

Treewidth, Crushing, and Hyperbolic Volume

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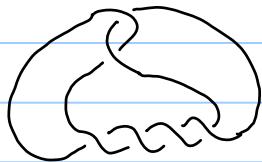
Discrete Maths Seminar, 3 October 2018



I. Motivation: Computation in 3-dimensional topology

3-manifold can be described by a triangulation:

example: The complement of the 6_1 knot



can be obtained by gluing
five tetrahedra:

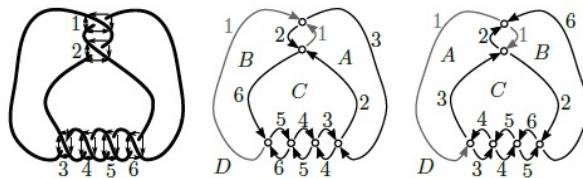


FIGURE 4.1. Left to right: The 6_1 knot, the top polyhedron, the bottom polyhedron

Collapse all bigons, identifying edges 1 and 2, and 3 through 6. New edges and orientations are shown in figure 4.2.

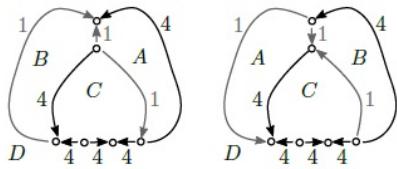


FIGURE 4.2. Polyhedra for 6_1 knot with bigons collapsed

We cone the top polyhedron to the vertex in the center. This subdivides faces C and D into triangles, shown in figure 4.3 in both top and bottom polyhedra.

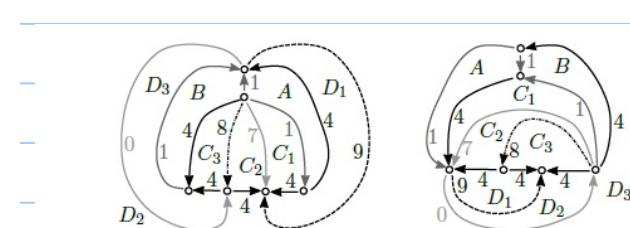


FIGURE 4.3. A subdivision of faces C and D in the top polyhedron (left) leads to a subdivision of the bottom (right)

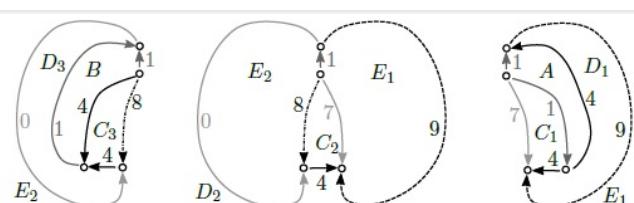


FIGURE 4.4. The top polyhedron splits into the three tetrahedra shown

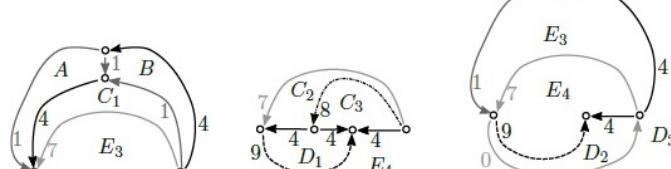
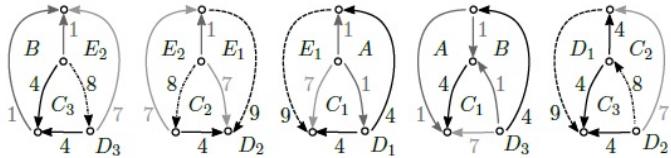


FIGURE 4.5. Splitting off two tetrahedra in the bottom polyhedron



Computer programs accept triangulation, return geometric/
topological information.

* SnapPy : Computes hyperbolic geometry



~ 1980

* Regina : Finds embedded surfaces.



Regina
Software for
low-dimensional topology

~ 2004

Question: How efficient are these programs?

Answer: Depends on how complicated the triangulation is.

Running time: Depends on "simplicity" of gluings of tetrahedra.

Measuring simplicity of triangulations:

(1) Count number of tetrahedra.

... Does not seem to give a good indication of computational complexity for many algorithms.

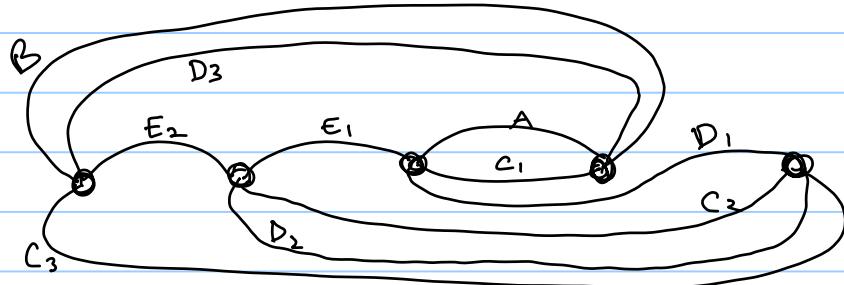
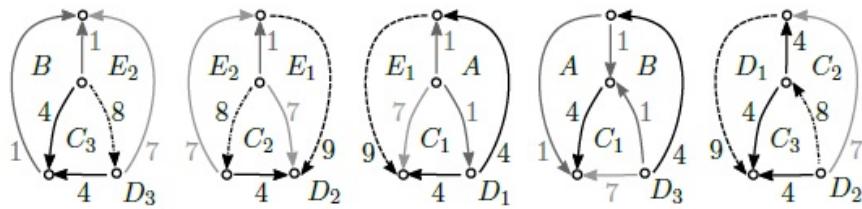
(2) Encode more information: How they connect.

Form (multi) graph: Nodes = tetrahedra

Edge between nodes \iff assoc. tetrahedra

glue along a face.

Example:



Terminology:
Nodes/arcs in graph Vertices/edges in tetrahedra

Apply measures of difficulty from graph theory:

Tree width

Carving width

(Defined below)

Courcelle's thm (9): Many graph theory problems can be described
in linear time in tree width of a graph.

→ Extends to 3-mfdS: (Burton-Downey 2017)

∃ highly efficient algorithms recently developed for triangulations w/
low tree width (Burton, Spreer, Maria, ...)

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Question: When does a 3-manifold have a triangulation of high treewidth?

Answer 1: When I make bad choices.

Could I retriangulate to reduce treewidth?

Answer 2: When the underlying 3-manifold is sufficiently complicated
(Huszár-Spreer-Wagner preprint: \exists mflds w/ large treewidth)

Goals:

- * Produce 3-manifold triangulations w/ treewidth governed by topology / geometry
- * Bound treewidth in terms of intrinsic topology / geometry.

II. Geometric / topological properties of a 3-manifold

Geometrisation theorem (Perelman, '03). Every closed, orientable 3-manifold can be cut along spheres and tori into pieces that admit one of eight geometries.

Hyperbolic geometry: among the most prevalent

Associated to hyperbolic geometry:

- * Volume
- * Lengths of geodesics
- * Injectivity radius $(= \inf_{x \in M} \{ R \mid \underbrace{B(x, R)}_{\text{Ball centred at } x} \text{ embeds in } M \})$
- * etc.

radius R

Theorem (Morgan-Purcell) There exists a universal constant $C > 0$ such that a hyperbolic 3-manifold M with volume $\text{vol}(M)$ admits a triangulation w/ treewidth at most $C \cdot \text{vol}(M)$.

Notes: ① Converse is false: There are families of 3mflds w/ bounded treewidth \nexists unbounded volume. (Thm (Morgan-P))

② Proof considers carving width.

We show that crushing (common tool to simplify triangulation) does not increase carving width \implies treewidth bounds.

III. Carving-width (or congestion) (due to Seymour & Thomas 1994)

Def. G a graph (allow loops, multi-arcs)

T an unrooted binary tree.

π an injective mapping from nodes of G to leaves of T

Suppose u, v nodes of G that are endpoints of an arc.

$\exists!$ path $p(\pi(u), \pi(v))$

$$\text{Congestion of } \pi : \text{cng}(\pi) = \max_{\text{arc of } T} |\{(u, v) \in G \mid a \in p(\pi(u), \pi(v))\}|$$

Carving width of G :

$$\text{cng}(G) = \min_{\pi} \text{cng}(\pi)$$

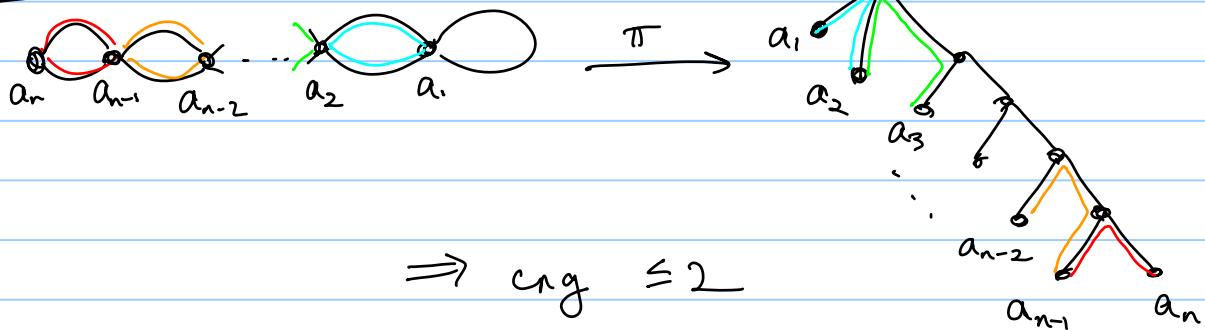
Carving width of T triangulation = $\text{cng}(G)$
 \nwarrow dual graph

Example: Daisy chain graph



Lemma. Daisy chain has carving width two.

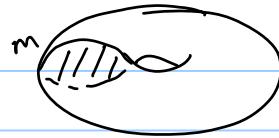
Proof:



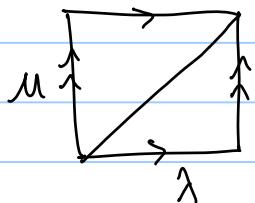
Fact: $\text{cng} \geq \text{max degree of a node after identifying multiple edges.}$

This case: 2. □

Example. Solid torus



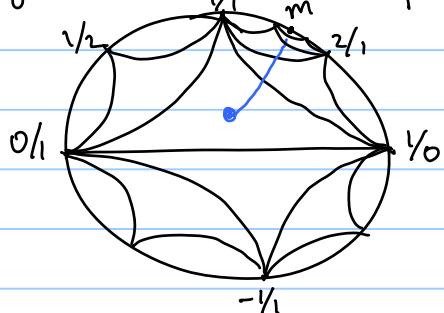
Boundary: will be torus made of 2 triangles, 1 vertex:



$$m = p\mu + q\lambda \quad \text{meridian}$$

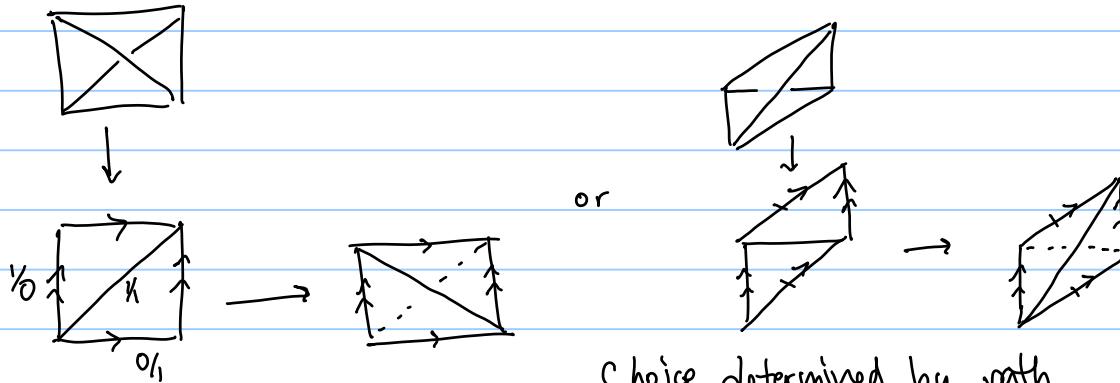
Write $\mu = 1/0$, $\lambda = 0/1$, $m = p/q$
 $\{\text{Simple closed curves on torus}\} = \mathbb{Q} \cup \{1/0\}$

Triangulations of torus parametrised by Farey triangulation \mathbb{H}^2



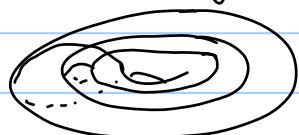
Triangulation of solid torus will
be built following path from
 $(0/1, 1/0, 1/1) \rightarrow (r, s, m)$

"Layer" a tetrahedron onto initial torus:



Choice determined by path
in Farey triangulation.

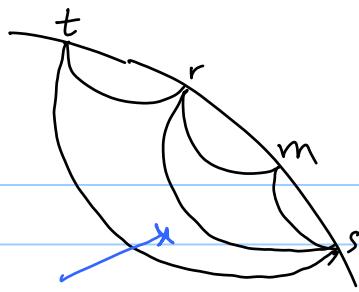
Continue layering. Get stack of tetrahedra forming $T^2 \times I$.



Boundaries: two tori

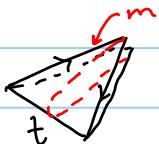
Each made of 2 triangles, 1 vertex

Close up to solid torus:



Continue in Farey graph
until triangle before m.

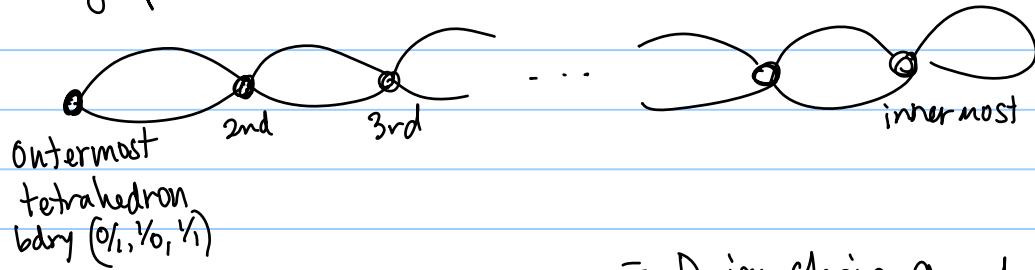
Identify 2 triangles on inside boundary by folding
into Möbius band:



Result is solid torus



Dual graph:



Cor. Layered solid torus (LST) has cng = 2.

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Def. Let G be a graph w/ loops & multiple arcs.

A tree decomposition $(X, \{B_t\})$ of G consists of a tree X and bags B_t of nodes of G for each node t of X , for which

- (1) each node u in G belongs to some bag B_t
- (2) \forall arc (u, v) in G , \exists bag B_t containing $u \not\in v$
- (3) \forall node u in G , the bags containing u form a connected subtree of X .

The width of the tree decomp. is $\max_{t \in X} |B_t| - 1$.

The treewidth of G $tw(G)$ is the smallest width of any tree decomposition.

Thm. Let G have max degree d .

$$\frac{2}{3}(\text{tw}(G) + 1) \leq \text{cngr}(G) \leq d(\text{tw}(G) + 1).$$

3-mfld triangulation: $d = 4$.

We will show

Thm: A closed hyperbolic 3-mfld w/ volume $\text{vol}(M)$ admits a triangulation with $\text{cngr} = O(\text{vol}(M))$.

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II. Hyperbolic manifolds.

Def. M is hyperbolic if $M \cong \mathbb{H}^3/\Gamma$

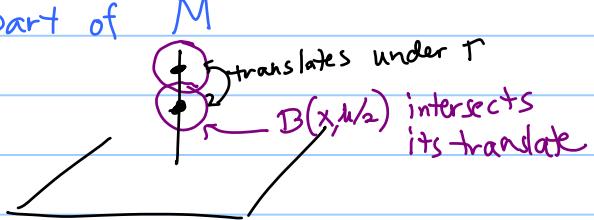
i.e. Universal cover is \mathbb{H}^3 , $\pi_1(M) \cong \Gamma$ discrete sgp of isometries

Def. Let $\mu > 0$, M a hyperbolic 3-manifold.

$$M^{>\mu} \quad \mu\text{-thick part of } M = \{x \in M \mid B_{\mathbb{H}^3}(x, \mu/2) \text{ embeds in } M\}$$

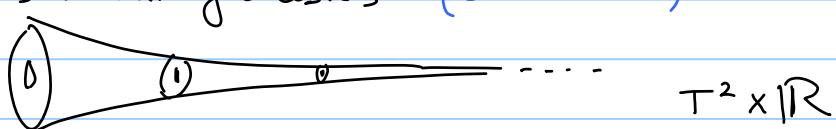
$$M - M^{>\mu} = M^{<\mu} \quad \mu\text{-thin part of } M$$

E.g. Short geodesic lies in $M^{<\mu}$:



Thm (Margulis) \exists universal constant ε_3 s.t. for any finite volume hyperbolic 3-mfld M , if any $\mu < \varepsilon_3$, the μ -thin part $M^{<\mu}$ consists only of

- Disjoint tubes about geodesics (solid tori)
- Cusps



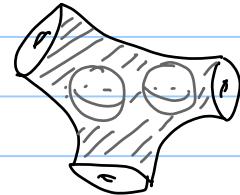
Thm. Fix $\mu > 0$. \exists a constant $M > 0$ such that $\forall M$ closed hyperbolic 3-mfld with volume $\text{vol}(M)$, the thick part $M^{\geq \mu}$ admits a triangulation w/ at most $\frac{\text{vol}(M)}{v(\mu)} = O(\text{vol}(M))$ tetrahedra.

Proof:

Build an ε -net: Set $\varepsilon \leq \mu$, $P = \emptyset$.

While $\exists x \in M^{\geq \mu} - \bigcup_{p \in P} B_{M^{\geq \mu}}(p, \varepsilon)$

Set $P = P \cup \{x\}$.



At any stage, balls of radius $\varepsilon/2$ w/ centres in P are disjoint \nsubseteq embedded in M

$$\Rightarrow |P| \cdot \underbrace{\text{vol}(B_{\mathbb{H}^3}(\varepsilon/2))}_{\text{constant}} \leq \text{vol}(M)$$

\Rightarrow process terminates.

(Boissonnat, Dyer, Ghosh 2017) \Rightarrow can perturb P s.t. Delaunay triangulation gives triangulation of $M^{\geq \mu}$.

(Breslin 2009) \Rightarrow can perturb vertices of Delaunay triangulation to obtain only "thick" tetrahedra:

Dihedral angles $\geq \Theta(\mu)$

edge lengths in interval $[a(\mu), b(\mu)]$.

Hyperbolic volumes of Breslin's tetrahedra are bounded below by $v(\mu)$. \square

Breslin's thick triangulation of $M^{\geq \mu}$: T_B .

$$\# |T_B| = O(\text{vol}(M)) \Rightarrow \text{crg}(T_B) = O(\text{vol}(M))$$

To obtain M from $M^{3, \mu}$, attach $M^{\leq \mu}$, solid tori

Dehn filling

We have $\textcircled{1} T_B$ triangulation $M^{3, \mu}$, $\textcircled{2}$ triangulation of solid torus
Would like to combine.

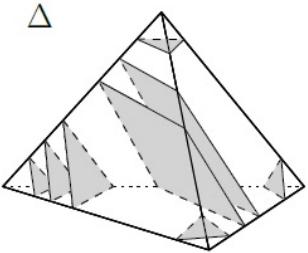
Problem: Layered solid torus has 2 triangles, 1 vertex
on boundary

$M^{3, \mu}$ likely has many more triangles.

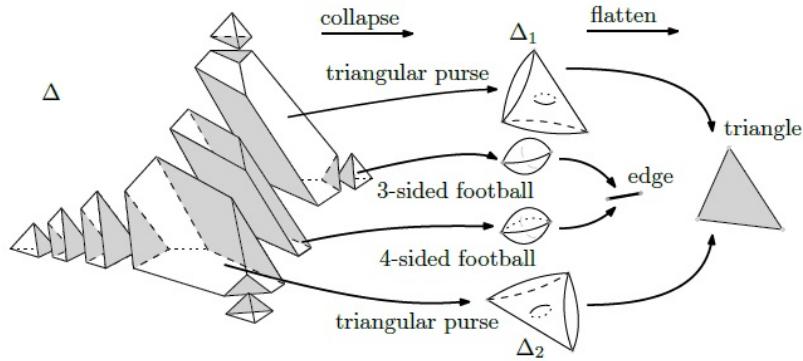


II. Crushing

Normal surface: meets tetrahedra in only triangles & quads:



Simplify by cut - collapse - flatten = Crushing



(Jaco-Rubinstein
1990's,
Burton 2000's)

Effect on dual graph: immersion.

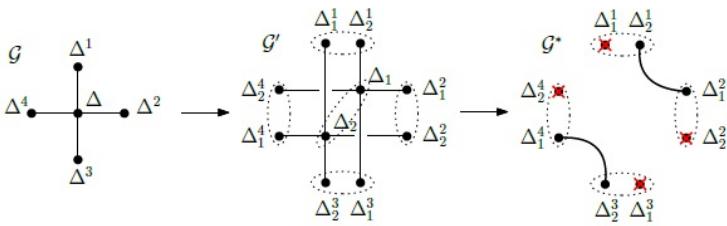


FIGURE 5. Local transformation of the dual graph at tetrahedron Δ from Figure 4 when crushing iteratively at Δ . \mathcal{G} is the dual graph before crushing, \mathcal{G}' is the dual graph after cutting, collapsing, and then flattening only the 3 and 4-sided footballs from Δ (Δ_1 and Δ_2 stand for the two triangular purses), and \mathcal{G}^* is the dual graph after flattening the triangular purses. Note that some of the nodes Δ_i^j in \mathcal{G}' may have already been removed when flattening adjacent footballs, and some of the bigon faces of triangular purses may already be collapsed. This does not change the analysis as it only removes nodes and edges from the dual graph. \mathcal{G}^* is obtained from \mathcal{G} by lifting $\Delta^1\Delta\Delta^2$ and $\Delta^3\Delta\Delta^4$, then removing the node Δ . Because their corresponding cells have bigonal faces, and hence cannot be tetrahedra, the crossed out nodes on \mathcal{G}^* will be removed from the graph when flattening adjacent cells. The immersion of \mathcal{G}^* into \mathcal{G} is obtained by mapping the (non-removed) nodes Δ_i^j in \mathcal{G}^* to Δ^j in \mathcal{G} .

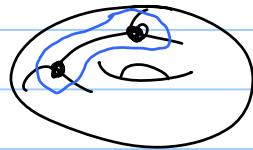
Effect of immersion on cng : Only decreases.

Crushing T_B :

Suppose ∂T has \geq two vertices.

Nbhd of edge between distinct vertices gives disk

Make it normal \rightsquigarrow



Crush.

tetrahedra decreases, cng decreases.

$$\leq O(\text{vol}(M))$$

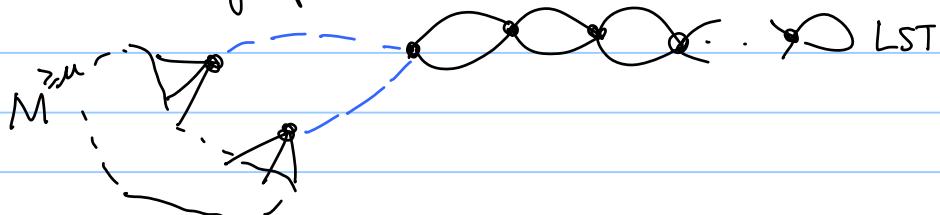
$\leq O(\text{vol}(M))$

Repeat until 1 vertex \Rightarrow 2 triangles on $\partial M^{\geq n}$

Call this triangulation T_{JR} (Jaco-Rubinstein)

Attach layered solid torus.

Dual graph:



$$\begin{aligned}
 \text{cng} &\leq \max\{\text{cng}(\mathcal{T}_{JR})+1, \text{cng}(\text{LT})+1, \max_{\text{degrees}}\} \\
 &\quad \underbrace{2+1}_{4} \\
 &= \max\{O(\text{vol}(M))+1, 4\} = O(\text{vol}(M)). \quad \square.
 \end{aligned}$$

Concludes proof that \exists triang. w/ $\text{tw} \leq O(\text{vol}(M))$.

VII. Converse is false.

Thm \exists sequence of hyperbolic 3-mflds M_n with
 $\lim_{n \rightarrow \infty} \text{vol}(M_n) \rightarrow \infty$, $\text{tw}(M_n) \leq 7$.

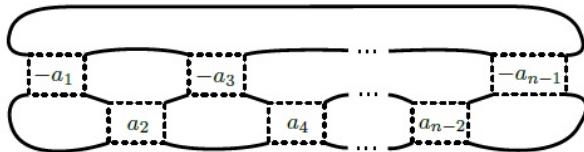


FIGURE 6. The diagram of $K[a_{n-1}, \dots, a_1]$, for n even. The box labelled $\pm a_i$ denotes a horizontal twist region with $|a_i|$ crossings, with the sign of the crossing equal to the sign of $\pm a_i$. The crossing number is $C = |a_{n-1}| + \dots + |a_1|$.

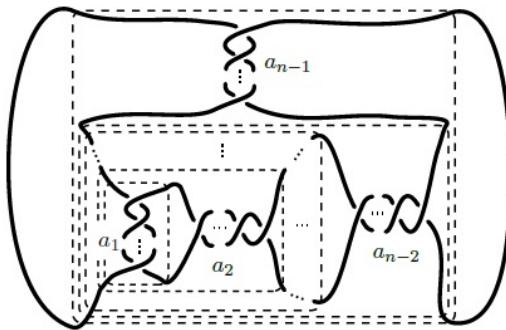


FIGURE 7. Another diagram of a 2-bridge knot.

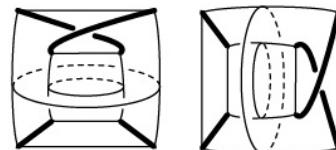


FIGURE 8. Vertical (left) and horizontal (right) blocks of the form $S \times I$. The 4-punctured spheres on the outside and inside correspond to $S \times \{1\}$ and $S \times \{0\}$, respectively. Figure from [32].

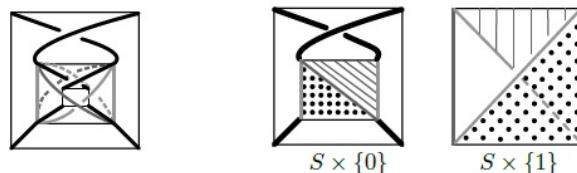


FIGURE 9. On the far left, the edges of the tetrahedron are shown. Middle: two faces of the tetrahedron T_i^1 lying on the surface $S \times \{0\} \subset S \times I$ for the $(i+1)$ -st block. Right: Position of those two faces when isotoped to $S \times \{1\}$ on the $(i+1)$ -st block. Figure from [32].

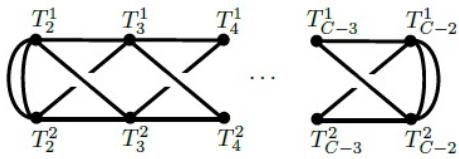


FIGURE 10. The form of the dual graph to a 2-bridge knot triangulation.

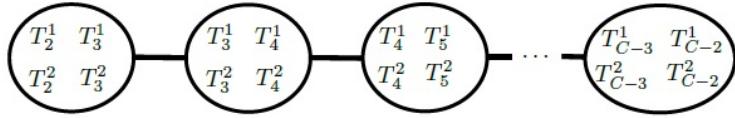


FIGURE 11. A tree decomposition of an ideal triangulation of a 2-bridge knot.

$$\Rightarrow \tau_W \leq 3.$$

Volume $\rightarrow \infty$ as $n \rightarrow \infty$ ($n = \# \text{ twist regions } \approx \infty$)