A DUAL OF TARSKI'S PLANK PROBLEM

– USING TOPOLOGY

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Uwagi o stopniu równoważności wielokątów.

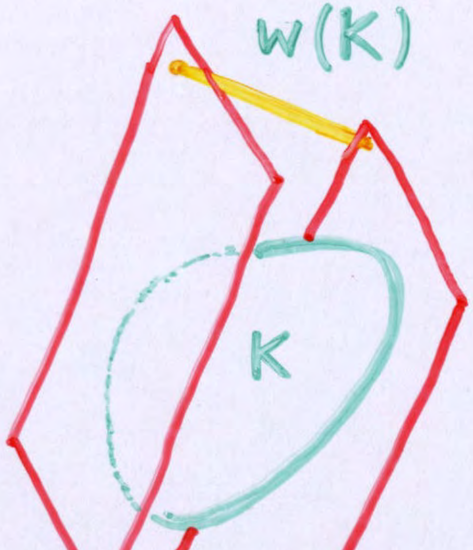
W artykule, ogłoszonym w „*Młodym Matematyku*"[1]), postawiłem szereg zagadnień, dotyczących stopnia równoważności wielokątów. Artykuł napisany był widocznie „szczęśliwą ręką": temat, który poruszyłem, zainteresował kilku matematyków; dzięki ich poszukiwaniom różne przypuszczenia, wypowiedziane przeze mnie, zostały potwierdzone lub obalone, i jeśli chodzi o główne zagadnienie wysunięte we wspomnianym artykule, — o wyczepujące zbadanie funkcji $\tau(x)$, to niewiele już brak w chwili obecnej do definitywnego jego rozstrzygnięcia.

W uwagach poniższych, opierając się na poprzedzającym artykule p. H. Moesego w tymże zeszycie „*Parametra*" oraz na znanych mi a dotąd nieopublikowanych rozważaniach pp. dr. A. Lindenbauma i mg. Z. Waraszkiewicza, pragnę zestawić wszystkie uzyskane do chwili obecnej wyniki, odnoszące się do funkcji $\tau(x)$, a przytem wydobyć z nich pewne fakty ogólniejszej natury; ponadto zamierzam wysunąć kilka dalszych zagadnień z tego samego zakresu.

TARSKI'S PLANK PROBLEM (1932)

If a convex body K is covered by a finite number of slabs (planks) $S_1, S_2, \ldots, S_n$, then their minimal widths satisfy

$$\sum_{i=1}^{n} w(S_i) \geq w(K)$$



Bang (1950, 1951) +

Gardner (1988) −

Ball (1991) relative width

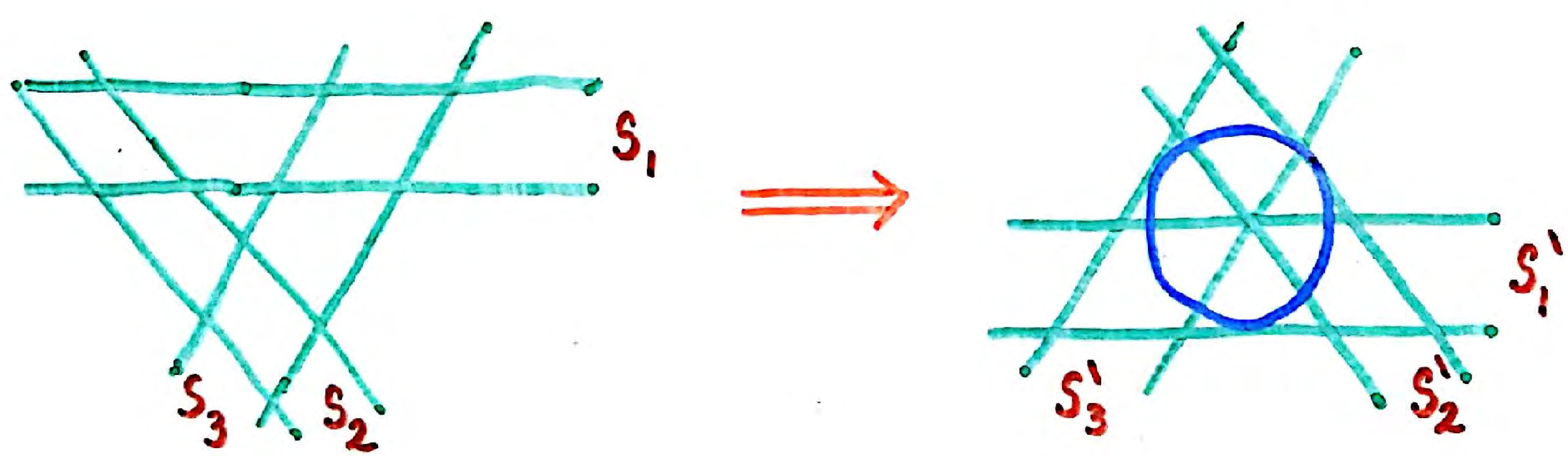
$K = -K$

πw
w

TRANSLATIVE COVERING CONJECTURE

slab (strip) = part of $\mathbb{R}^d$ between two parallel hyperplanes at distance w (width)

Conjecture (Makai - P. 1983)
Let $S_1, S_2, \dots$ be a sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \dots$.
If $\sum_{i=1}^{\infty} w_i = \infty$, then there exist translates S_i' of S_i such that $\bigcup_{i=1}^{\infty} S_i'$ covers the unit ball.



Theorem (Kupavskii - P. '16)
Let $w_1 \geq w_2 \geq \cdots$ be a sequence of positive numbers satisfying

$$\limsup_{n \to \infty} \frac{w_1 + w_2 + \cdots + w_n}{\log(1/w_n)} > 0 .$$

Then any sequence of slabs with widths $w_1, w_2, \cdots$ permits a translative covering of $\mathbb{R}^d$.

Lemma. Let $w_1 \geq w_2 \geq \cdots \geq w_n > 0$ satisfy
$$w_1 + w_2 + \cdots + w_n \geq 3d \log \frac{2}{w_n}.$$
Then any sequence of slabs with these widths permits a translative covering of a ball of diameter $1 - \frac{w_n}{2}$ in $\mathbb{R}^d$.

Proof

1. Replace S_i of width w_i by a slab S_i' of width $\frac{w_i}{2}$.
2. Translate $S_1', S_2', \ldots$ successively so as to minimize the uncovered area in each step.

   $$\mathrm{Vol}(\text{Uncovered}) \leq \mathrm{Vol}(B) \prod_{i=1}^{n} \left(1 - \frac{w_i}{3}\right) < \mathrm{Vol}(B) \exp\left(-\frac{1}{3} \sum_{i=1}^{n} w_i\right)$$
   $$< \mathrm{Vol}(B) \left(\frac{w_n}{2}\right)^d$$

   $\Rightarrow$ $\forall$ Uncovered point is at distance $\leq \frac{w_n}{4}$ from a slab or from $\mathrm{Bd}(B)$
3. Expand each slab by a factor of 2.

CONTROLLING FUNCTION CLASSES

Given a class of functions $\mathcal{F}$, a sequence of positive numbers $0 < x_1 \leq x_2 \leq \ldots$ is called $\mathcal{F}$-controlling if there exists another sequence $y_1, y_2, \ldots$ with the property that

$$\forall f(x) \in \mathcal{F} \quad \exists i \quad \text{such that} \quad |f(x_i) - y_i| \leq 1.$$

Examples

$\mathcal{F} = \{ ax+b : a,b \in \mathbb{R} \}$, $\{ a_0 + a_1 x + \ldots + a_d x^d : a_0, a_1, \ldots, a_d \in \mathbb{R} \}$

Lipschitz functions

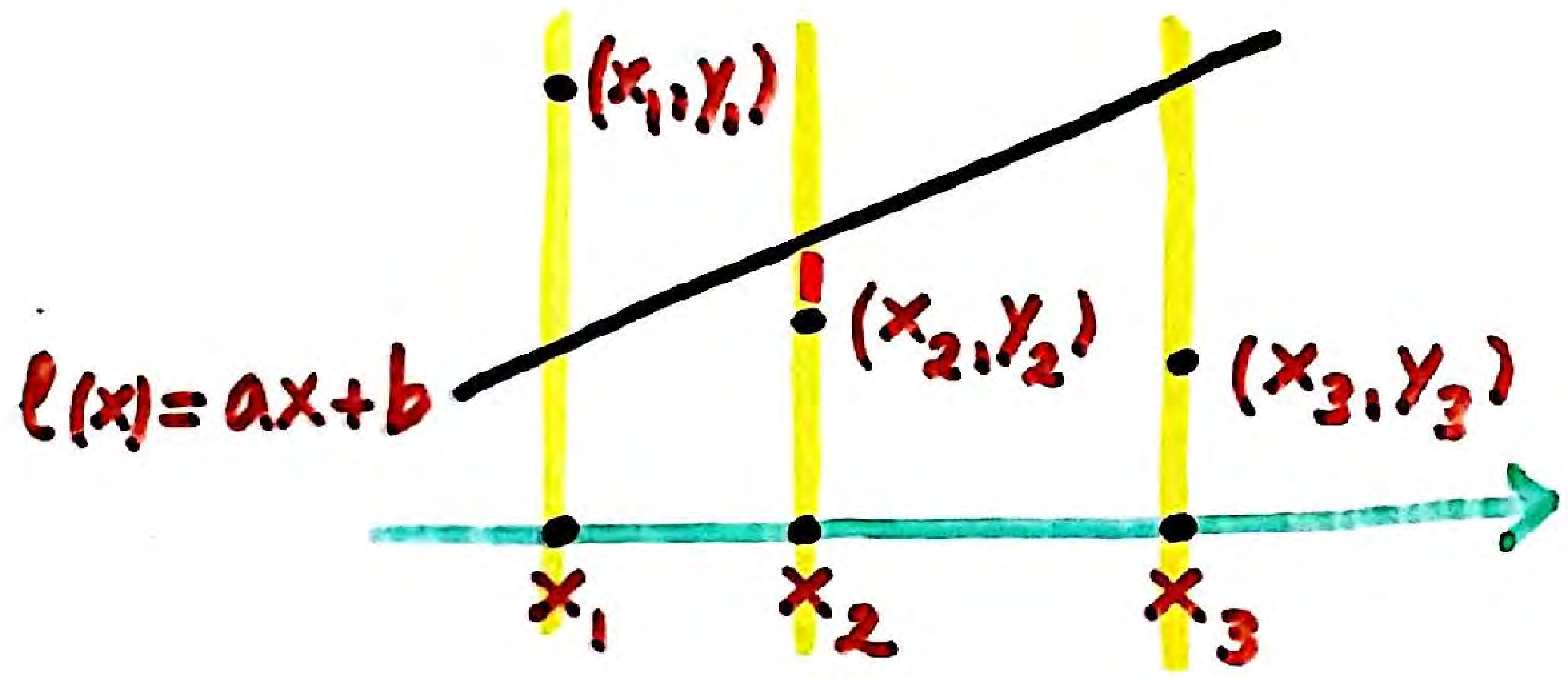
Problem (Makai - P., 1983)

How "sparse" can an $\mathcal{F}$-controlling sequence of numbers be?

EXAMPLE 1: LINEAR FUNCTIONS

$\mathcal{L} = \{ ax+b : a,b \in \mathbb{R} \}$

$x_1, x_2, \dots$ is $\mathcal{L}$-controlling : $\exists\, y_1, y_2, \dots$ such that for every $\ell(x) \in \mathcal{L}$ there exists i with $|\ell(x_i) - y_i| \le 1$



Theorem (Makai - P. 1983)

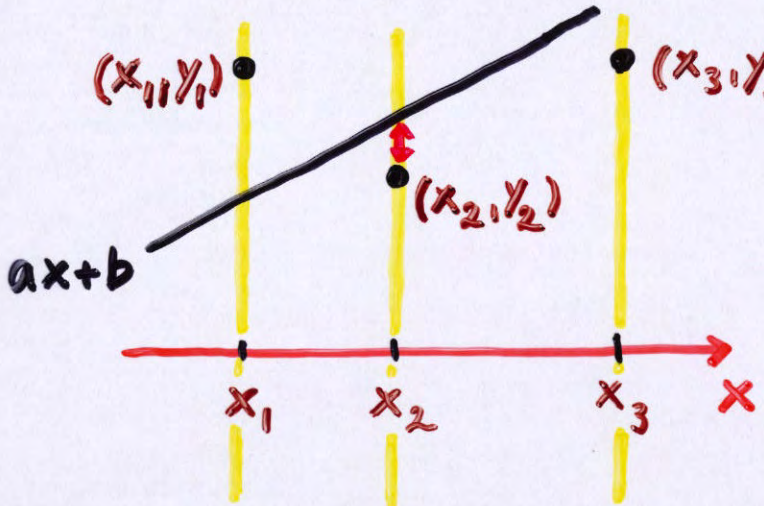
$x_1, x_2, \dots > 0$ is $\mathcal{L}$-controlling $\Longleftrightarrow \sum_{i=1}^{\infty} \frac{1}{x_i} = \infty$

Translative covering conjecture for $d=2$

LINEAR FUNCTIONS $\mathcal{L} = \{ ax + b : a, b \in \mathbb{R} \}$

$x_1 \leq x_2 \leq \dots$ $\mathcal{L}$-controlling : there exist $y_1, y_2, \dots$ such that

$(x_1, y_1), (x_2, y_2), \dots$ "control" every $\ell(x) = ax + b \in \mathcal{L}$



$$|ax_i + b - y_i| \leq 1$$

$$y_i - 1 \leq ax_i + b \leq y_i + 1$$

set of lines controlled by (x_i, y_i) $= \{(a,b) \in \mathbb{R}^2 : y_i - 1 \leq ax_i + b \leq y_i + 1\}$ = strip with normal vector $(x_i, 1)$ and width $\frac{2}{\sqrt{x_i^2+1}} \approx \frac{2}{x_i}$

EXAMPLE 2: POLYNOMIALS OF DEGREE d

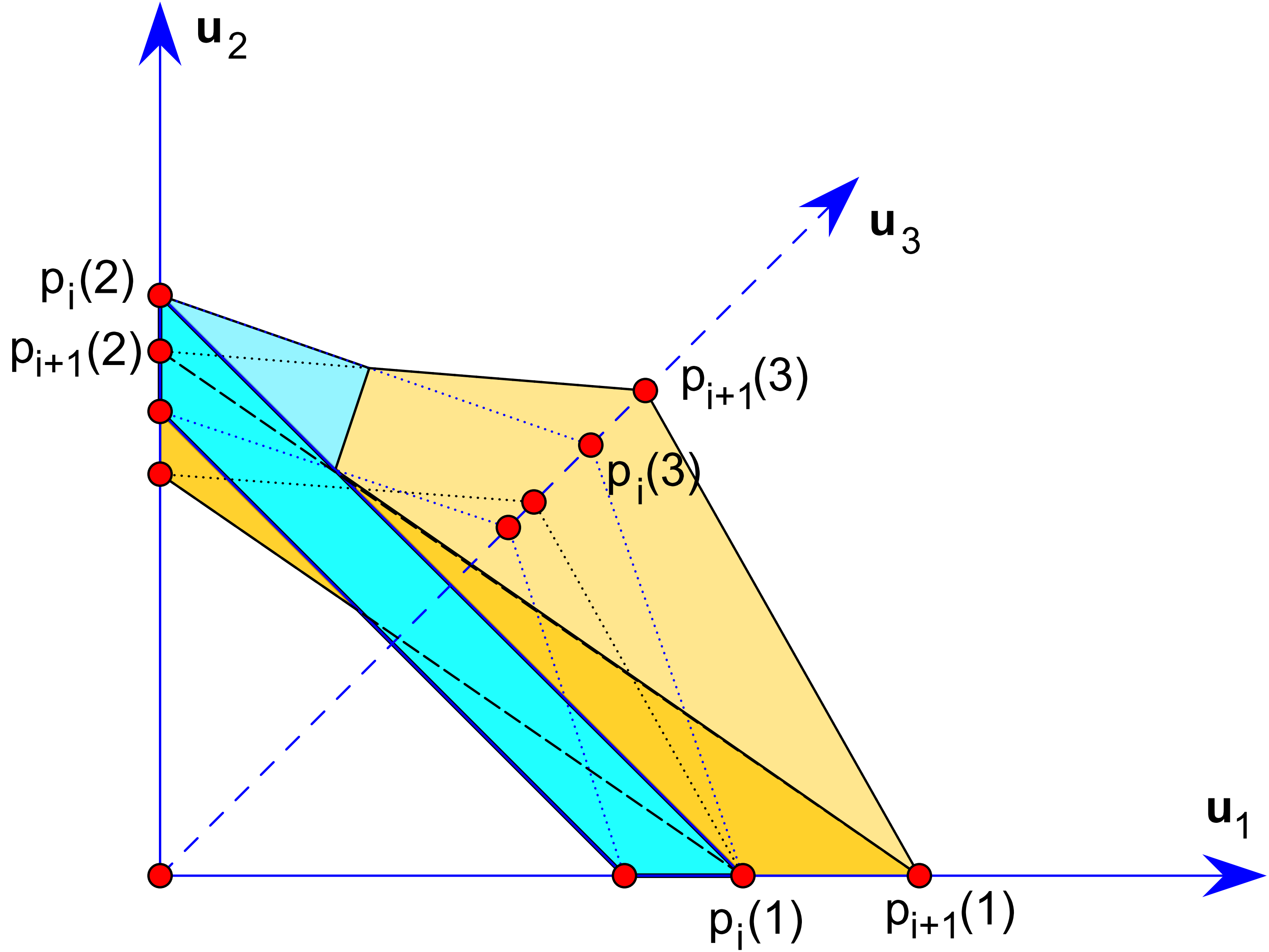
$\mathcal{P}_d = \{ a_0 + a_1 x + \dots + a_d x^d \ : \ a_0, \dots, a_d \in \mathbb{R} \}$

$x_1, x_2, \dots$ is $\mathcal{P}_d$-controlling : $\exists\, y_1, y_2, \dots$ such that
for every $p(x) \in \mathcal{P}_d$ there exists i with $|p(x_i) - y_i| \le 1$

Conjecture (Makai - P. 1983)

$x_1, x_2, \dots > 0$ is $\mathcal{P}_d$-controlling $\iff \displaystyle\sum_{i=1}^{\infty} \frac{1}{x_i^d} = \infty$

Translative covering conjecture in $\mathbb{R}^{d+1}$



u 2
u 3
u 1
p i (2)
p i+1 (2)
p i+1 (3)
p i (3)
p i (1)
p i+1 (1)

EXAMPLE 3: LIPSCHITZ FUNCTIONS

$$\mathrm{Lip}(\mathbb{R},\mathbb{R}^d)=\left\{f:\mathbb{R}\to\mathbb{R}^d \;\middle|\; \exists C \text{ with } |f(x)-f(x')|\le C|x-x'| \text{ for } \forall x,x'\right\}$$

Theorem (Makai - P. 1983)

If $x_1, x_2, \dots$ is $\mathrm{Lip}(\mathbb{R},\mathbb{R}^d)$-controlling, then

$$\sup_{n\in\mathbb{N}} \frac{|\{i : |x_i|\le n\}|}{n^d} = \infty .$$

Theorem (Kupavskii - P. - Tardos 2017)

If $x_1, x_2, \dots$ satisfy the above condition, then the sequence is $\mathrm{Lip}(\mathbb{R},\mathbb{R}^d)$- controlling.

LIPSCHITZ FUNCTIONS $\mathbb{R}^m \to \mathbb{R}^d$

Corollary (of the case $m=1$)
If a sequence $x_1, x_2, \dots \in \mathbb{R}^m$ is $\mathrm{Lip}(\mathbb{R}^m, \mathbb{R}^d)$-controlling, then

$$\sup_{n \in \mathbb{N}} \frac{|\{i : |x_i| \le n\}|}{n^d} = \infty .$$

Conjecture for the case $m \le d$ (Kupavskii-P.-Tardos)
If a sequence $x_1, x_2, \dots \in \mathbb{R}^m$ satisfies the above condition, then it is $\mathrm{Lip}(\mathbb{R}^m, \mathbb{R}^d)$-controlling.

Theorem (Kupavskii - P. - Tardos 2017)

Let $m \le d$. If a sequence $x_1, x_2, \dots \in \mathbb{R}^m$ satisfies the condition that for every $\alpha > 0$ and for every $x \in \mathbb{R}^m$ with $|x| \ge c_\alpha$

$$\left|\{\, i : |x - x_i| < \alpha \,\}\right| \ge |x|^{d-m},$$

then the sequence is $\mathrm{Lip}(\mathbb{R}^m, \mathbb{R}^d)$ controlling.

The above condition implies that

$$\sup_{n \in \mathbb{N}} \frac{\left|\{\, i : |x_i| \le n \,\}\right|}{n^d} = \infty,$$

but not vice versa.