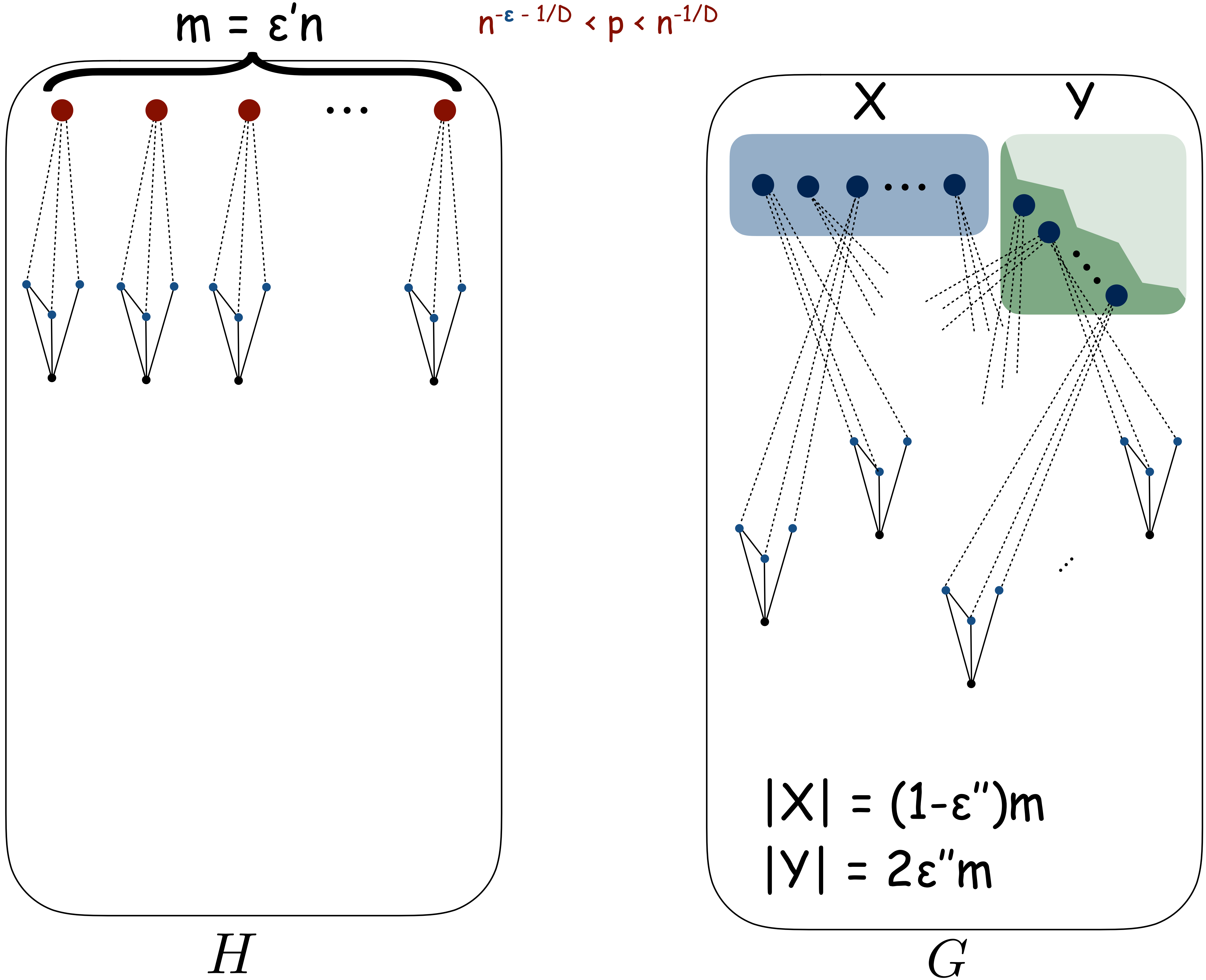
$m = \varepsilon' n$
$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
H
G

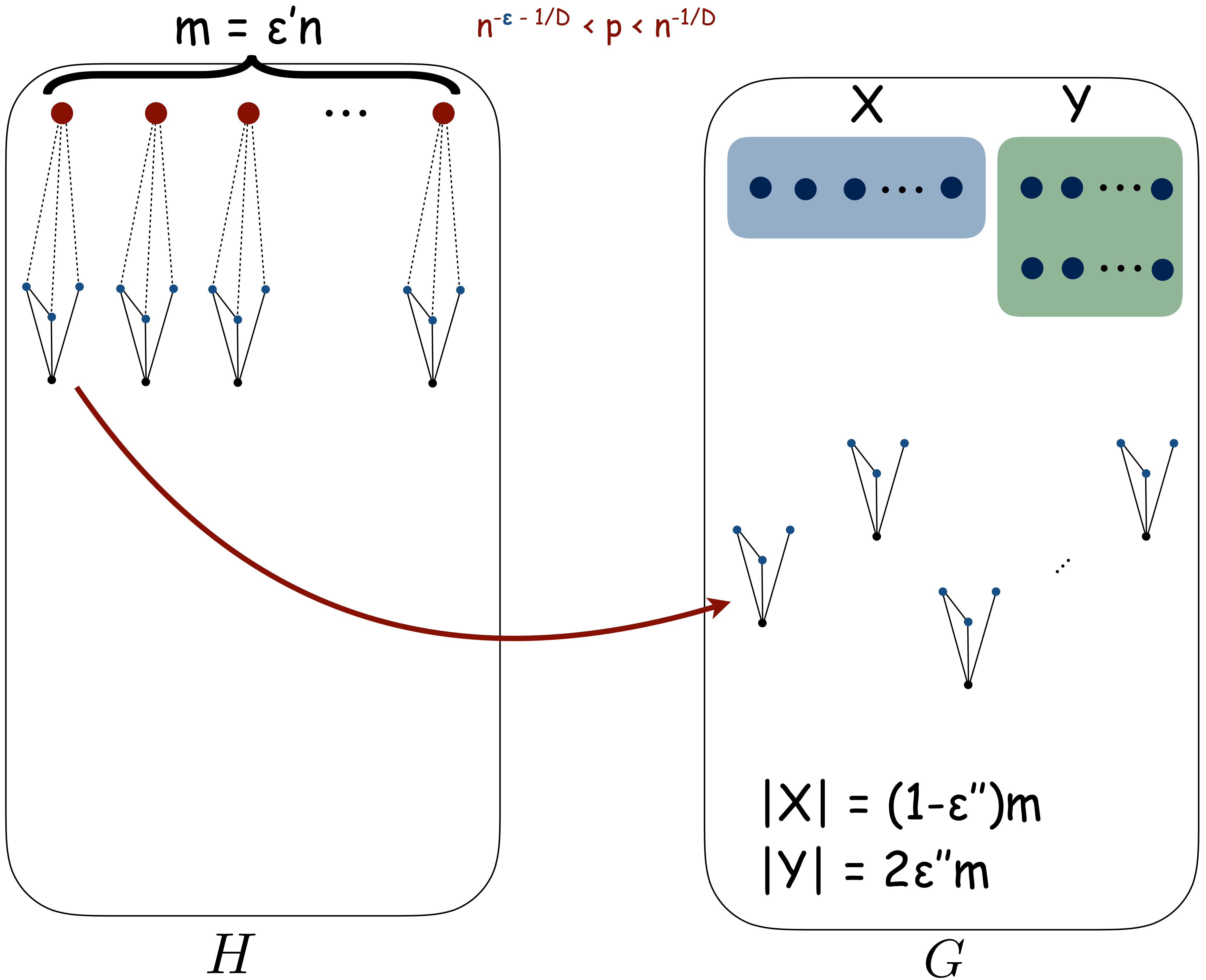


m = ε'n
n^{-ε - 1/D} < p < n^{-1/D}
X
Y
|X| = (1-ε'')m
|Y| = 2ε''m
H
G

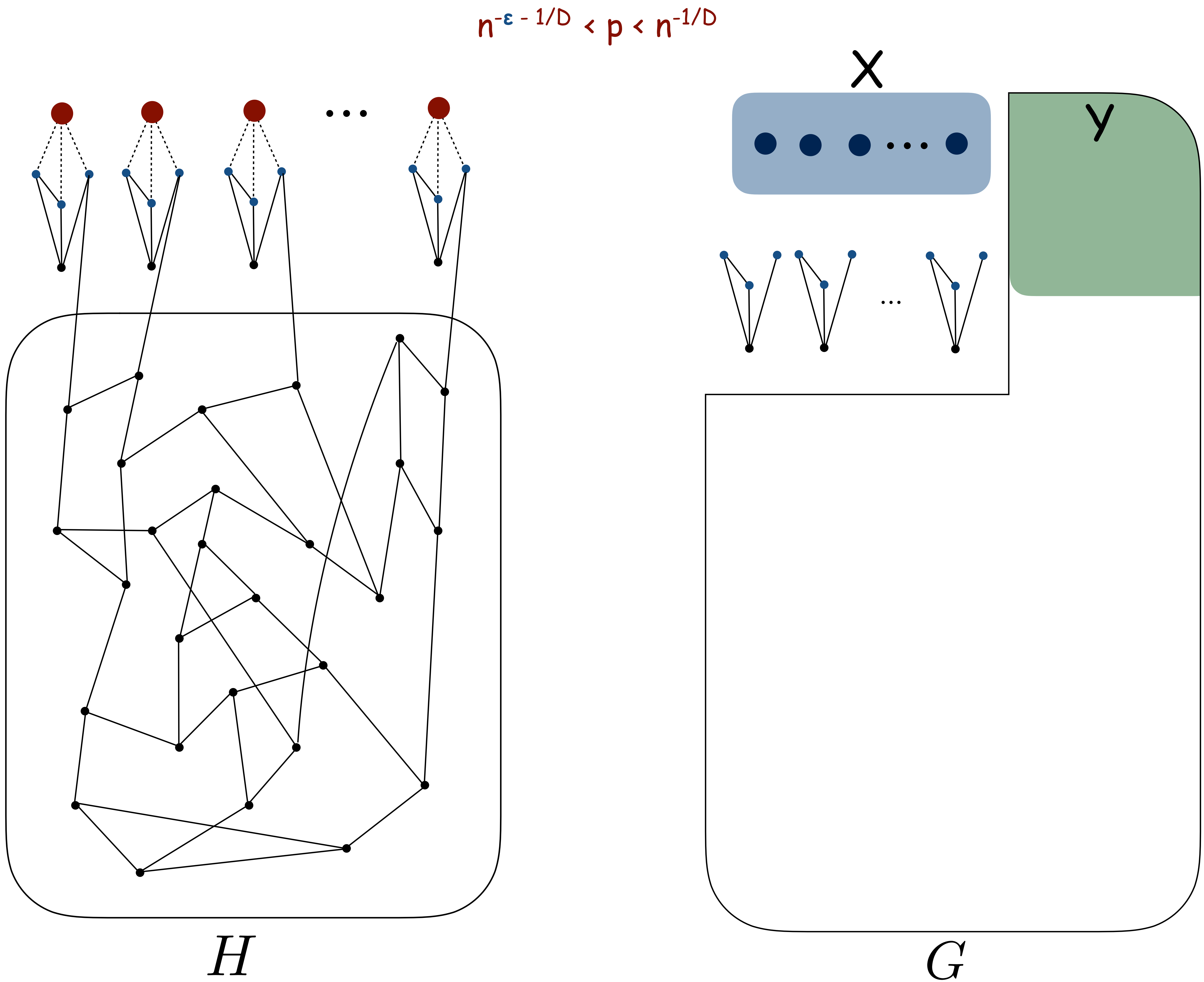


m = ε'n
n^{-ε - 1/D} < p < n^{-1/D}
X
Y
|X| = (1-ε'')m
|Y| = 2ε''m
H
G

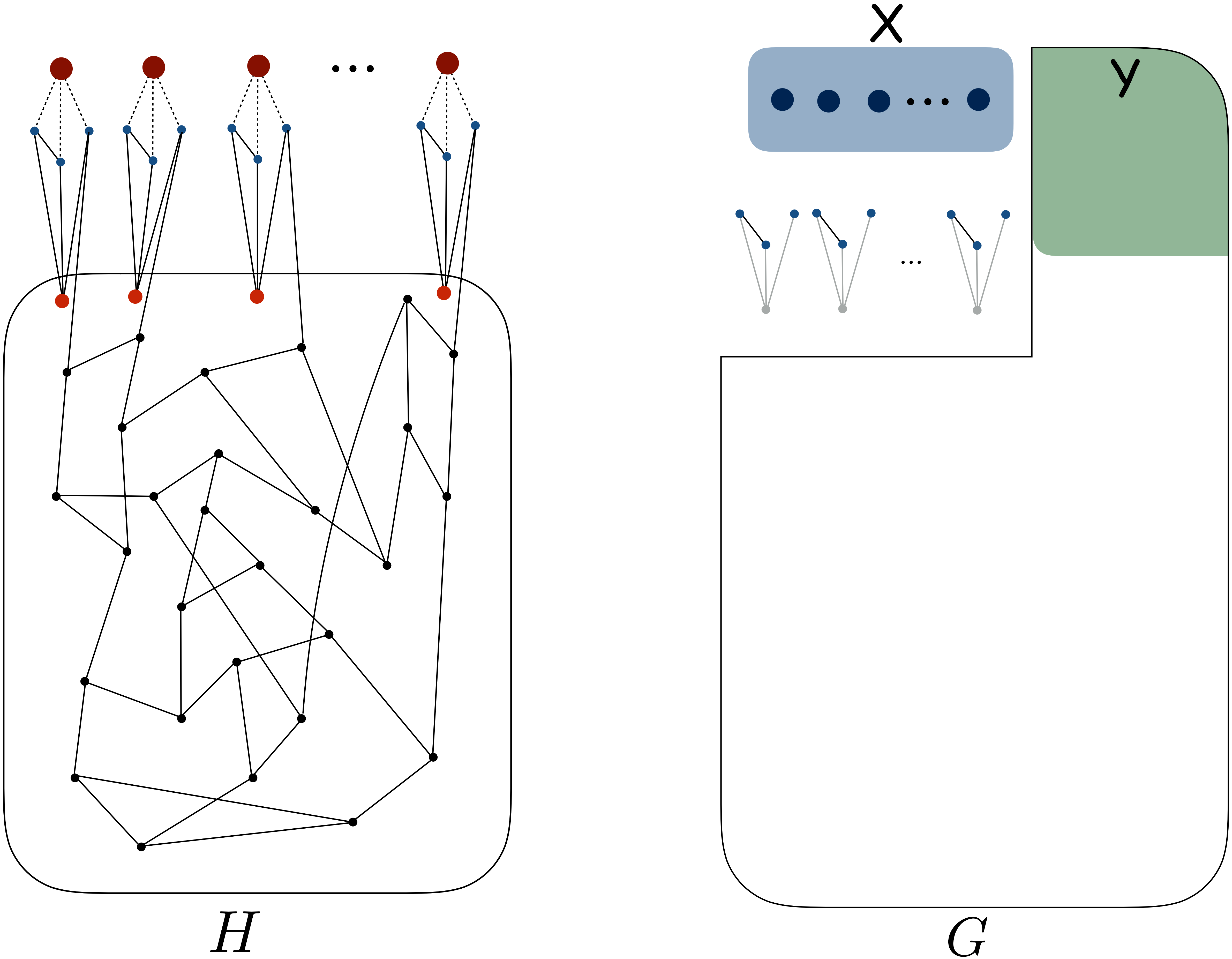

m = ε'n
n^{-ε - 1/D} < p < n^{-1/D}
X
Y
|X| = (1-ε'')m
|Y| = 2ε''m
H
G

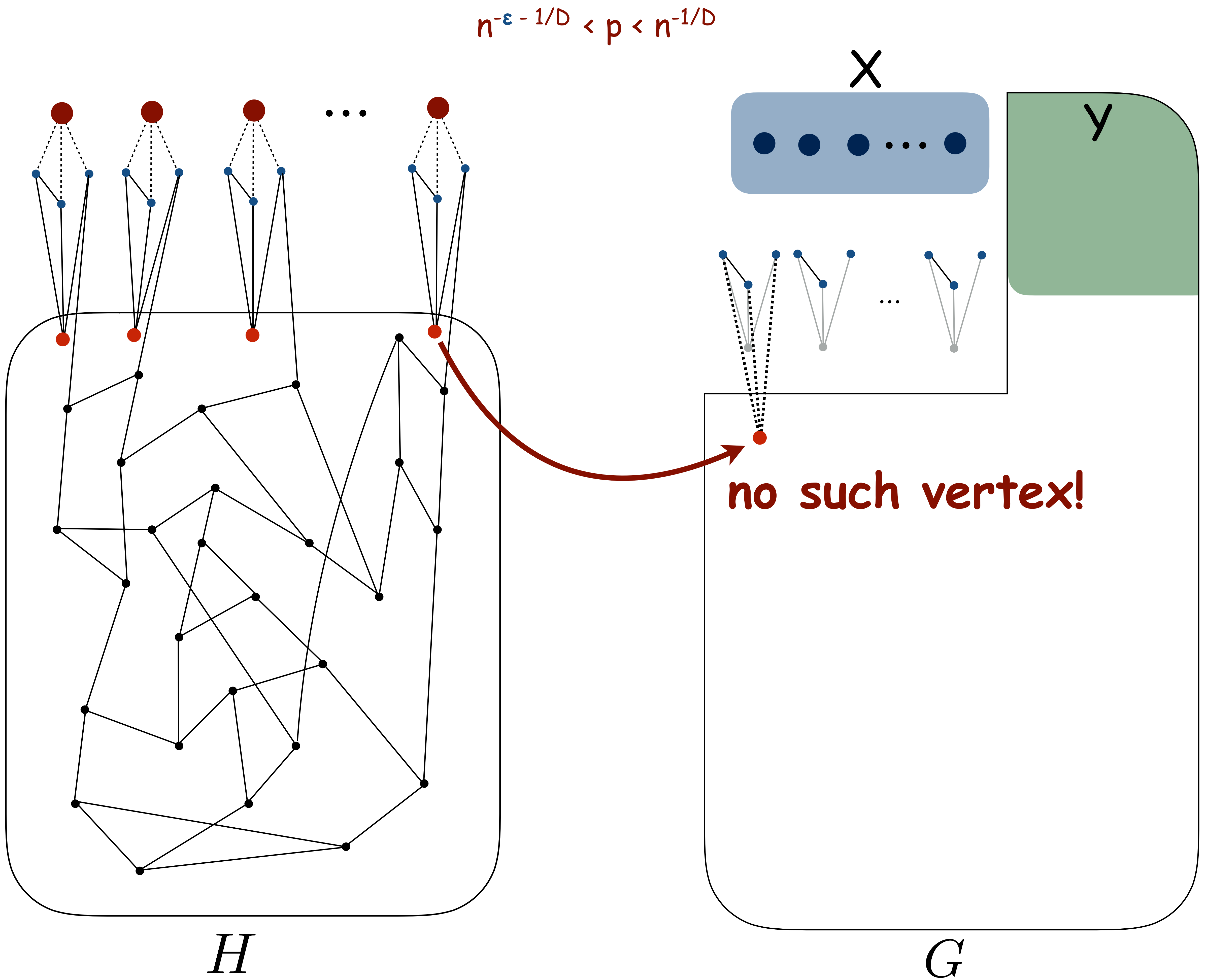


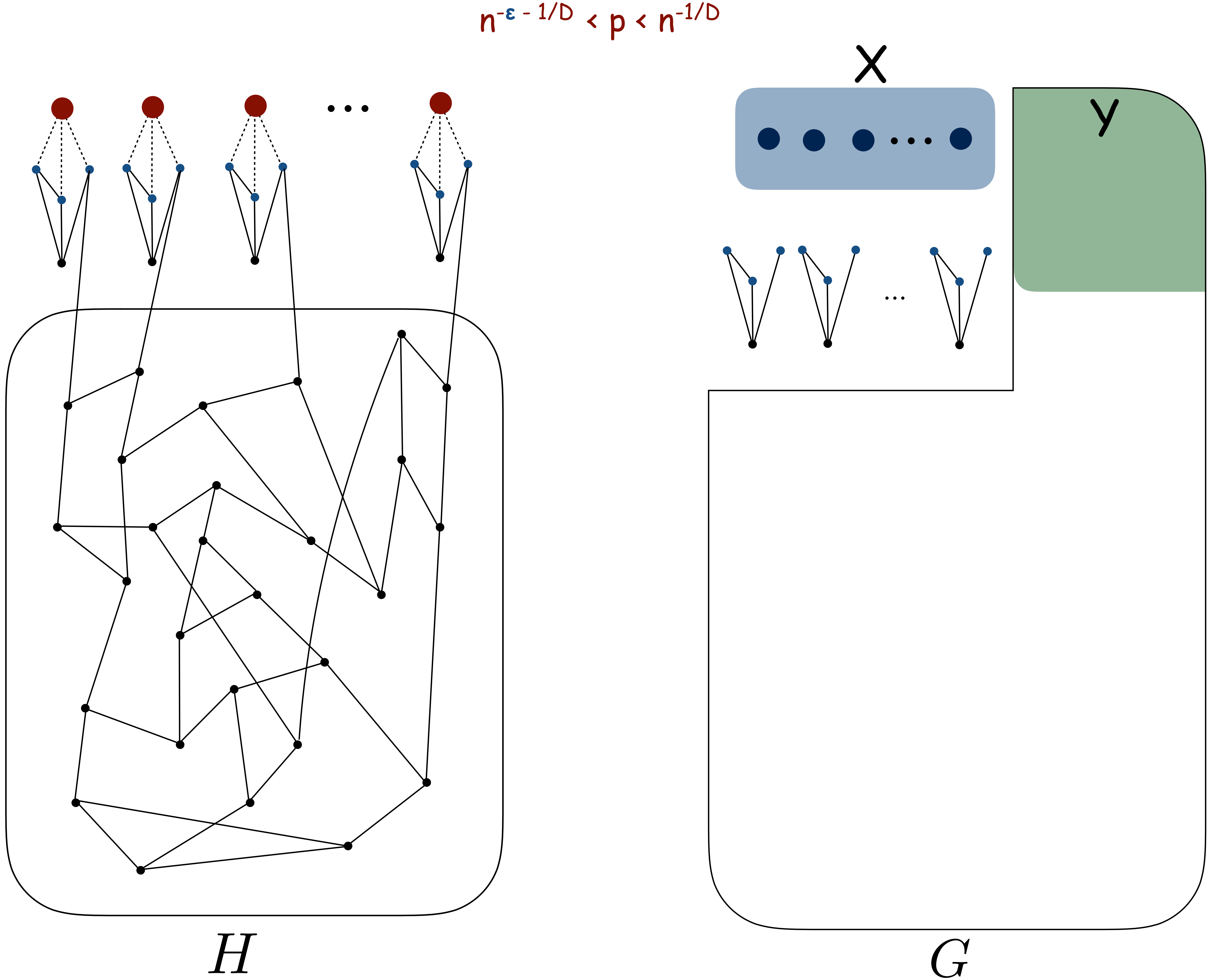
m = ε′n
n^{-ε - 1/D} < p < n^{-1/D}
X
Y
|X| = (1-ε″)m
|Y| = 2ε″m
H
G


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
H
G

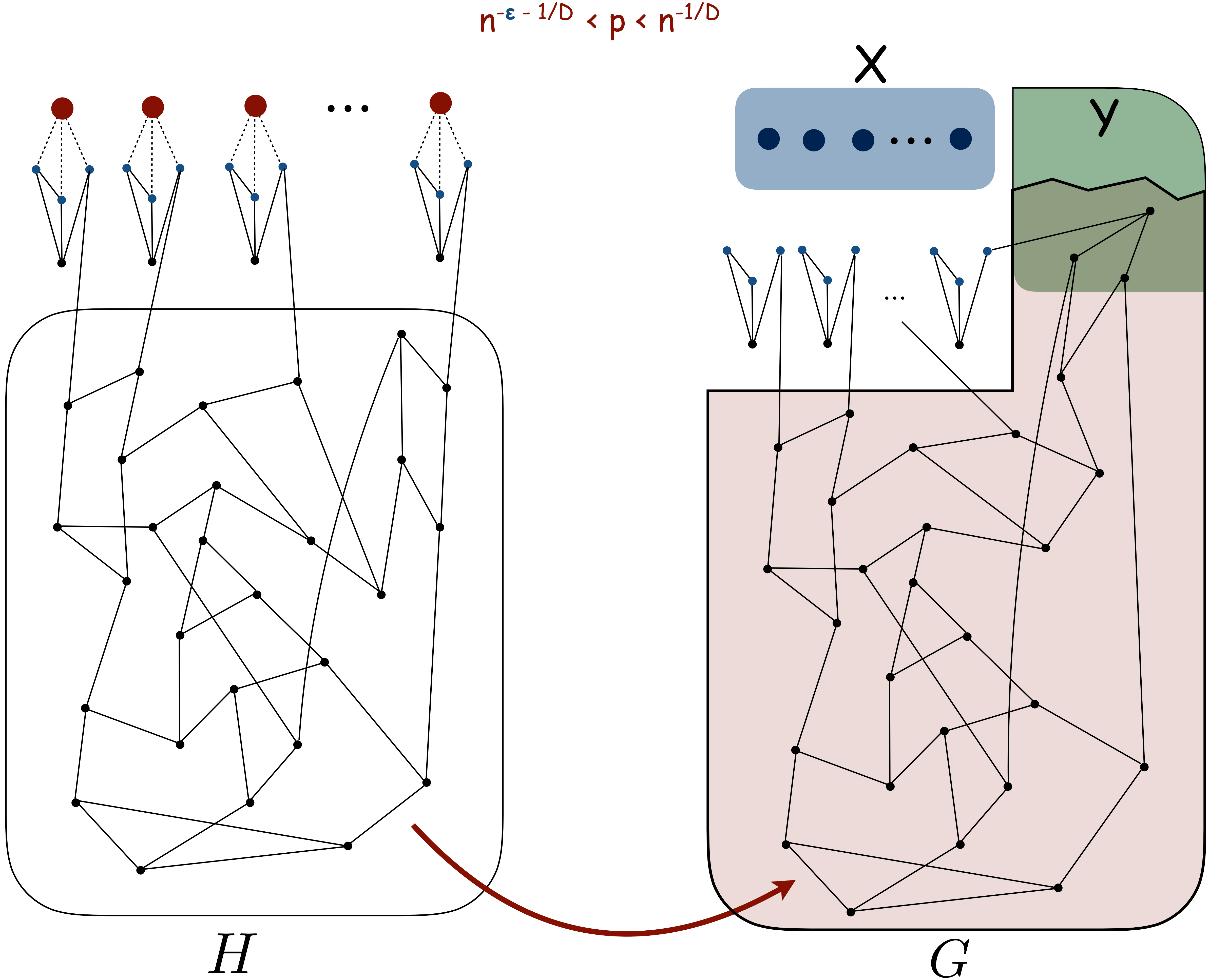
$n^{-\varepsilon - 1/D} < p < n^{-1/D}$



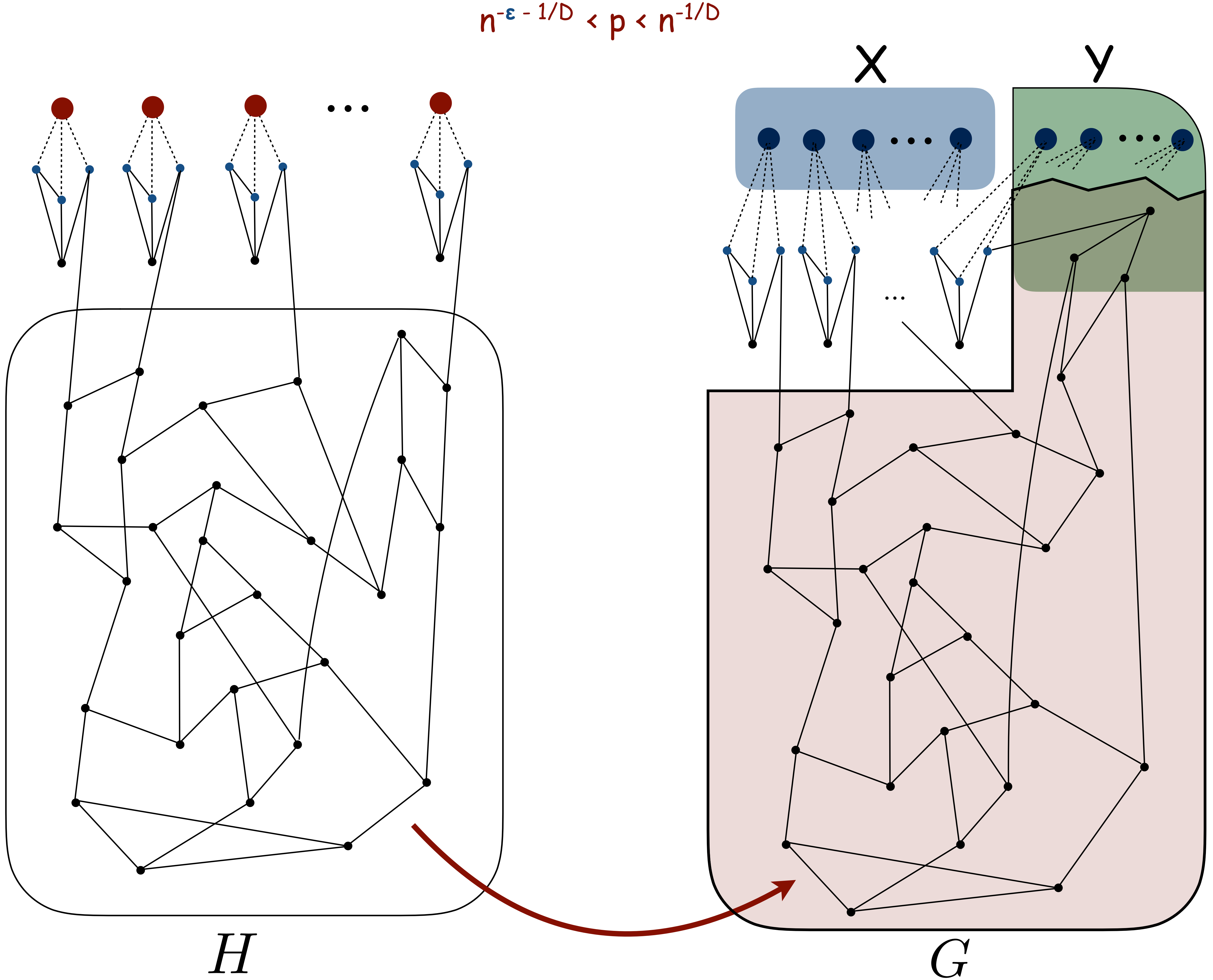

$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
y
no such vertex!
H
G



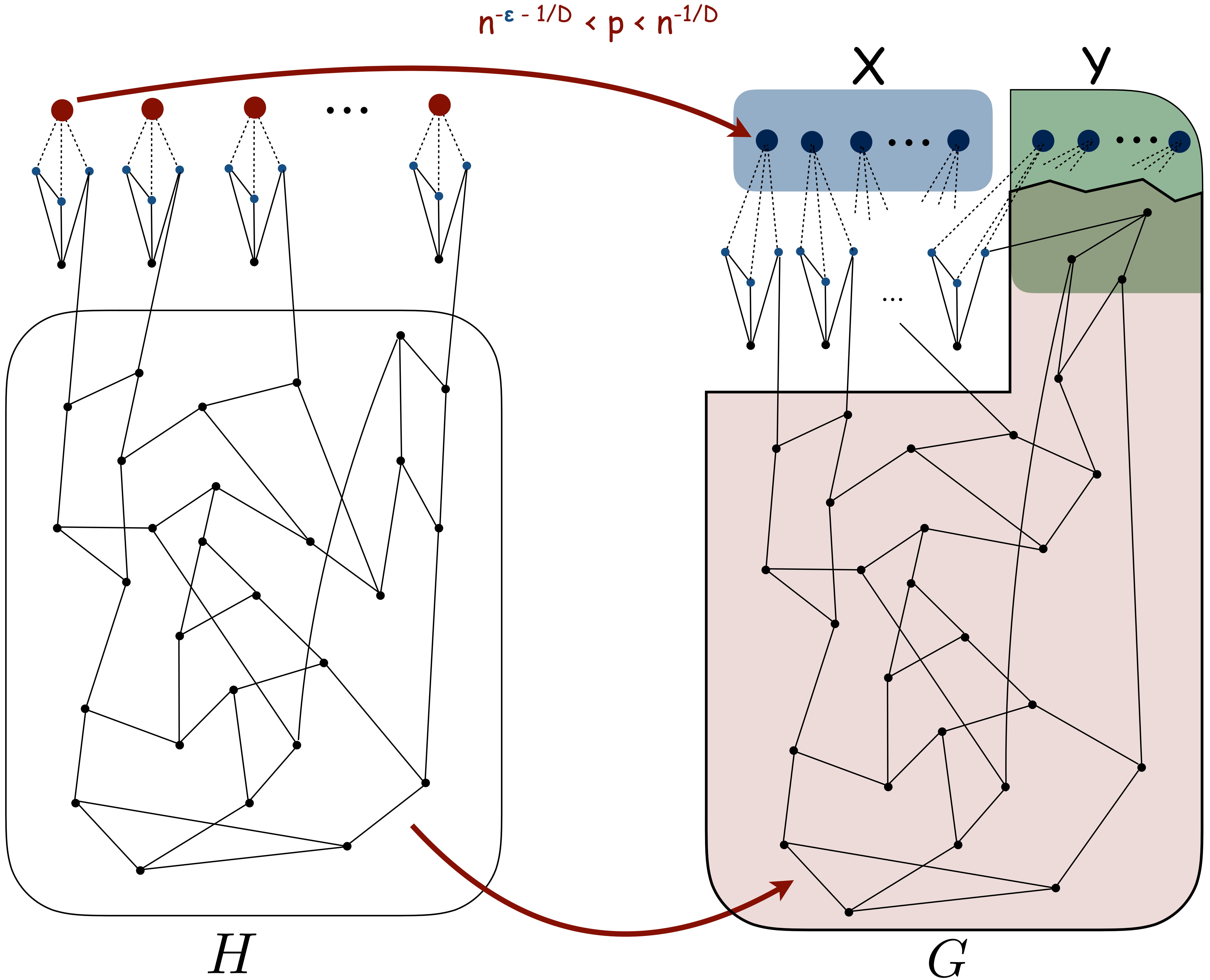
$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
H
G



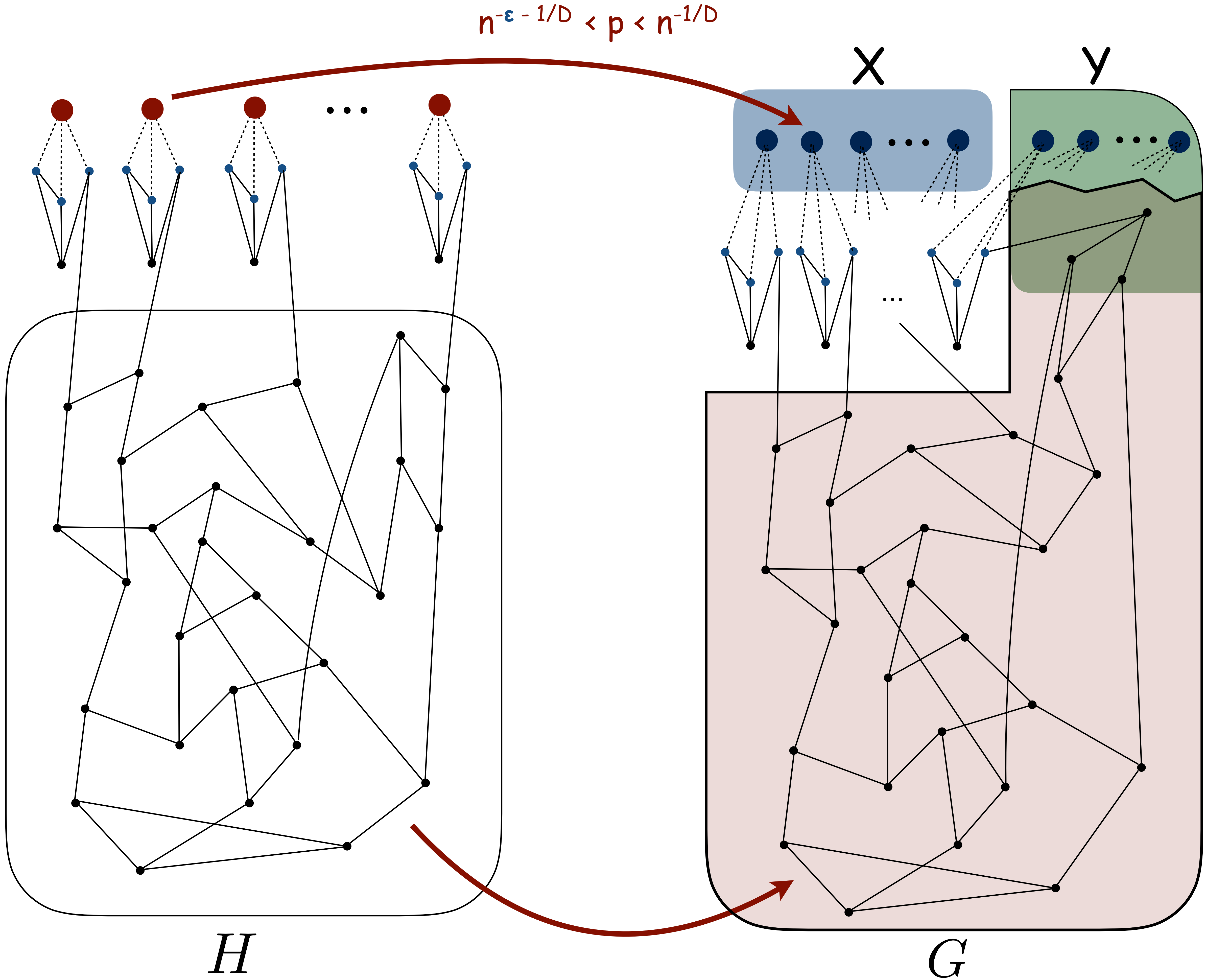
$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
H
G



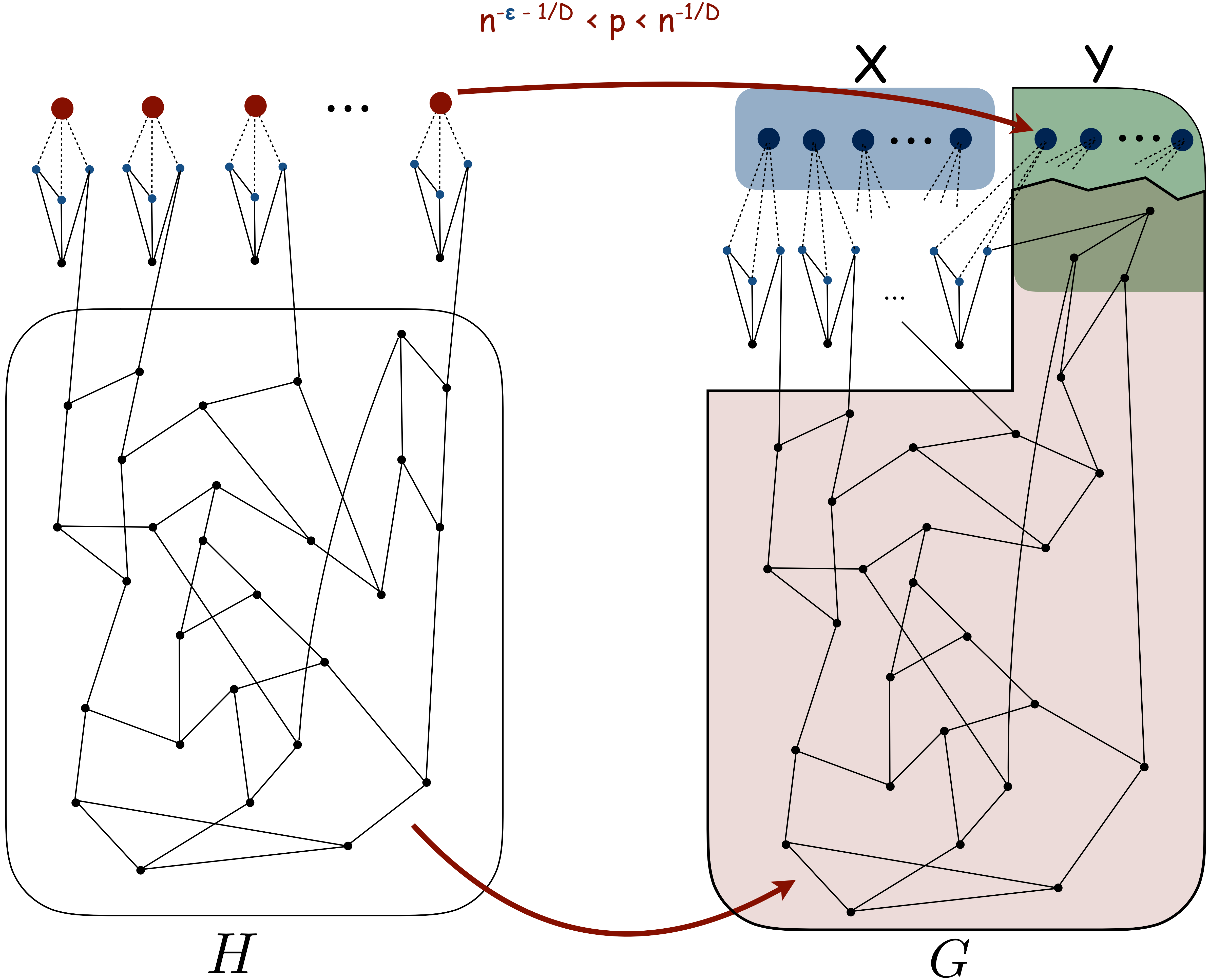
$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
H
G



$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
H
G

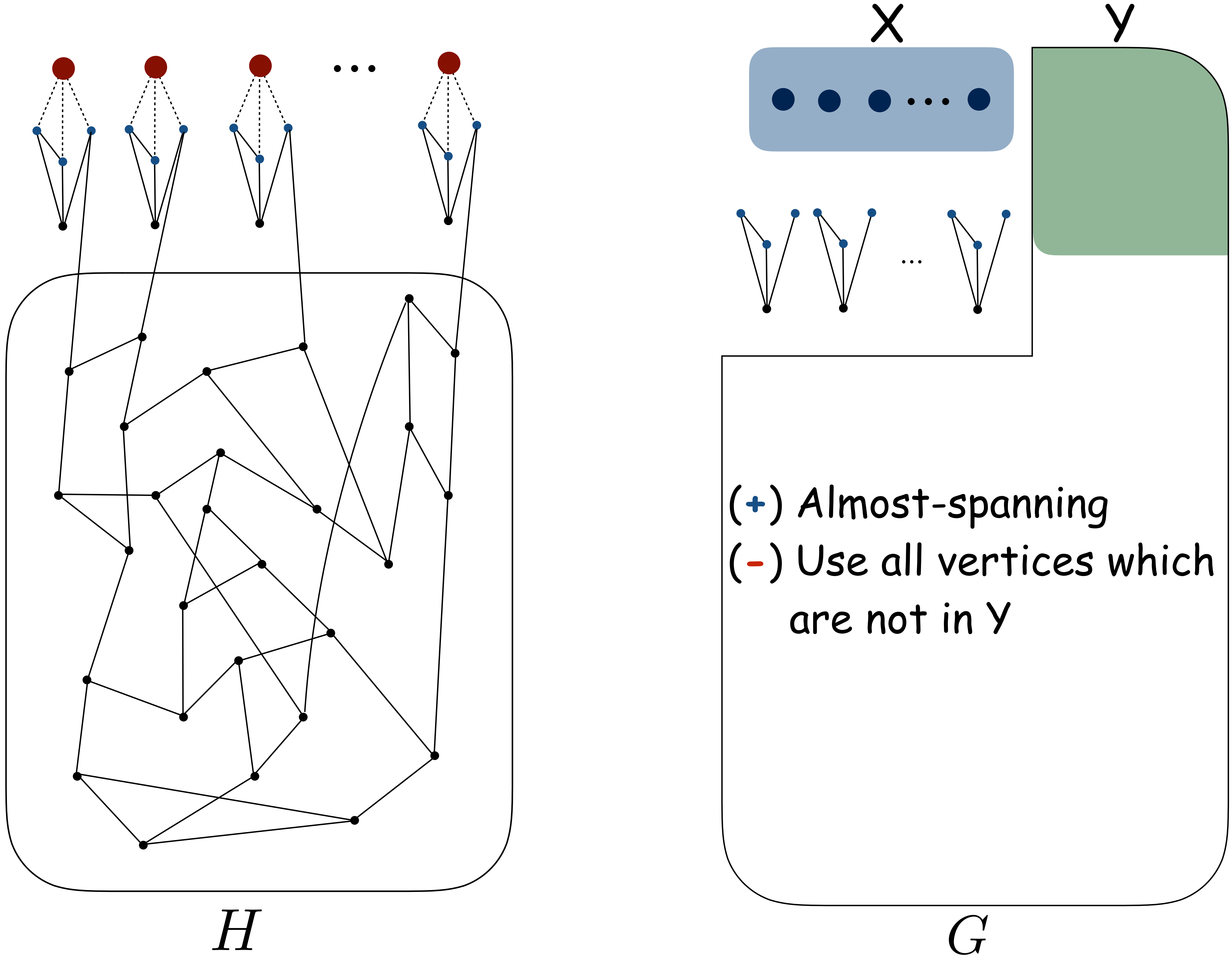


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
H
G

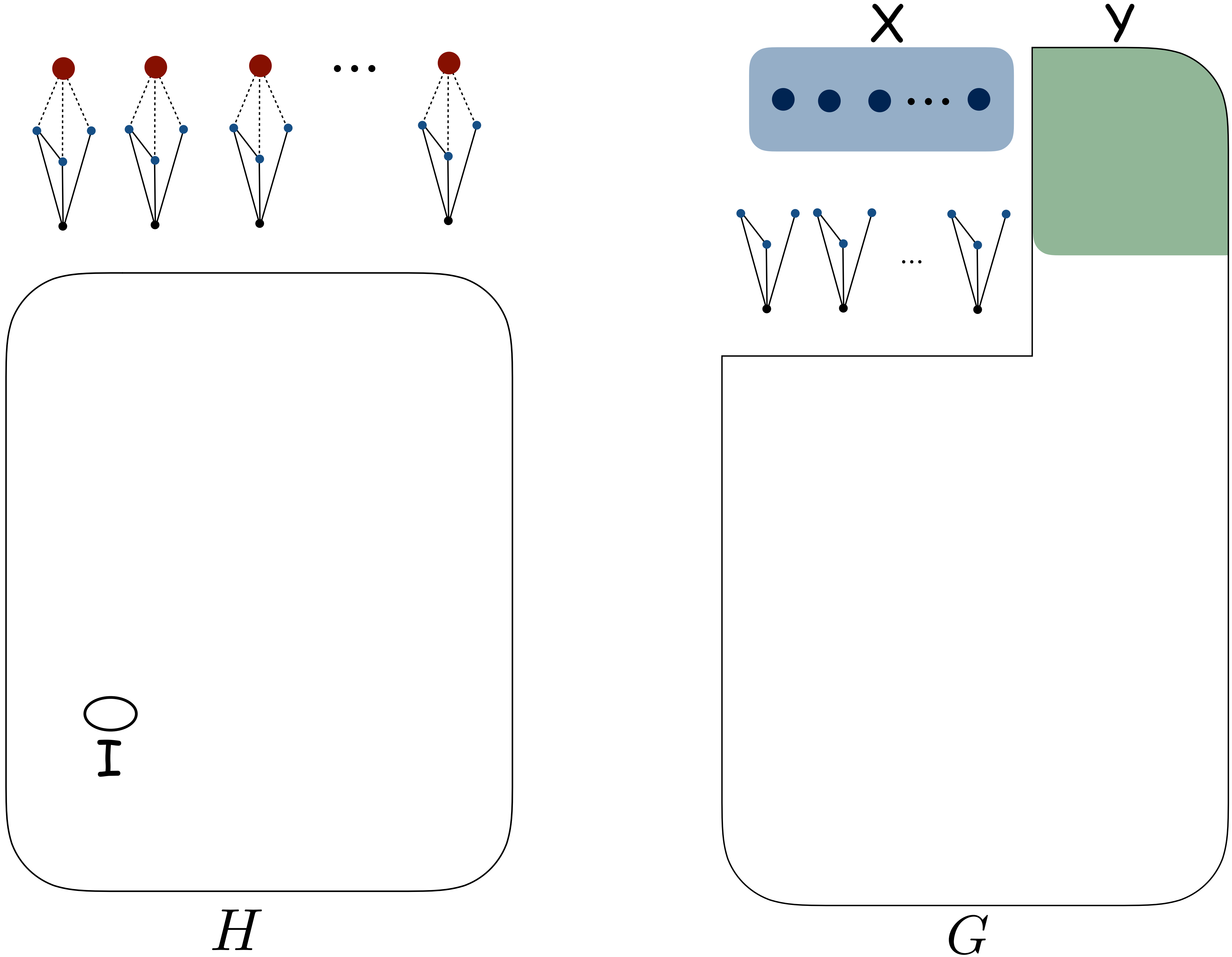


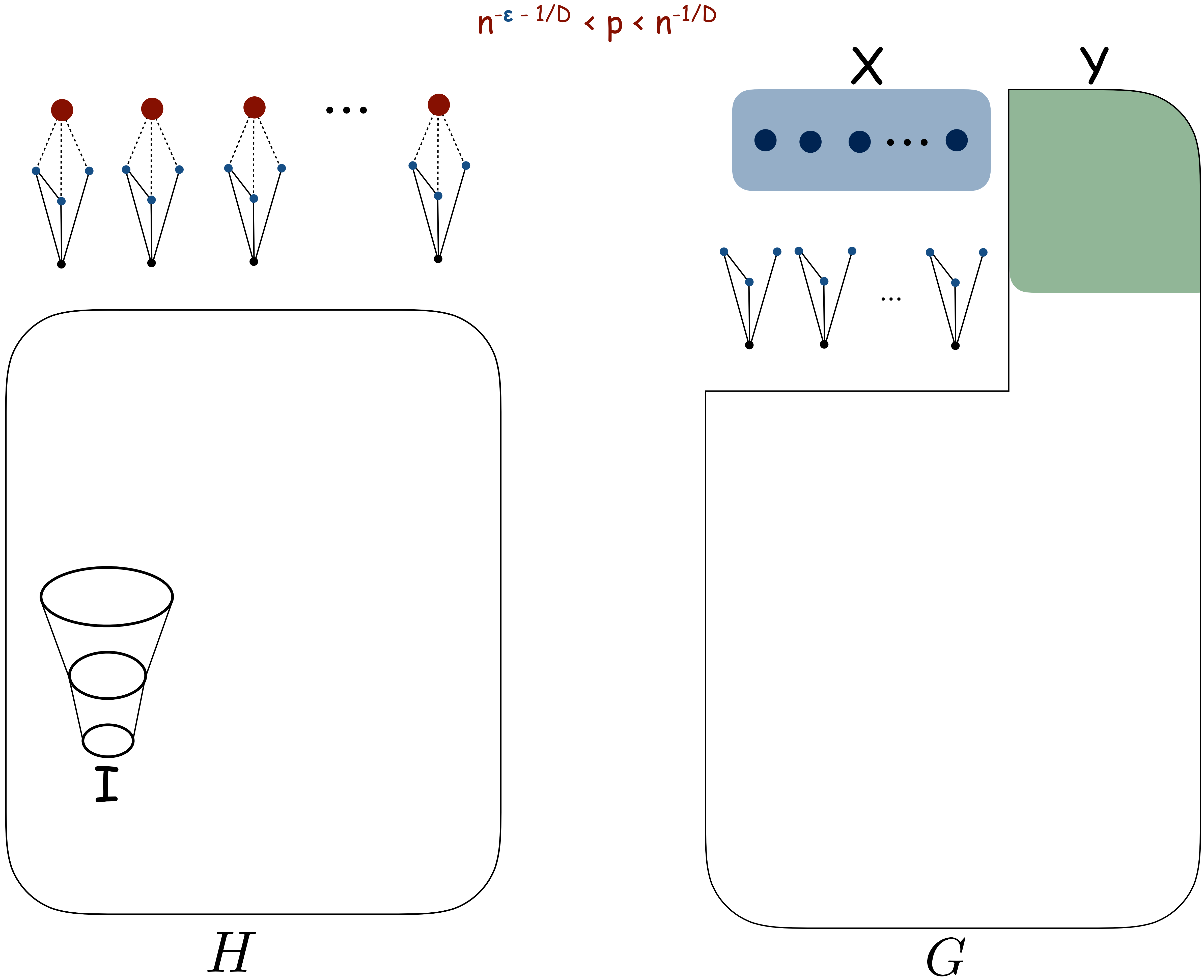
$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
H
G

$n^{-\varepsilon - 1/D} < p < n^{-1/D}$

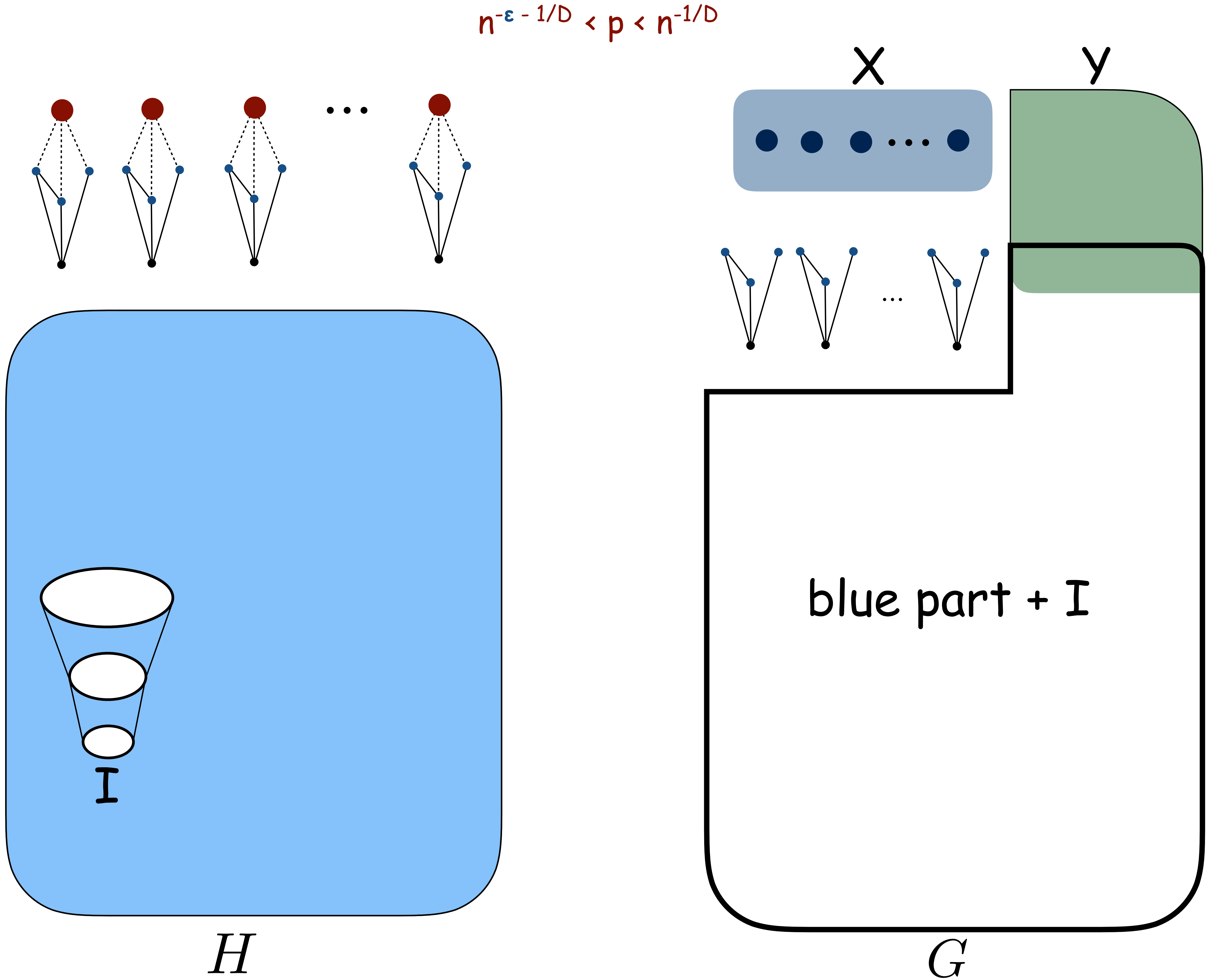


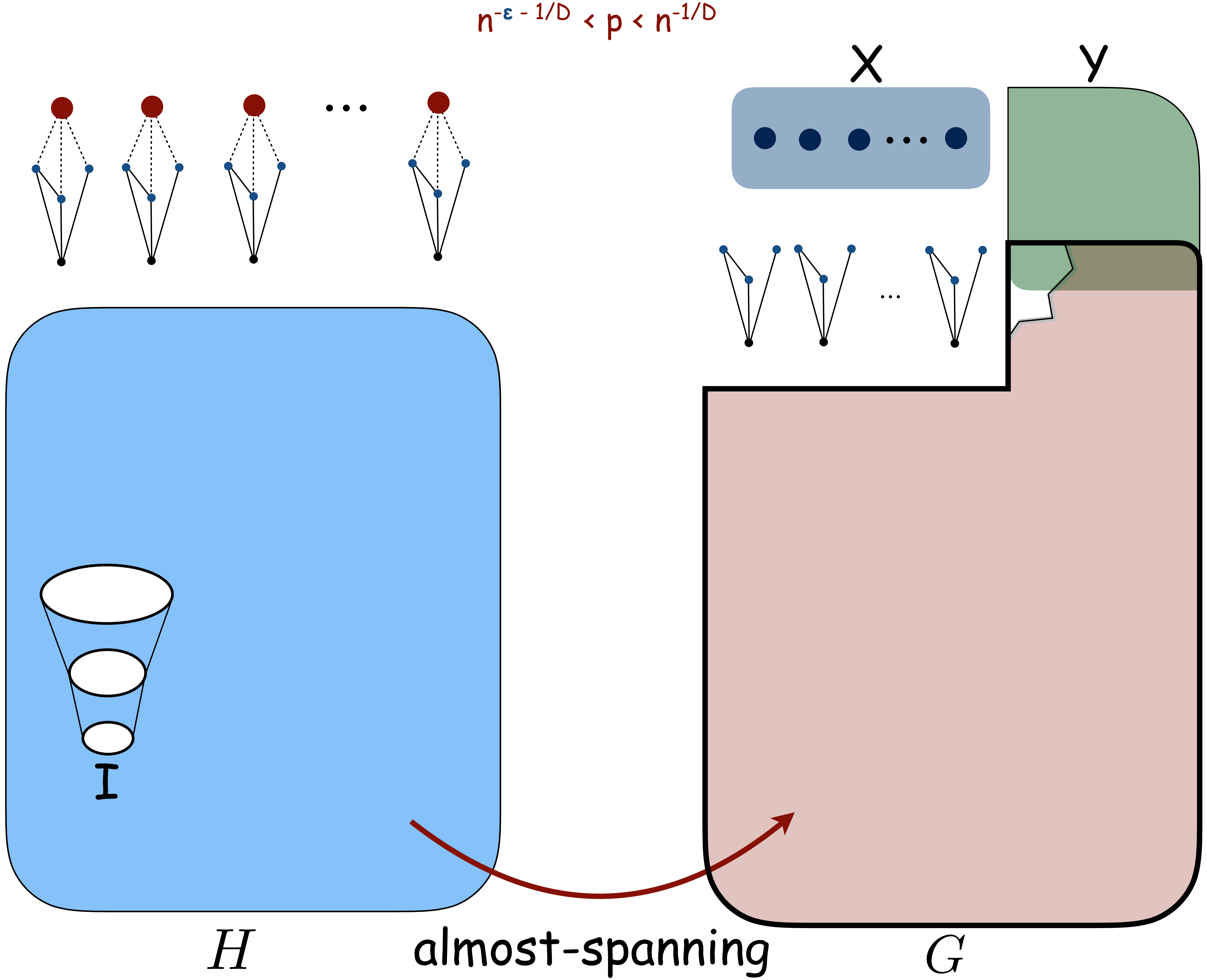
$n^{-\varepsilon - 1/D} < p < n^{-1/D}$

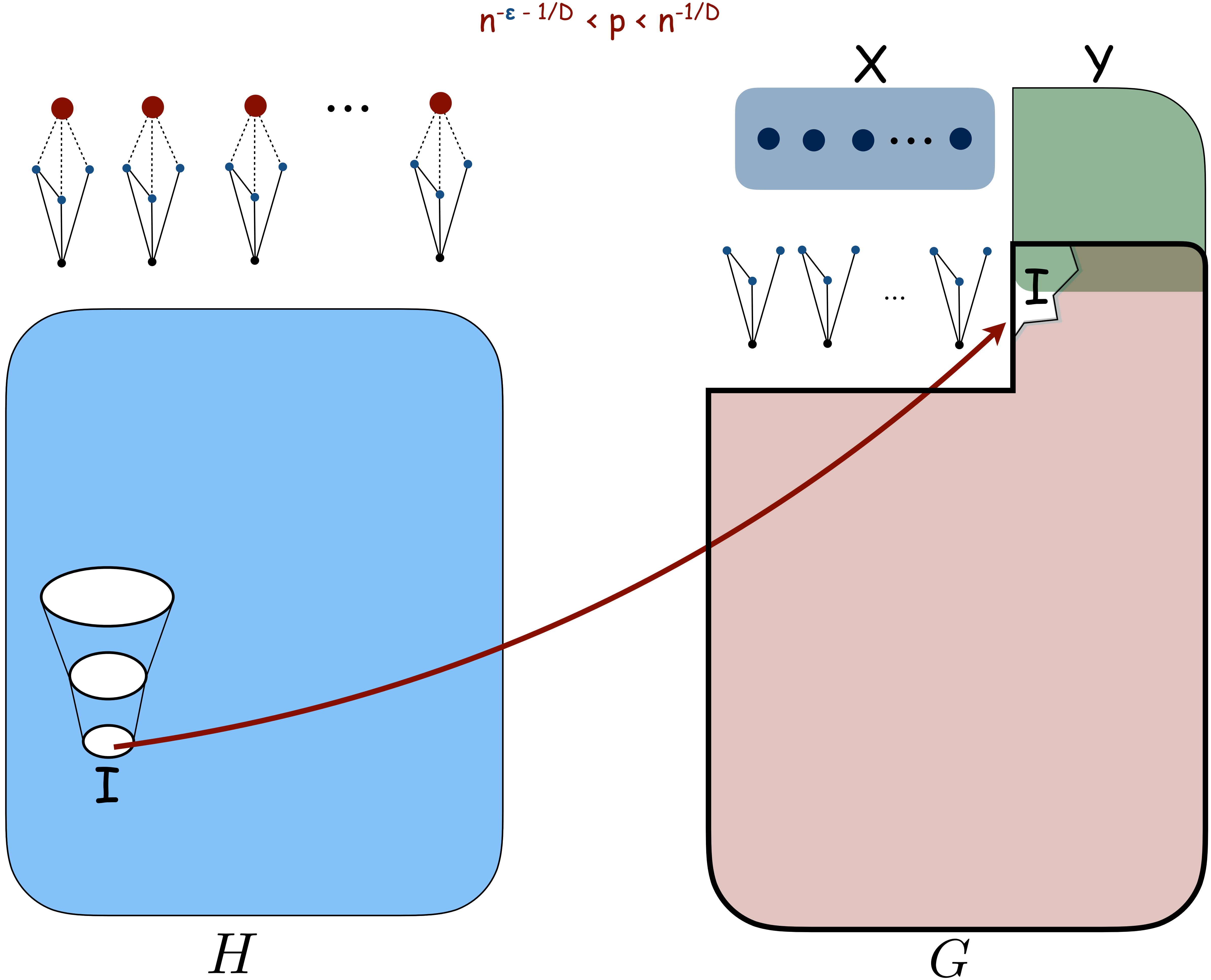


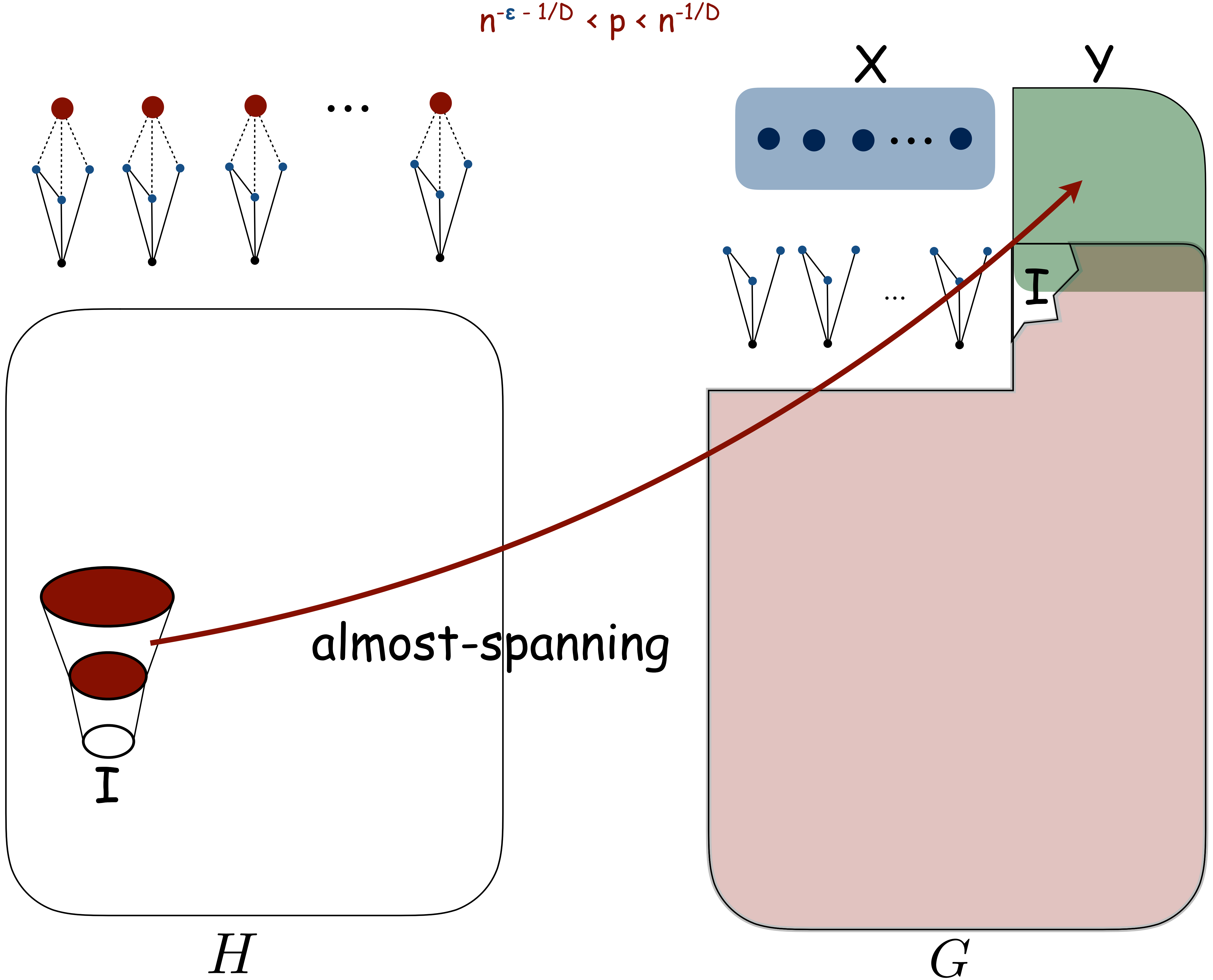


n-ε - 1/D < p < n-1/D
X
Y
I
H
G


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
I
blue part + I
H
G


n^{-ε - 1/D} < p < n^{-1/D}
X
Y
I
H
almost-spanning
G


$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
I
I
H
G



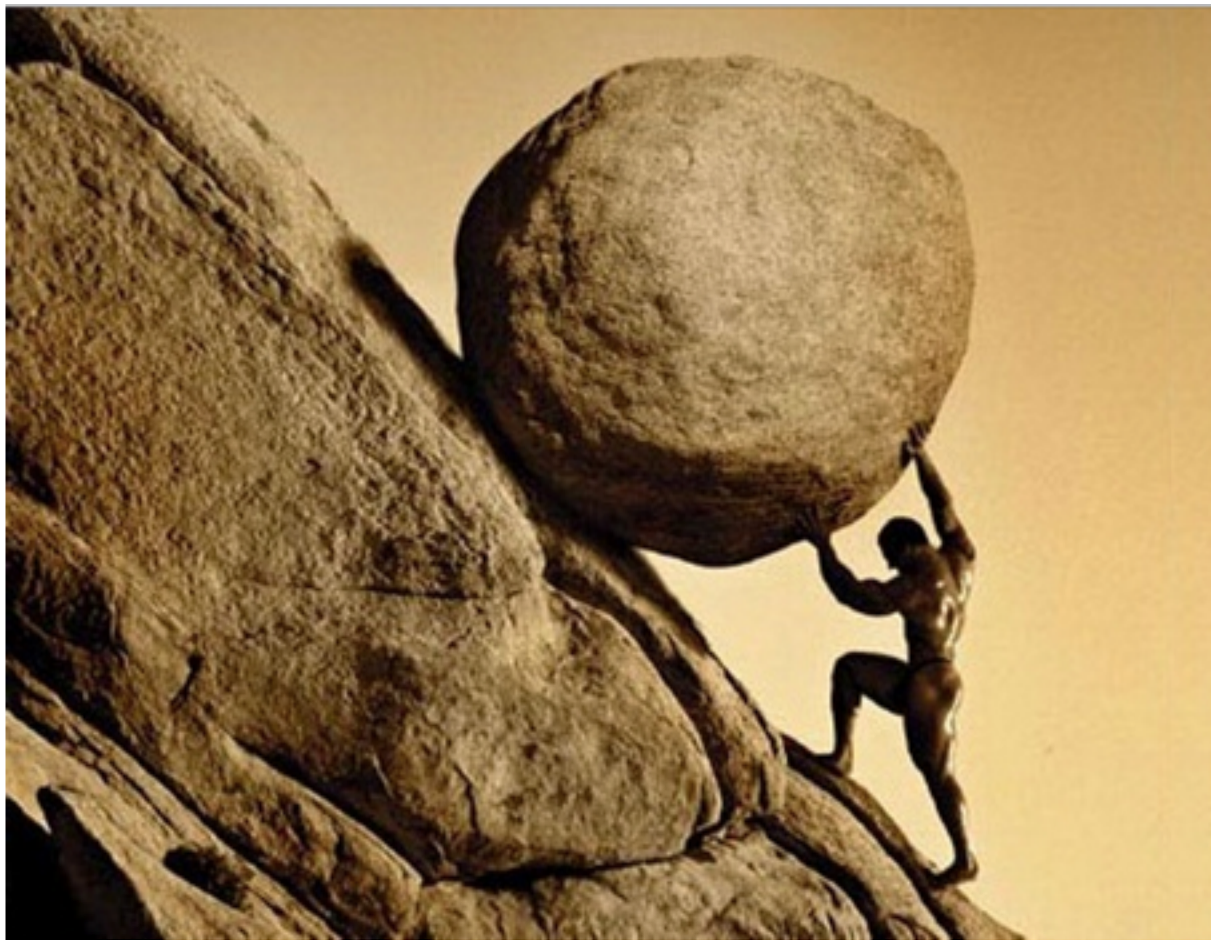
$n^{-\varepsilon - 1/D} < p < n^{-1/D}$
X
Y
I
I
almost-spanning
H
G

$$n^{-\varepsilon - 1/D} < p < n^{-1/D}$$

Future directions

$n^{-1/(D-1)} < p < n^{-1/(D-0.5)}$

- optimal for **D=3**
- matches the best known almost-spanning result



Future directions

$n^{-1/(D-1)} < p < n^{-1/(D-0.5)}$

- optimal for **D=3**
- matches the best known almost-spanning result

$n^{-\varepsilon-1/(D-1)} < p < n^{-1/(D-1)}$

- open even for almost-spanning

?

Thank you!