Full Diversity Unitary Precoded Integer-Forcing

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Definitions

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- Every lattice does have a bases and every lattice point is an integer linear combinations of bases vectors.
- A lattice Λ can be represented with a generator matrix $\mathbf{G}$ by stacking its n-dimensional bases vectors as rows of $\mathbf{G}$.

Successive minimas

- For an n-dimensional lattice $\Lambda_{\mathbf{G}}$, we define the m-th successive minima, for $1 \leq m \leq n$ as

$$\epsilon_m(\Lambda_{\mathbf{G}}) \triangleq \inf\left\{r: \dim\left(\text{span}\left(\Lambda_{\mathbf{G}} \cap \mathcal{B}_r(\mathbf{0})\right)\right) \geq m\right\}.$$

The m-th successive minima of $\Lambda_{\mathbf{G}}$ is the infimum of the numbers r such that there are m independent vectors of $\Lambda_{\mathbf{G}}$ in $\mathcal{B}_r(\mathbf{0})$.

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  The m-th successive minima of $\Lambda_{\mathbf{G}}$ is the infimum of the numbers r such that there are m independent vectors of $\Lambda_{\mathbf{G}}$ in $\mathcal{B}_r(\mathbf{0})$.
- The quantity ϵ_1 is also called the minimum distance of $\Lambda_{\mathbf{G}}$.

Full-diversity lattices and minimum product distance

- An n-dimensional lattice $\Lambda_{\mathbf{G}}$ is called full-diversity if for all disjoint $\mathbf{x}, \mathbf{y} \in \Lambda_{\mathbf{G}}$, the number of elements in

$$\{m \colon [\mathbf{x}]_m \neq [\mathbf{y}]_m\}$$

be exactly n.

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- The minimum product distance of a full-diversity lattice $\Lambda_{\mathbf{G}}$ is denoted by $d_{p,\min}(\Lambda_{\mathbf{G}})$ and is defined by:

$$d_{p,\min}(\Lambda_{\mathbf{G}}) \triangleq \min_{\mathbf{0} \neq \mathbf{x} \in \Lambda_{\mathbf{G}}} \prod_{m} |[\mathbf{x}]_m| \,.$$

Lattice Codes

- For any point $\mathbf{x} \in \Lambda$ the Voronoi cell $\mathcal{V}(\mathbf{x})$ is

$$\left\{ \mathbf{v} = \sum_{m=1}^{k} \alpha_m \boldsymbol{\ell}_m \colon \|\mathbf{v} - \mathbf{x}\| \leq \|\mathbf{v} - \mathbf{y}\|, \ \forall \mathbf{y} \in \Lambda, \ \alpha_m \in \mathbb{C} \right\}.$$

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- A lattice code $\mathcal{C} \subseteq \Lambda$ is a finite set of points of Λ.
- A subset $\Lambda' \subseteq \Lambda$ is called a sublattice if Λ' is a lattice itself.
- Given a sublattice Λ', we define the quotient Λ/Λ' as a lattice code. The notions of coding lattice and shaping lattice.

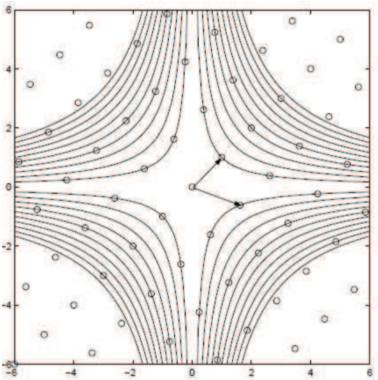

Figure: A full-diversity non-vanishing minimum product distance lattice with its bases vectors.

MIMO Channel Model 1

- We consider a quasi-static flat-fading $n \times n$ MIMO channel as above Figure with both CSIT and CSIR.

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- The channel matrix is $\mathbf{H} \in \mathbb{C}^{n \times n}$ with entries distributed independently and identically as $\mathcal{CN}(0, 1)$.
- An n-layer lattice coding scheme is used. For $1 \leq m \leq n$, the m-th layer is equipped with a lattice encoder

$$\mathcal{E} : \mathcal{R}^k \rightarrow \Lambda/\Lambda' \subset \mathbb{C}^n$$

$$\mathbf{s}_m \mapsto \mathbf{x}_m.$$

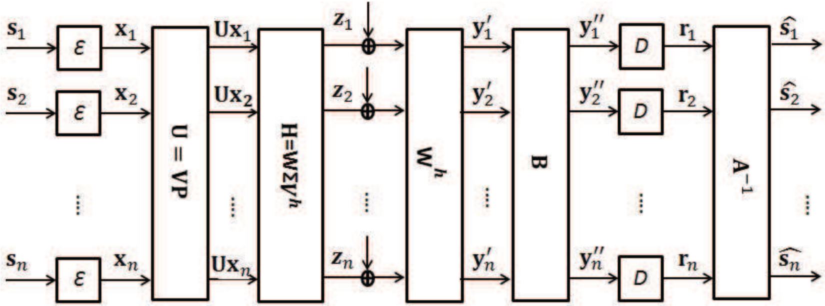

Figure: System model block diagram.

MIMO Channel Model 2

- Let $\mathbf{W}\mathbf{\Sigma}\mathbf{V}^h$ be the singular value decomposition (SVD) of $\mathbf{H}$, i.e.
  - $\mathbf{W}, \mathbf{V} \in \mathbb{C}^{n \times n}$ are two unitary matrices,
  - $\mathbf{\Sigma}$ is a diagonal matrix given by $\mathbf{\Sigma} = \text{diag}(\sigma_1, \ldots, \sigma_n)$ for which $\sigma_1 \geq \cdots \geq \sigma_n$.

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- We assume that the entries of $\mathbf{Z}$ are i.i.d. as $\mathcal{CN}(0, 1)$.
- Let $\mathbf{X} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_n^T]^T$, then the received signal $\mathbf{Y}$ is given by

$$\mathbf{Y} = \sqrt{\rho} \cdot \mathbf{H}\mathbf{U}\mathbf{X} + \mathbf{Z},$$

where $\rho = \frac{\text{SNR}}{n}$.

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  where $\mathbf{Y}' = \mathbf{W}^h\mathbf{Y}$ and $\mathbf{Z}' = \mathbf{W}^h\mathbf{Z}$.
- Note that $\mathbf{Z}'$ continues to be distributed as $\mathcal{CN}(0, 1)$ because the product of a unitary matrix by a Gaussian matrix is a Gaussian matrix.

IF Linear Receiver

- The goal of integer-forcing linear receiver is to project $\mathbf{\Sigma P}$ (by left multiplying it with a receiver filtering matrix $\mathbf{B}$) onto a non-singular integer matrix $\mathbf{A}$.
- In order to uniquely recover the information symbols, the matrix $\mathbf{A}$ must be invertible over the ring $\mathcal{R}$. Thus, we have

$$\mathbf{Y}'' = \mathbf{B}\mathbf{Y}' = \sqrt{\rho} \cdot \mathbf{B\Sigma P X} + \mathbf{B}\mathbf{Z}'.$$

Unitary Precoded IF

- A suitable signal model is

$$\begin{aligned}\mathbf{Y}'' &= \sqrt{\rho}\cdot\mathbf{A}\mathbf{X} + \sqrt{\rho}\cdot(\mathbf{B}\boldsymbol{\Sigma}\mathbf{P} - \mathbf{A})\mathbf{X} + \mathbf{B}\mathbf{Z}' \\ &= \sqrt{\rho}\cdot\mathbf{A}\mathbf{X} + \mathbf{E}\end{aligned}$$

- We let $P_e(m, \boldsymbol{\Sigma}\mathbf{P}, \Lambda) = \mathsf{Pr}(\mathbf{s}_m \neq \widehat{\mathbf{s}_m})$.

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- The average energy of effective noise $\mathbf{E}$, denoted by $\mathbf{e}_m$, along with the m-th row of $\mathbf{Y}''$ is defined as

$$G(\mathbf{a}_m,\mathbf{b}_m)\triangleq\rho\|\mathbf{b}_m\boldsymbol{\Sigma}\mathbf{P}-\mathbf{a}_m\|^2+\|\mathbf{b}_m\|^2.$$

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$$G(\mathbf{a}_m, \mathbf{b}_m) \triangleq \rho\|\mathbf{b}_m\boldsymbol{\Sigma}\mathbf{P} - \mathbf{a}_m\|^2 + \|\mathbf{b}_m\|^2.$$

- We refer to the above signal model as *Unitary Precoded Integer-Forcing*.

Our Approach

The optimum value of $\mathbf{b}_m$ that minimizes the rate is given $\mathbf{a}_m$ is

$$\mathbf{b}_m = \rho \cdot \mathbf{a}_m \mathbf{\Sigma} \mathbf{P}^h \left(\mathbf{I}_n + \rho \cdot \mathbf{\Sigma} \mathbf{P} \left(\mathbf{\Sigma} \mathbf{P} \right)^h \right)^{-1} \triangleq \rho \cdot \mathbf{a}_m \left(\mathbf{\Sigma} \mathbf{P} \right)^h \mathbf{S}^{-1}.$$

With this, the quantization noise term along the m-th layer is

$$\begin{aligned} G(\mathbf{a}_m, \mathbf{b}_m) &= \rho \|\mathbf{b}_m \mathbf{\Sigma} \mathbf{P} - \mathbf{a}_m\|^2 + \|\mathbf{b}_m\|^2 \\ &= \rho \cdot \mathbf{a}_m (\mathbf{I} - (\mathbf{\Sigma} \mathbf{P})^h \mathbf{S}^{-1} \mathbf{\Sigma} \mathbf{P}) \mathbf{a}_m^h \\ &= \rho \cdot \mathbf{a}_m \left(\mathbf{I} + \rho \cdot (\mathbf{\Sigma} \mathbf{P})^h \mathbf{\Sigma} \mathbf{P} \right)^{-1} \mathbf{a}_m^h \\ &= \rho \cdot \mathbf{a}_m \mathbf{P}^h \left(\mathbf{I} + \rho \cdot \mathbf{\Sigma}^h \mathbf{\Sigma} \right)^{-1} \mathbf{P} \mathbf{a}_m^h \\ &= \rho \cdot \mathbf{a}_m \mathbf{P}^h \mathbf{L} \mathbf{L}^h \mathbf{P} \mathbf{a}_m^h \\ &\triangleq \rho \cdot \mathbf{a}_m \mathbf{L}_p \mathbf{L}_p^h \mathbf{a}_m^h, \end{aligned}$$

Upper Bound on Probability of Error

Theorem

The probability of error for decoding the m-th layer in $\mathbb{Z}[i]$ is upper bounded as

$$P_e(m, \mathbf{\Sigma P}, \mathbb{Z}[i]) \leq \exp\left(-c\epsilon^2_{2n-m+1}(\Lambda_{\mathbf{L}_p^{-1}})\right),$$

where c is some constant independent of ρ.

Sketch of Proof 1

Since the minimum Euclidean distance of $\mathbb{Z}$ is unity, an error is declared if $\mathbf{e}_m \geq \frac{\sqrt{\rho}}{2}$. The $P_e\left(m, \mathbf{\Sigma}\mathbf{P}, \mathbb{Z}^{2n}\right)$ equals

$$
\begin{aligned}
&= \text{Pr}\left(|\mathbf{e}_m| \geq \frac{\sqrt{\rho}}{2}\right) = 2\text{Pr}\left(\mathbf{e}_m \geq \frac{\sqrt{\rho}}{2}\right) \\
&\leq 2\min_{t>0} \frac{\mathbb{E}(\exp(t\mathbf{e}_m))}{\exp\left(\frac{\sqrt{\rho}t}{2}\right)} \\
&= 2\min_{t>0} \frac{\mathbb{E}(\exp\left(t\sqrt{\rho}\cdot\langle\mathbf{b}_m\mathbf{\Sigma}\mathbf{P}-\mathbf{a}_m, \mathbf{x}_m\rangle + t\cdot\langle\mathbf{b}_m, \mathbf{z}'_m\rangle\right))}{\exp\left(\frac{\sqrt{\rho}t}{2}\right)} \\
&= \min_{t>0} \frac{\mathbb{E}(\exp(t\sqrt{\rho}\cdot\langle\mathbf{b}_m\mathbf{\Sigma}\mathbf{P}-\mathbf{a}_m, \mathbf{x}_m\rangle))\mathbb{E}(\exp\left(t\cdot\langle\mathbf{b}_m, \mathbf{z}'_m\rangle\right))}{\frac{1}{2}\exp\left(\frac{\sqrt{\rho}t}{2}\right)}.
\end{aligned}
$$

Sketch of Proof 2

- Since $\mathbf{z}'_m \sim \mathcal{N}(0,1)$, we have

$$\mathbb{E}\left(\exp\left(t\cdot\langle\mathbf{b}_m,\mathbf{z}'_m\rangle\right)\right)\leq\exp\left(\frac{t^2\|\mathbf{b}_m\|^2}{2}\right).$$

- Let $\mathbf{q}_m \triangleq t\sqrt{\rho}\cdot(\mathbf{b}_m\mathbf{\Sigma}\mathbf{P}-\mathbf{a}_m)$.

$$\mathbb{E}\left(\exp(t\sqrt{\rho}\cdot\langle\mathbf{q}_m,\mathbf{x}_m\rangle)\right)=\prod_{j=1}^{2n}\mathbb{E}\left(\exp\left(t\sqrt{\rho}\cdot[\mathbf{q}_m]_j[\mathbf{x}_m]_j\right)\right)$$
$$\leq\prod_{j=1}^{2n}\frac{\sinh\left(t\sqrt{\rho}|[\mathbf{q}_m]_j[\mathbf{x}_m]_j|\right)}{t\sqrt{\rho}|[\mathbf{q}_m]_j[\mathbf{x}_m]_j|}$$
$$\leq\prod_{j=1}^{2n}\exp\left(\frac{t^2\rho|[\mathbf{q}_m]_j|^2}{6}\right)\leq\exp\left(\frac{t^2\rho\|\mathbf{q}_m\|^2}{2}\right.$$

Sketch of Proof 3

Overall we get $P_e(m, \mathbf{\Sigma P}, \mathbb{Z})$ less than or equal to

$$\begin{aligned} &\leq 2\min_{t>0} \frac{\exp\left(\frac{t^2\rho\|\mathbf{b}_m\mathbf{\Sigma P}-\mathbf{a}_m\|^2}{2}\right)\exp\left(\frac{t^2\|\mathbf{b}_m\|^2}{2}\right)}{\exp\left(\frac{\sqrt{\rho}t}{2}\right)} \\ &= 2\min_{t>0} \frac{\exp\left(\frac{t^2 G(\mathbf{a}_m,\mathbf{b}_m)}{2}\right)}{\exp\left(\frac{\sqrt{\rho}t}{2}\right)} \\ &= 2\exp\left(\frac{-\rho}{4G(\mathbf{a}_m, \mathbf{b}_m)}\right). \end{aligned}$$

Sketch of Proof 4

By appropriately choosing $\mathbf{a}_m$ and $\mathbf{b}_m$, we get

$$\frac{G(\mathbf{a}_m, \mathbf{b}_m)}{\rho} = \epsilon_m^2(\Lambda_{\mathbf{L}_p^h}),$$

and

$$\epsilon_m^2(\Lambda_{\mathbf{L}_p^h}) \leq \frac{(2n)^3 + (3n)^2}{\epsilon_{2n-m+1}^2(\Lambda_{\mathbf{L}_p^h}^*)} = \frac{(2n)^3 + (3n)^2}{\epsilon_{2n-m+1}^2(\Lambda_{\mathbf{L}_p^{-1}})}.$$

Therefore, we have

$$\frac{\rho}{G(\mathbf{a}_m, \mathbf{b}_m)} \geq \frac{\epsilon_{2n-m+1}^2(\Lambda_{\mathbf{L}_p^{-1}})}{c_0}.$$

and

$$P_e\left(m, \mathbf{\Sigma P}, \mathbb{Z}^{2n}\right) \leq \exp\left(-c\epsilon_{2n-m+1}^2(\Lambda_{\mathbf{L}_p^{-1}})\right).$$

Diversity Analysis

Overall for the worst layer

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Definition

Let the average probability

$$P_e = \mathbb{E}_{\mathbf{H}}\left(P_e(\mathbf{\Sigma P}, \mathbb{Z})\right),$$

where the expectation is taken over all channel matrices $\mathbf{H}$. In an $2n \times 2n$ MIMO system and at a high SNR, if P_e is approximated by $(c.\text{SNR})^{-\delta}$, then δ is called the diversity gain (or diversity order). For a MIMO system with precoding, if $\delta = (2n)^2$, then, we say that the precoder achieves full-diversity order.

Main Theorem 1

Theorem

Let the precoding matrix $\mathbf{P}$ *be such that* $[\mathbf{P}\mathbf{v}]_1 \neq 0$*, where* $\mathbf{v} \in \mathbb{Z}^{2n}$ *is the vector satisfying* $\epsilon_1^2(\Lambda_{\mathbf{L}_p^{-1}}) = \|\mathbf{L}_p^{-1}\mathbf{v}\|^2$*, then the unitary precoded integer-forcing achieves full-diversity* $(2n)^2$*.*

Main Theorem 2

Theorem

Let the precoding matrix $\mathbf{P}$ *be such that* $d_{p,\min}(\Lambda_{\mathbf{P}}) \neq 0$*, then the achievable diversity of the unitary precoded integer-forcing is* $(2n)^2$*.*

Type I UPIF:Definition

Based on the first main Theorem, the optimal Type I UPIF is as follows:

$$\mathbf{P}_{1,\text{opt}} = \arg \max_{\mathbf{P} \in \mathcal{O}_{2n}} \min_{\substack{\mathbf{v} \in \mathbb{Z}^{2n} \setminus \{\mathbf{0}\} \\ [\mathbf{P}\mathbf{v}]_1 \neq 0}} \|\mathbf{L}^{-1}\mathbf{P}\mathbf{v}\|^2$$

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In other words, we should design a precoder matrix $\mathbf{P}$ such that the minimum distance of the lattice $\Lambda_{\mathbf{L}_p^{-1}}$ with generator matrix $\mathbf{L}^{-1}\mathbf{P}$ is maximized.

Type II UPIF: 2×2 Case

- We numerically search for

$$\mathbf{P}_{1,\text{opt}}^{(\mathbb{R})} = \arg \max_{\mathbf{P}(\theta) \in \mathcal{O}_2} \min_{[\mathbf{P}(\theta)\mathbf{v}]_1 \neq 0} \|\mathbf{L}^{-1}\mathbf{P}(\theta)\mathbf{v}\|^2,$$

for

$$\mathbf{P}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in [0 : 0.0001 : \pi/4]$$

- It follows that

$$\mathbf{L}^{-1}\mathbf{P}(\theta) = \xi \begin{pmatrix} \cos\eta & 0 \\ 0 & \sin\eta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

with

$$\xi = \sqrt{2 + \rho(\sigma_1^2 + \sigma_2^2)}, \qquad \eta = \tan^{-1}\left(\frac{\sqrt{1+\rho\sigma_2^2}}{\sqrt{1+\rho\sigma_1^2}}\right).$$

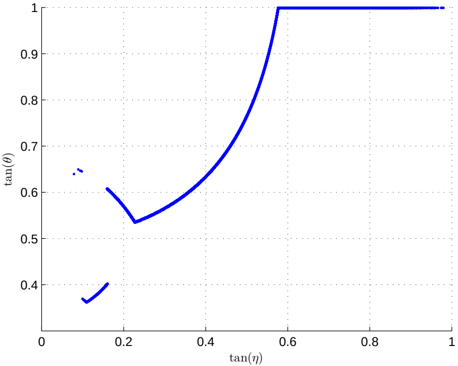

Figure: The variation of $\tan\theta$ based on the variation of $\tan\eta$ in a 2×2 complex MIMO Channel using real Type I UPIF.

Coding Gain

The coding gain formula is:

$$\gamma(\Lambda_{\mathbf{L}_p^{-1}}) = \frac{\epsilon_1^2(\Lambda_{\mathbf{L}_p^{-1}})}{\det\left(\mathbf{L}_p^{-1}\right)^{\frac{2}{2n}}}.$$

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The coding gain measures the increase in density of $\Lambda_{\mathbf{L}_p^{-1}}$ over the integer lattice $\mathbb{Z}^{2n}$ with $\gamma\left(\mathbb{Z}^{2n}\right) = 1$.

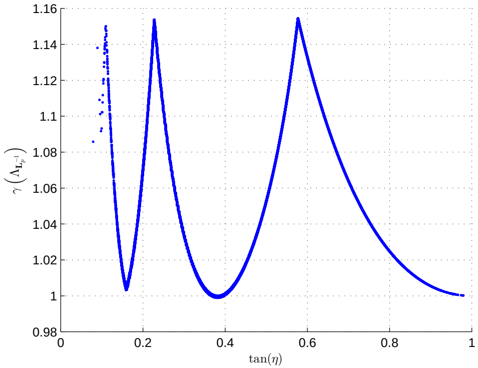

Figure: The variation of $\gamma(\Lambda_{\mathbf{L}_p^{-1}})$ based on the variation of $\tan\eta$ in a 2×2 complex MIMO Channel using real Type I UPIF.

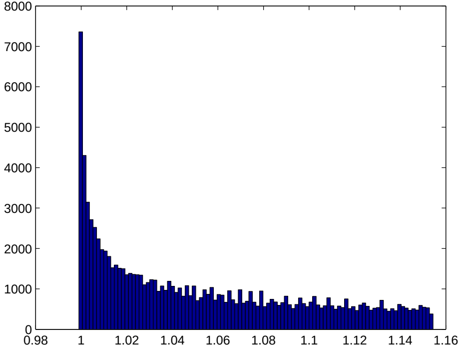

Figure: The histogram of $\gamma(\Lambda_{\mathbf{L}_p^{-1}})$ in a 2×2 complex MIMO Channel using real Type I UPIF.

Type II UPIF:Definition

Based on Theorem 4, the optimal Type II UPIF is as follows:

$$\mathbf{P}_{2,\text{opt}} = \arg\max_{\mathbf{P}\in\mathcal{O}_{2n}} d_{p\,\min}^{\frac{1}{n}}\left(\Lambda_{\mathbf{P}}\right).$$

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The solution for the above maximization is provided by OV04 as well as GBB97 using algebraic number theoretic lattices. A list of full-diversity algebraic rotations is available in Emanuele's Website.

Procedure: Modulo Lattice Decoding

1. **Infinite lattice decoding**: Each component of $\mathbf{B}\mathbf{y}'$ is decoded to the nearest point in $\mathbb{Z}[i]$ to get $\hat{\mathbf{y}}$. In particular, we use $\hat{\mathbf{y}} = \lfloor \mathbf{B}\mathbf{y}' \rceil$.

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3. **Decoupling the lattice codewords**: Further, we solve the system of linear equations $\mathbf{r} \equiv \mathbf{A}\mathbf{s} \pmod 2$ over the ring $\{0, 1\}$ to obtain the decoded vector $\hat{\mathbf{s}}$.

Comparison Cases: MIMO X-Codes and Y-Codes

The UPIF scheme and MIMO precoding X-codes and Y-codes share similar properties, which make them suitable for comparison:

- both schemes use SVD decomposition technique to transform the channel matrix into a diagonal one,
- the precoder matrices in both systems must be unitary/orthogonal matrices,
- both the detectors at the receiver side, *i.e.* lattice reduction based IF linear receiver and a combination of two 2-dimensional ML decoders, provide full receive diversity in 2×2 MIMO.

CER 2×2 MIMO Channel

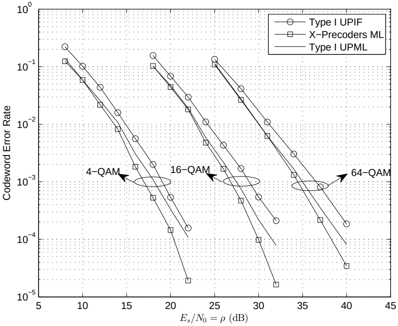

Figure: Type I UPIF in comparison with, X-Precoders decoded with sphere decoding algorithm, and Type II UPML in a 2×2 complex MIMO Channel.

CER for 2×2 MIMO Channel

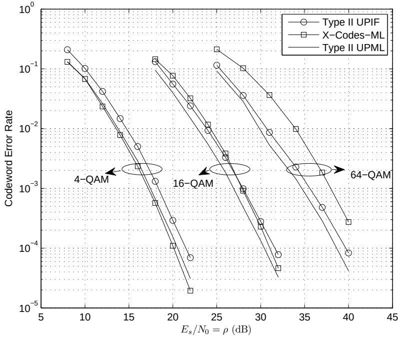

Figure: Type II UPIF in comparison with X-Codes and Type II UPML in a 2×2 complex MIMO Channel.

CER for 4×4 MIMO Channel

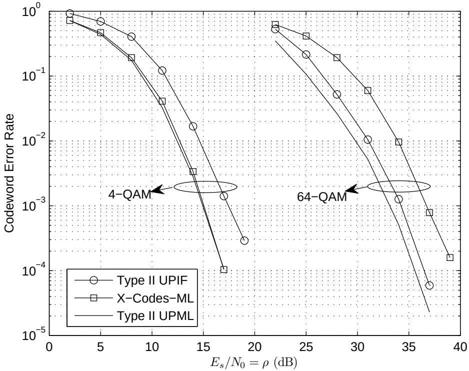

Figure: Type II UPIF in comparison with X-Codes and Type II UPML in a 4×4 complex MIMO Channel.

CER for 2×2 MIMO Channel

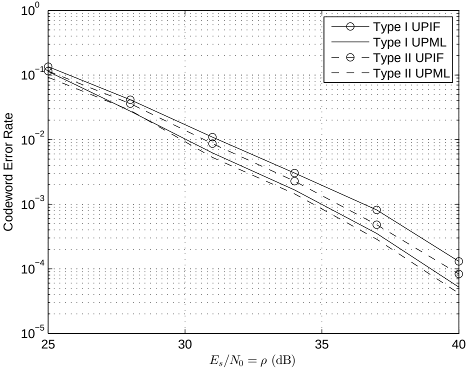

Figure: Type I versus Type II UPIF and UPML schemes in a 2×2 complex MIMO Channel.

Conclusions

- A unitary precoding scheme has been introduced to be employed at the transmitter of a flat-fading MIMO channel in the presence of both CSIT and CSIR, where an IF linear receiver is employed.
- The diversity gains of the proposed approach has been analyzed both theoretically and numerically.

Further Research Topics

- Designing full-diversity unitary precoders with IF receiver at the destination without having CSIT is of interest.
- Let the transmitter have access to limited feedback over a delay-free link from the IF receiver. Designing a suitable codebook of unitary precoding matrices which attains higher rates and obtain higher coding gains seems to be a promising research topic.

Thank you!