Structural Sparsity

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— Melbourne $\binom{2^{\binom{2^2}{2}}}{2}$ —

The Dense–Sparse Dichotomy

Overview

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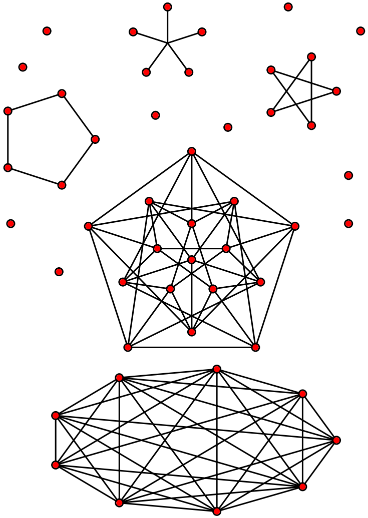
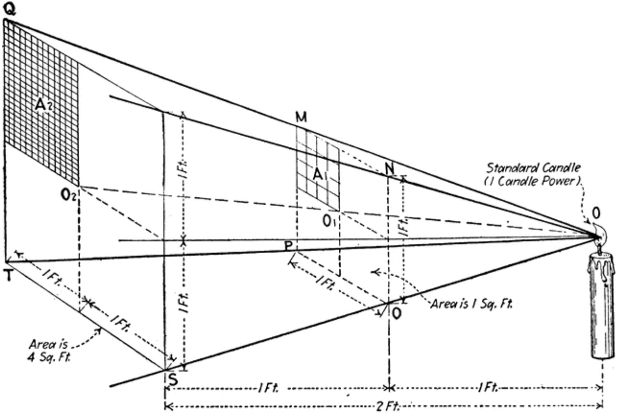
What is a Sparse Structure?

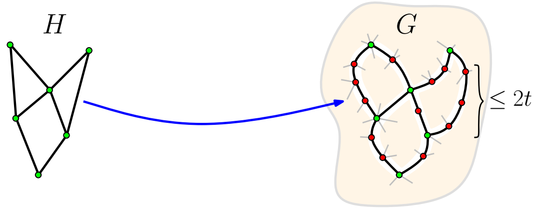


Class Resolutions



Topological resolution of a class $\mathcal{C}$

Shallow topological minors at depth t:



$$\mathcal{C}\,\widetilde{\nabla}\,t = \{H : \text{some} \le 2t\text{-subdivision of } H \text{ is a subgraph of some } G \in \mathcal{C}\}.$$

* * *

Topological resolution:

$$\mathcal{C} \subseteq \mathcal{C}\,\widetilde{\nabla}\,0 \subseteq \mathcal{C}\,\widetilde{\nabla}\,1 \subseteq \dots \subseteq \mathcal{C}\,\widetilde{\nabla}\,t \subseteq \dots \subseteq \mathcal{C}\,\widetilde{\nabla}\,\infty$$

$$\xrightarrow[\textit{time}]{\hspace{10em}}$$

The Somewhere dense — Nowhere dense dichotomy

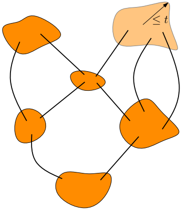
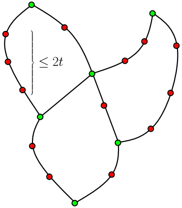
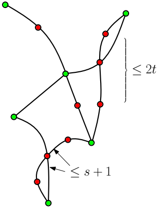
A class $\mathcal{C}$ is *somewhere dense* if there exists τ such that $\mathcal{C} \,\widetilde{\nabla}\, \tau$ contains all graphs.

$$\iff \qquad (\exists \tau)\ \omega(\mathcal{C} \,\widetilde{\nabla}\, \tau) = \infty.$$

A class $\mathcal{C}$ is *nowhere dense* otherwise.

$$\iff \qquad (\forall \tau)\ \omega(\mathcal{C} \,\widetilde{\nabla}\, \tau) < \infty.$$

Every kind of shallow minors

Minor	*Topological minor*	*Immersion*
		

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	ω
Minors			
Topological minors	Bounded expansion		Nowhere dense
Immersions			

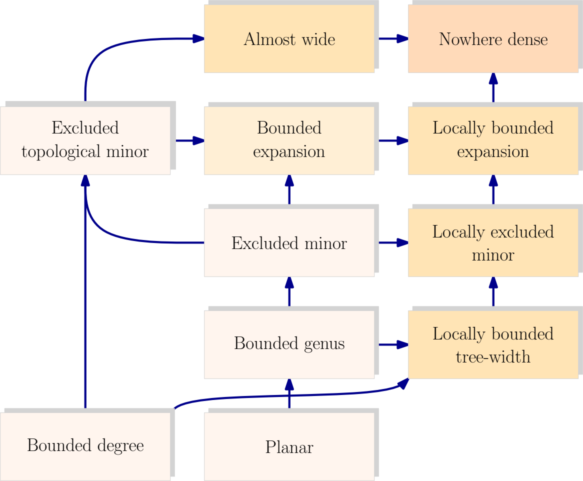
Definition

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	ω
Minors	Bounded expansion	Bounded expansion	Nowhere dense
Topological minors	Bounded expansion	Bounded expansion	Nowhere dense
Immersions	Bounded expansion	Bounded expansion	Nowhere dense

Theorem (Nešetřil, POM 2012)

The Nowhere Dense World



Almost wide

Nowhere dense

Excluded topological minor

Bounded expansion

Locally bounded expansion

Excluded minor

Locally excluded minor

Bounded genus

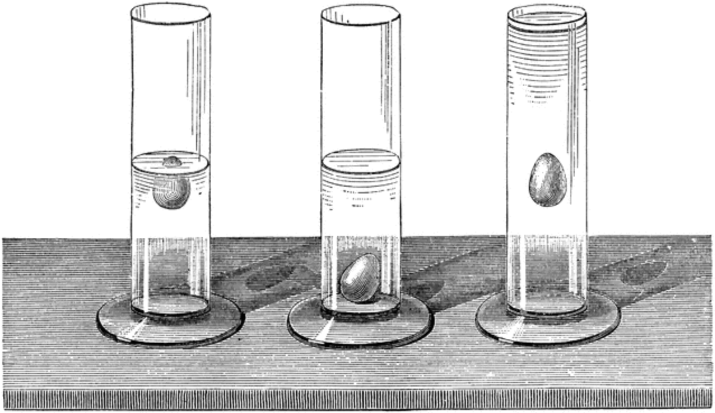
Locally bounded tree-width

Bounded degree

Planar

More...

Density



What is unavoidable in dense graphs?

Theorem (Erdős, Simonovits, Stone 1966)

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

Theorem (Bukh, Jiang 2016)

$$\mathrm{ex}(n, C_{2k}) \leq 80\sqrt{k \log k}\; n^{1+\frac{1}{k}} + O(n).$$

Theorem (Jiang 2010)

$$\mathrm{ex}(n, K_t^{(\leq p)}) = O(n^{1+\frac{10}{p}}).$$

Concentration

Theorem (Jiang 2010)

$$\mathrm{ex}(n, K_t^{(\leq p)}) = O(n^{1+\frac{10}{p}}).$$

$$\begin{tikzcd}
\mathcal{C} & \subseteq & \mathcal{C}\,\widetilde{\nabla}\,0 & \subseteq & \dots & \subseteq & \mathcal{C}\,\widetilde{\nabla}\,t & \subseteq & \dots & \subseteq & \mathcal{C}\,\widetilde{\nabla}\,\frac{10t}{\epsilon} & \subseteq & \dots & \subseteq & \mathcal{C}\,\widetilde{\nabla}\,\infty \\
 & & & & & & \|G\| > C_t\,|G|^{1+\epsilon} \arrow[u] & & & & K_t \arrow[u] & & & &
\end{tikzcd}$$

$\|G\|$= number of edges $\qquad$ $|G|$= number of vertices

Hence:

$$\limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\,t} \frac{\log\|G\|}{\log|G|} > 1+\epsilon \quad\Longrightarrow\quad \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\,\frac{10t}{\epsilon}} \frac{\log\|G\|}{\log|G|} = 2.$$

Classification by logarithmic density

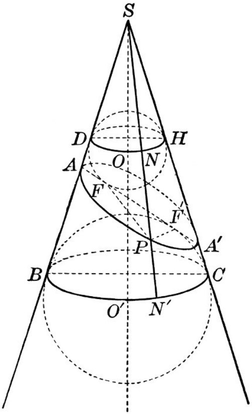
Theorem (Class trichotomy — Nešetřil and POM)

Let $\mathcal{C}$ be an infinite class of graphs. Then

$$\sup_{t} \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\,t} \frac{\log\|G\|}{\log|G|} \in \{-\infty, 0, 1, 2\}.$$

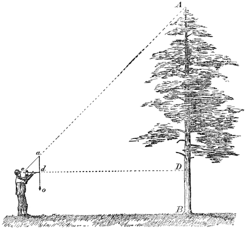
- *bounded size* class $\iff$ $-\infty$ or 0;
- *nowhere dense* class $\iff$ $-\infty, 0$ or 1;
- *somewhere dense* class $\iff$ 2.

Decomposing



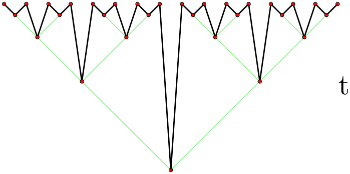
Tree-depth

Definition



The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest Y s.t.

$$G \subseteq \mathrm{Closure}(Y).$$



$$\mathrm{td}(P_n) = \log_2(n+1)$$

Low tree-depth decompositions

$\chi_p(G)$ is the minimum of colors such that every subset I of $\leq p$ colors induces a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.

* * *

Theorem (Nešetřil and POM; 2006, 2010)

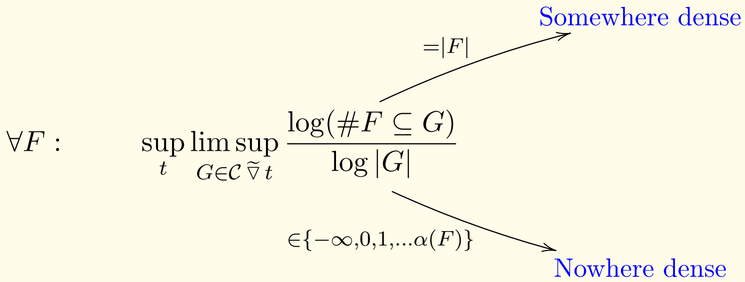
$$\forall p,\ \sup_{G\in\mathcal{C}} \chi_p(G) < \infty \qquad \iff \qquad \mathcal{C} \text{ has bounded expansion.}$$

$$\forall p,\ \limsup_{G\in\mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} = 0 \qquad \iff \qquad \mathcal{C} \text{ is nowhere dense.}$$

(extends DeVos, Ding, Oporowski, Sanders, Reed, Seymour, Vertigan on low tree-width decomposition of proper minor closed classes, 2004)

Logarithmic density (again)

Theorem (Nešetřil and POM)



$$\forall F : \quad \sup_{t} \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\,t} \frac{\log(\#F \subseteq G)}{\log|G|}$$

$=|F|$ → Somewhere dense

$\in\{-\infty,0,1,\dots\alpha(F)\}$ → Nowhere dense

Remark

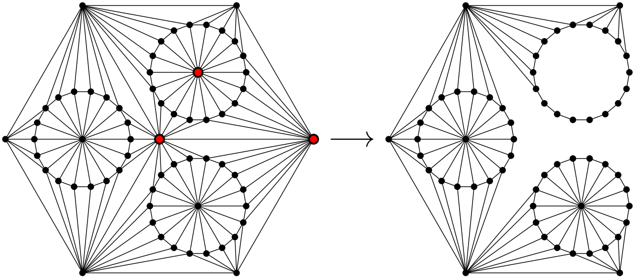
Proof based on Low Tree-Depth Decompositions and regularity properties of bounded height trees. ▸ Details

Flatening

Quasi-wide classes

A class $\mathcal{C}$ of graphs is *quasi-wide* if
$\forall d\ \exists s\ \forall m\ \exists N$: $\forall G \in \mathcal{C}, |G| \geq N$, $\exists S, A \subseteq V(G)$ with

- $|S| \leq s, |A| \geq m$,
- $\forall x \neq y \in A \setminus S,\ \mathrm{dist}_{G-S}(x, y) > d$.



Quasi-wide classes

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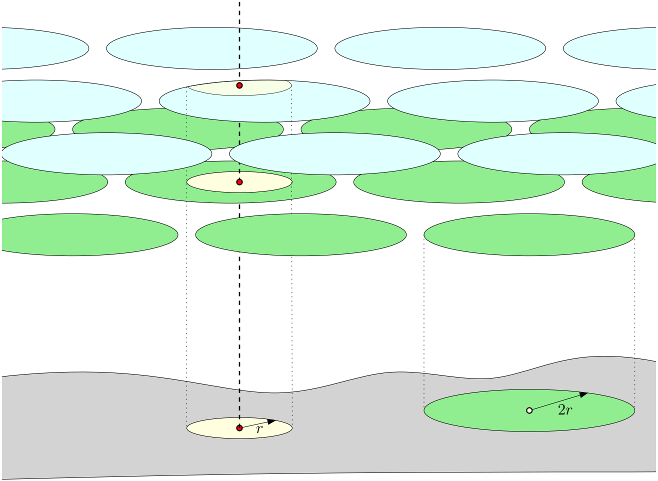
- $|S| \leq s, |A| \geq m$,
- $\forall x \neq y \in A \setminus S,\ \mathrm{dist}_{G-S}(x, y) > d$.

Theorem (Nešetril and POM)

A hereditary class of graphs is quasi-wide if and only if it is nowhere dense.

r-neighbourhood covers



r-neighbourhood covers

Theorem (Grohe, Kreutzer, Siebertz 2013)

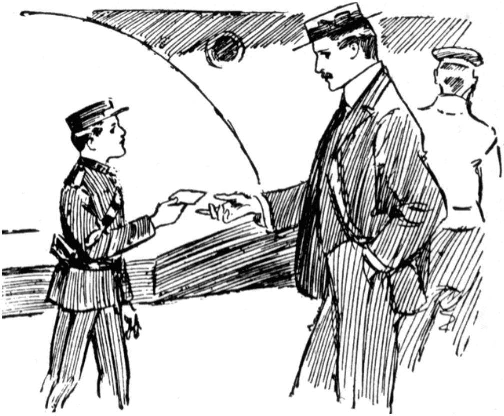
For every $\mathcal{C}$ nowhere dense (resp. bounded expansion) class of graphs there is f s.t.
$\forall r \in \mathbb{N}$, $\epsilon > 0$, and $G \in \mathcal{C}$ with $|G| \geq f(r, \epsilon)$ there exists a family $\mathcal{X}$ of subgraphs of G s.t.

- the maximum radius of $H \in \mathcal{X}$ is $\leq 2r$;
- every $v \in G$ has all its r-neighborhood in some $H \in \mathcal{X}$;
- every $v \in G$ belongs to at most $|G|^\epsilon$ (resp. $K(\mathcal{C}, r, \epsilon)$) subgraphs in $\mathcal{X}$.

Remark

Actually a characterization of nowhere dense and bounded expansion monotone classes.

Model Checking



Model checking

Theorem (Dvořák, Kráľ, Thomas 2010)

For every class $\mathcal{C}$ with bounded expansion, every property of graphs definable in first-order logic can be decided in time $O(n)$ on $\mathcal{C}$.

Theorem (Kazana, Segoufin 2013)

For every class $\mathcal{C}$ with bounded expansion, every first-order definable subset can be enumerated in lexicographic order in constant time between consecutive outputs and linear time preprocessing time.

Model checking

Theorem (Grohe, Kreutzer, Siebertz 2014)

For every nowhere dense class $\mathcal{C}$ and every $\epsilon > 0$, every property of graphs definable in first-order logic can be decided in time $O(n^{1+\epsilon})$ on $\mathcal{C}$.

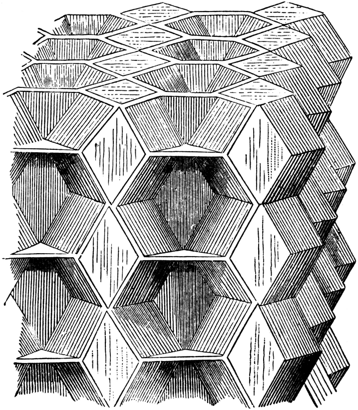
Theorem (Dvořák, Kráľ, Thomas 2010; Kreutzer 2011)

if a monotone class $\mathcal{C}$ is somewhere dense, then deciding first-order properties of graphs in $\mathcal{C}$ is not fixed-parameter tractable (unless FPT $=$ W[1].

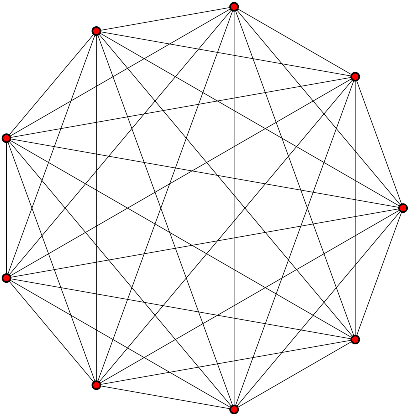
Remark

Hence a characterization of nowhere dense/somewhere dense dichotomy in terms of algorithmic complexity.

Excluded Structures

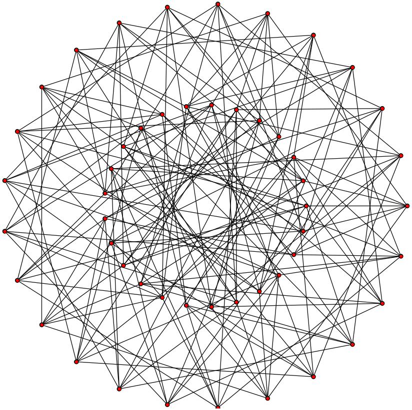


Nowhere dense



$$\text{At each depth, an excluded } \begin{cases} \text{topological minor} \\ \text{minor} \\ \text{immersion} \end{cases}$$

Bounded expansion



$$\text{At each depth, bounded } \left\{\begin{array}{l}\text{average degree}\\ \text{chromatic number}\end{array}\right\} \text{ of } \left\{\begin{array}{l}\text{topological minors}\\ \text{minors}\\ \text{immersions}\end{array}\right.$$

Forbidden structure for bounded expansion?

Conjecture

Every monotone nowhere dense class of graphs $\mathcal{C}$

- either has bounded expansion,
- or contains, for some $k \in \mathbb{N}$, the k-th subdivisions of graphs with arbitrarily large girth and chromatic number.

Conjecture (Thomassen)

$$\delta(G) \geq F_\delta(d, g)$$

$$\Downarrow$$

$$\exists H \subseteq G : \begin{cases} \delta(H) \geq d, \\ \text{girth}(H) \geq g \end{cases}$$

($g = 6$: Kühn, Osthus 2002)

Conjecture (Erdős–Hajnal)

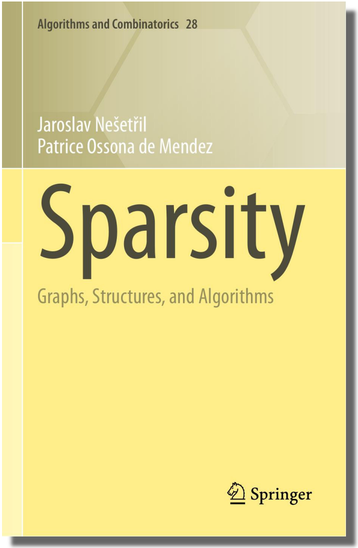
$$\chi(G) \geq F_\chi(c, g)$$

$$\Downarrow$$

$$\exists H \subseteq G : \begin{cases} \chi(H) \geq c, \\ \text{girth}(H) \geq g \end{cases}$$

($g = 4$: Rödl 1977)

Commercial Break



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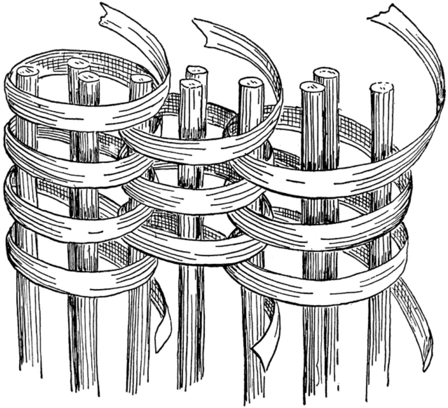
Sparsity

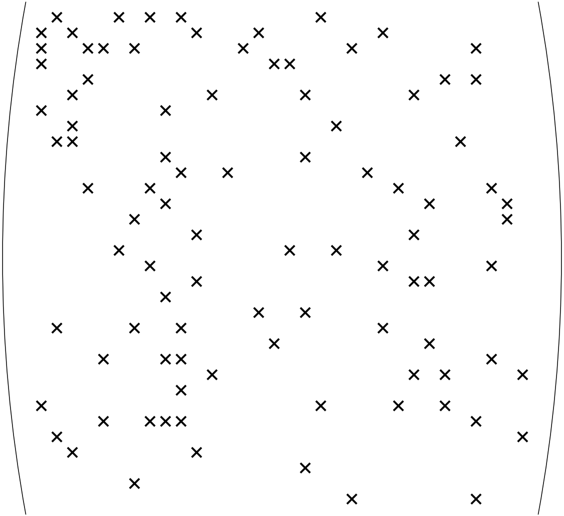
Graphs, Structures, and Algorithms

Springer

Structurally sparse classes



What is a Sparse Structure? (the return)



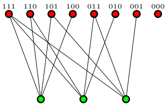
Structurally Sparse Classes

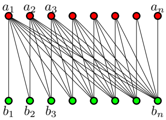
Definition (outline)

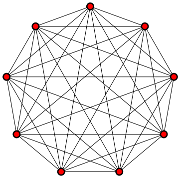
A class is *structurally sparse* if it can be “interpreted” in a sparse class.

- small classes;
- random-free classes;
- classes with few (local) types of vertices;
- classes excluding some special (generic) structures.

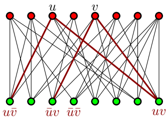
Special Generic Structures



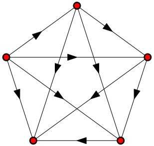




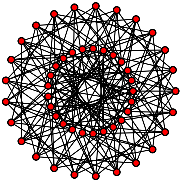
Nowhere dense



NIP

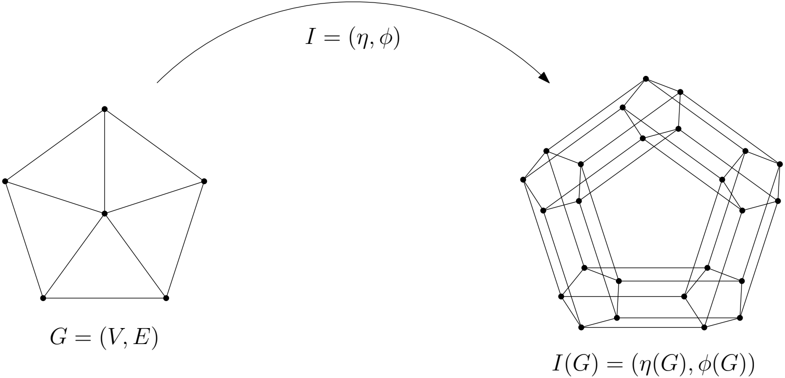


Stability



Bounded expansion

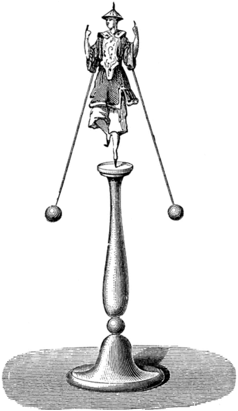
Interpretation



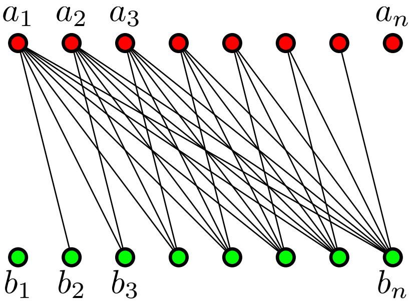
$$\eta(x_1, x_2) := (\deg(x_1) = 3) \wedge (\deg(x_2) = 3)$$

$$\phi(x_1, x_2; y_1, y_2) := ((x_1 \sim y_1) \wedge (x_2 = y_2)) \vee ((x_1 = y_1) \wedge (x_2 \sim y_2))$$

Stability & NIP

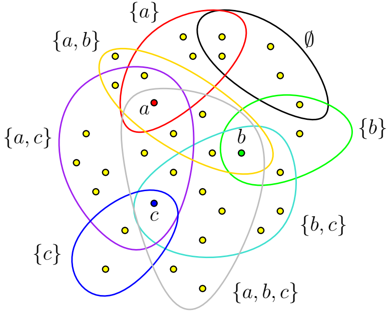


Stability and Order property



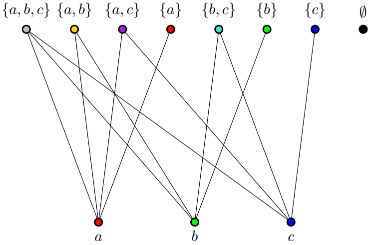
$$G \models \phi(\bar{a}_i, \bar{b}_j) \quad \iff \quad i < j$$

NIP and VC-dimension



$$\phi(G, \bar{y}) = \{\bar{x} : \ G \models \phi(\bar{x}, \bar{y})\}$$

$$\mathcal{K}(\phi, G) = \{\phi(G, \bar{y}) : \ \bar{y} \in G\}$$



Bounded VC-dimension

Theorem (Grohe, Turán 2004)

For any monotone graph class $\mathcal{C}$, the following are equivalent:

1. MSO has bounded VC dimension on $\mathcal{C}$;
2. $\mathcal{C}$ has bounded treewidth.

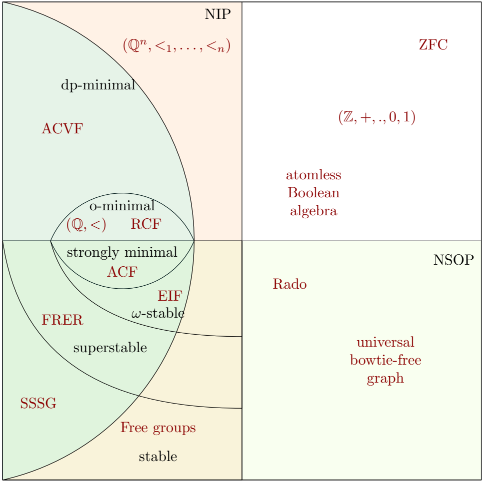
Theorem (Adler, Adler 2010; Laskowski 1992)

For any monotone graph class $\mathcal{C}$, the following are equivalent:

1. FO has bounded VC dimension on $\mathcal{C}$ (NIP);
2. FO has bounded order property on $\mathcal{C}$ (Stability);
3. $\mathcal{C}$ is nowhere dense.

▸ Example

A Glimpse at Model Theory World



(based on model theory universe by Gabriel Conant)

ACVF	Algebraically Closed Value Fields
RCF	Real Closed Field
ACF	Algebraically Closed Field
EIF	Everywhere Infinite Forest (Fraïssé limit of finite trees)
FRER	Finitely Refining Equivalence Relations
SSSG	Strictly Stable Superflat Graph

Shatter function

$$\pi_{\mathcal{S}}(n) = \max_{|A| \leq n} \left| \{C \cap A : \ C \in \mathcal{S}\} \right| \quad \textit{shatter function}$$

Theorem

Let $\mathcal{C}$ be a monotone class of graphs.
For $r \in \mathbb{N}$ let $\mathcal{S}_r = \{N_r(G, v) : \ v \in V(G), G \in \mathcal{C}\}$.

Then $\mathcal{C}$ is

- a somewhere dense class iff $(\exists r)\ \pi_{\mathcal{S}_r}(n) = 2^n$;
- a nowhere dense class iff $(\forall r)\ \pi_{\mathcal{S}_r}(n)$ is polynomial in n;
- a bounded expansion class iff $(\forall r)\ \pi_{\mathcal{S}_r}(n)$ is linear in n.

Proof.

Adler–Adler (2010) + Sauer–Shelah (1972) +
Reidl–Villaamil–Stavropoulos (2016) □

Structural Limits

Structural Limits

Definition (Stone pairing)

Let G be a graph and let ϕ be a first-order formula with p free variables.

$$\langle \phi, G \rangle = \frac{|\phi(G)|}{|G|^p} = \Pr(G \models \phi(X_1, \ldots, X_p))$$

for independently and uniformly distributed $X_i \in G$.

A sequence $(\mathbf{A}_n)$ is FO-convergent if, for every $\phi \in \mathrm{FO}$, the sequence $\langle \phi, \mathbf{A}_1 \rangle, \ldots, \langle \phi, \mathbf{A}_n \rangle, \ldots$ is convergent.

Representation theorem

Theorem (Nešetřil, POM 2012)

There are maps $\mathbf{A} \mapsto \mu_{\mathbf{A}}$ and $\phi \mapsto k(\phi)$, such that

- $\mathbf{A} \mapsto \mu_{\mathbf{A}}$ is injective;
- $\mu_{\mathbf{A}}$ is $\mathfrak{S}_\omega$-invariant;
- $\langle \phi, \mathbf{A} \rangle = \int_S k(\phi)\, \mathrm{d}\mu_{\mathbf{A}}$;
- a sequence $(\mathbf{A}_n)_{n \in \mathbb{N}}$ is FO-convergent iff $\mu_{\mathbf{A}_n}$ converges weakly.

Thus if $\mu_{\mathbf{A}_n} \Rightarrow \mu$, it holds

$$\langle \phi, \mu \rangle := \int_S k(\phi)\, \mathrm{d}\mu = \lim_{n \to \infty} \int_S k(\phi)\, \mathrm{d}\mu_{\mathbf{A}_n} = \lim_{n \to \infty} \langle \phi, \mathbf{A}_n \rangle.$$

Modelings

Definition

A modeling $\mathbf{A}$ is a graph on a standard probability space s.t. every first-order definable set is measurable.

$$\langle \phi, \mathbf{A} \rangle = \nu_{\mathbf{A}}^{\otimes p}(\phi(\mathbf{A})).$$

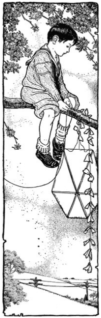
Theorem (Nešetřil, POM 2013)

If a monotone class $\mathcal{C}$ has modeling limits then $\mathcal{C}$ is nowhere dense.

▸ Proof

Conjecture

A monotone class $\mathcal{C}$ has modeling limits iff $\mathcal{C}$ is nowhere dense.



Thank you for your attention.

Hints: Random Graphs

Bounded Expansion Random Graphs

Demaine, Reidl, Rossmanith, Villaamil, Sikdar, Sullivan 2015

- Configuration Model and the Chung-Lu Model with specified asymptotic degree sequences

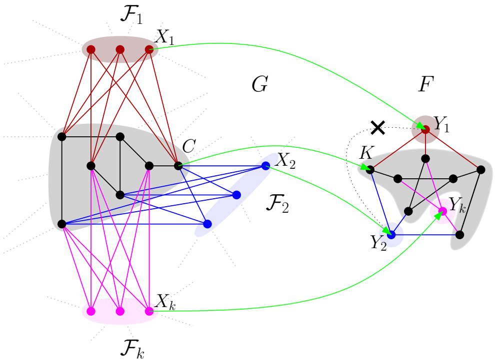
Power law	$d^{-\gamma}$	$\gamma > 2$
Power law w/ cutoff	$d^{-\gamma}e^{-\lambda d}$	$\gamma > 2, \lambda > 0$
Exponential	$e^{-\lambda d}$	$\lambda > 0$
Stretched exponential	$d^{\beta-1}e^{-\lambda d^{\beta}}$	$\lambda, \beta > 0$
Gaussian	$\exp\left(-\frac{(d-\mu)^2}{2\sigma^2}\right)$	μ, σ
Log-normal	$d^{-1}\exp\left(-\frac{(\log d-\mu)^2}{2\sigma^2}\right)$	μ, σ

- generalization of Erdős-Rényi graphs (perturbed bounded-degree graphs), which includes the stochastic block model with small probabilities.

◂ Back

Hints: Sunflowers

(k, F)-sunflower $(C, \mathcal{F}_1, \ldots, \mathcal{F}_k)$



$\forall X_1 \in \mathcal{F}_1, \ldots \forall X_k \in \mathcal{F}_k$

$G[C \cup X_1 \cup \cdots \cup X_k] \approx F$

$\Rightarrow k \leq \alpha(F)$ and

$$(\#F \subseteq G) \geq \prod_{i=1}^{k} |\mathcal{F}_i|.$$

Finding a large sunflower

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower $(C, \mathcal{F}_1, \dots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\chi_p(G)^{p/\epsilon}}\right)^{\tau(\epsilon,p)}.$$

Hence $\exists G' \subseteq G$ such that

$$|G'| \geq \left(\frac{|G|}{\chi_p(G)^{p/\epsilon}}\right)^{\tau(\epsilon,p)} \quad \text{and} \quad (\#F \subseteq G') \geq \left(\frac{|G'| - |F|}{k+1}\right)^{k+1}.$$

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Hints: VC dimension

Example

Problem

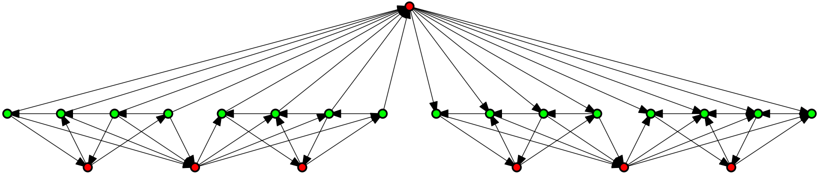
Prove that there exist functions f, g such that

$$\forall n, r \in \mathbb{N} \; \forall \text{graph } G$$

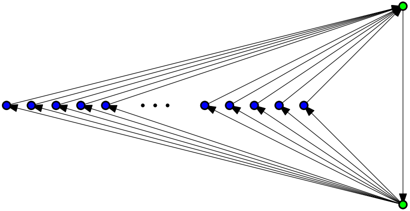
If there exists an orientation of G and a subset $X \subseteq V(G)$ with $|X| = f(n, r)$, such that $\forall u \neq v \in X$ there exists in oriented G

- either a directed path of length r from u to v;
- or a directed path of length r from v to u.

Then G contains a $g(r)$-subdivision of K_n.



But not



Example

Theorem

If there exists an orientation of G and a subset $X \subseteq V(G)$ with $|X| = f(n,r)$, such that $\forall u \neq v \in X$

- either there exists a directed path of length r from u to v;
- or there exists a directed path of length r from v to u.

Then G contains a $g(r)$-subdivision of K_n.

Proof.

Assume for contradiction $\exists \mathcal{C}$ (monotone) nowhere dense with graphs with arbitrarily large X, and let $\eta(x,y) := \exists$ directed path of length r from u to v.

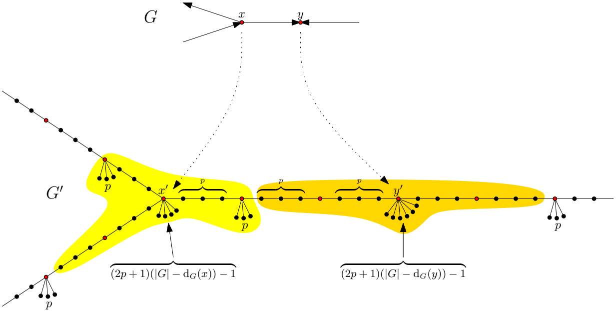
Unbounded tournaments $\Rightarrow$ Unbounded transitive tournaments $\Rightarrow$ Unbounded order property $\Rightarrow$ somewhere dense ↯ $\square$

Simple proof but does not give f and g!

Hints: Modelings

Proof (sketch)

- Assume $\mathcal{C}$ is somewhere dense. There exists $p \geq 1$ such that $\mathrm{Sub}_p(K_n) \in \mathcal{C}$ for all n;
- For an oriented graph G, define $G' \in \mathcal{C}$:



- $\exists$ basic interpretation I, such that for every graph G, $I(G') \cong G[k(G)] \overset{\text{def}}{=} G^+$, where $k(G) = (2p+1)|G|$.

Proof (sketch)

$$\begin{tikzcd}
G_n \arrow[r, "\mathrm{L}"] & 1/2 & \Longleftrightarrow & G_n^+ \arrow[r, "\mathrm{L}"] \arrow[dd, red] & 1/2 \\
G'_n \arrow[r, "\mathrm{FO}"] \arrow[d, "I"'] & \mathbf{A} \arrow[d, "I"] & & & \\
G_n^+ \arrow[r, "\mathrm{FO}"] & I(\mathbf{A}) & & \lightning & \\
& & & & \\
G_n^+ \arrow[r, "\mathrm{L}"] & W_{I(\mathbf{A})} \arrow[uurr, red] & & & \square
\arrow[from=3-1, to=5-2, phantom, "\Downarrow"]
\end{tikzcd}$$

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