Tight triangulations: a link between combinatorics and topology

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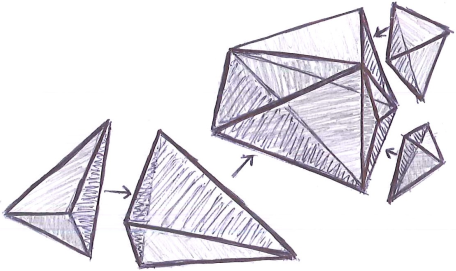
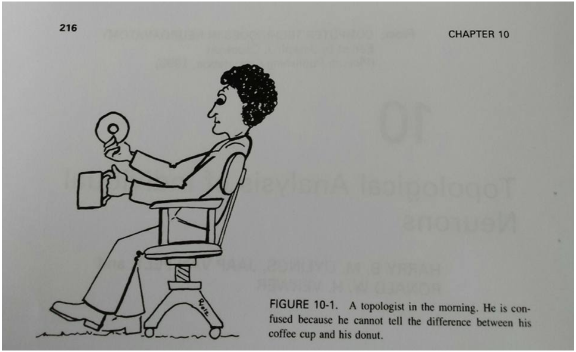
Melbourne, August 15, 2016



Topological manifolds

- (Geometric) Topology is study of manifolds (surfaces) up to continuous deformation



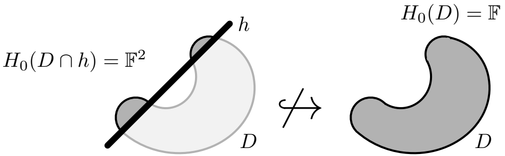
- Complicated deformations make recognition of a manifold difficult (if not impossible)
- Recognition might still be easy for "sufficiently nice" deformations

Convexity

- Topological balls / spheres $\Rightarrow$ convexity
- Limited use in geometry and topology: many objects are not balls / cannot be convex
- Intuitive notion: many objects look more convex than others

Tightness

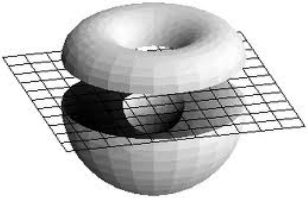
- An embedding of a topological space D into some Euclidean space $\mathbb{E}^d$ is said to be tight, if it is "as convex as possible" given its topological constraints
- D is tight if and only if intersecting D with a half space h does not introduce new topological features (eg. connected components or holes)

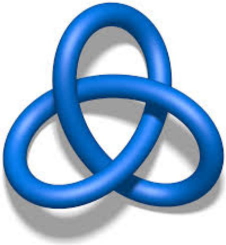


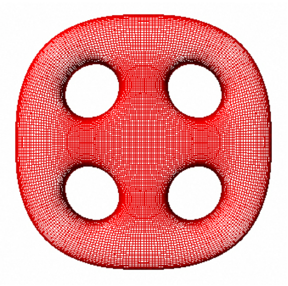
- Tightness generalises convexity, and minimises total absolute curvature.[1]

[1] Developed by Alexandrov, 1938; Milnor, Chern and Lashof, Kuiper, 1950's.

Tightness







Morse theory

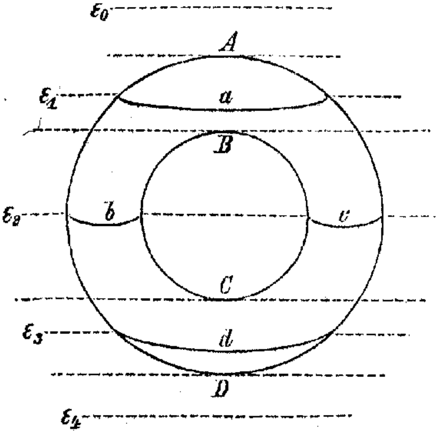


Fig. 5.

Möbius. Theorie der elementaren Verwandtschaft, 1863

Morse theory

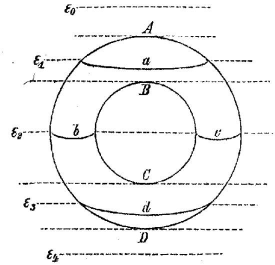


Fig. 5.

- Given a manifold M, look at a function $f : M \to \mathbb{R}$
- f has finitely many critical levels $f^{-1}(\alpha)$
- $f \;\to\;$ "height function", $\quad f^{-1}(M) \;\to\;$ "contour lines"
- Defines handle decomposition of $M \Rightarrow$ "finite" description of M
- Powerful: used to prove h-cobordism theorem

Morse theory and tightness

- Some height functions are better than others: fewer critical levels means more efficient handle decomposition
- How good can a Morse function be?

Theorem (Morse relations)

Let M be a manifold, and let $f : M \to \mathbb{R}$ be a Morse function. Then

$$\mu(f) \geq \sum_{i=0}^{d} \beta_i(M, \mathbb{Z}_2)$$

where $\beta_i(M, \mathbb{Z}_2)$ is the i-th Betti number of M with $\mathbb{Z}_2$-coefficients.

- If f attains equality, it is called perfect
- "Direction along which an object appears tight"
- Embedding is tight if and only if all projections are perfect Morse functions.
- Deciding whether such a function exist is **NP-complete**.[2]
- Is deciding whether "all Morse functions are perfect" **coNP-complete**?

[2] Joswig and Pfetsch 2006.

Finite description of manifolds

- Represent manifolds (surfaces) as simplicial complexes

$\langle\langle 1,2,4\rangle, \langle 1,2,6\rangle, \langle 1,3,4\rangle, \langle 1,3,7\rangle,$
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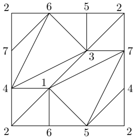
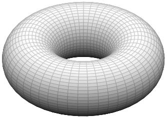
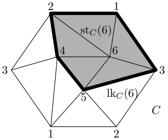
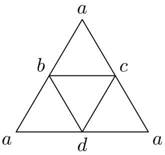




- A PL triangulation of a manifold M is a simplicial complex C such that every vertex link is PL homeomorphic to the standard sphere.





- Question: can we link combinatorial properties of C to topological properties of M?

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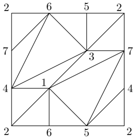

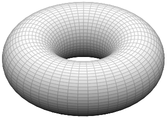

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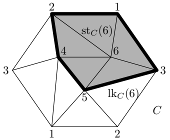

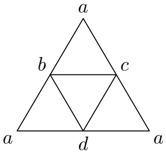

- Question: can we link combinatorial properties of *C* to topological properties of *M*?

- 🎉 Euler Characteristic 🎉

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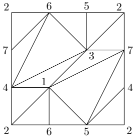

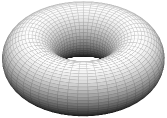

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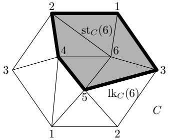
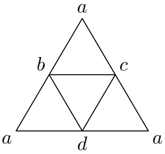




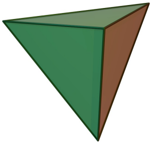
- Question: can we link combinatorial properties of C to topological properties of M?

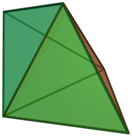
- 🎉Euler Characteristic🎉, first $\mathbb{Z}_2$-Betti number(?)

Abstract discrete version of tightness

Goal: intrinsic interaction between combinatorics and topology.

- C connected abstract simplicial complex with vertex set $V(C)$.
- $W \subset V(C)$, $C[W]$ subcomplex of C "induced by W".
- C is tight if and only if $\forall\ W \subset V(C)$, $C[W]$ does not introduce any new topological features.[3]





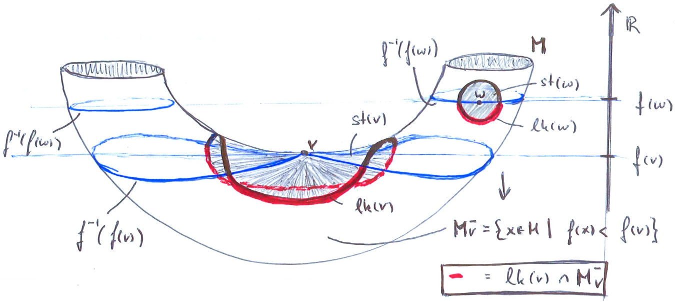
- Tight triangulations are conjectured to be strongly minimal[4]

[3] Banchoff 1970, Kühnel 1995

[4] Kühnel Lutz

Tightness → counting topological features

- New top. features $\Leftrightarrow$ non-perfect PL Morse function
- Compute average no. of critical levels per PL Morse function
  - For each subset $W \subset V(C)$, think of PL Morse functions $f : C \to \mathbb{R}$, s.t. $f(a) < f(b)$ for all pairs $a \in W$, $b \in V(C) \setminus W$
  - Count all such Morse functions
  - Count topological features of $\mathrm{lk}_C(v) \cap W$, for all $v \in V(C)$
  - Morse relation: Weighted sum (average) must equal sum of Betti numbers



Separation and μ index

Definition

C simplicial complex with n vertices, the μ index of C is given by

$$\mu(C) := \frac{1}{n} \sum_{v \in V(C)} \sigma(\mathrm{lk}_C(v)),$$

where

$$\sigma(\mathrm{lk}_C(v)) := \sum_{W \subseteq V(C)} \frac{\#(\mathrm{lk}_C(v)[W]) - 1}{\binom{n}{|W|}}$$

is called the separation index of $\mathrm{lk}_C(v)$.

Theorem (Combinatorial Morse relations)

C triangulation of (conn.) manifold M. Then $\mu(C) \geq \beta_1(M, \mathbb{Z}_2)$.

Known tight triangulations

- $\dim(C) \leq 2$: C tight $\Leftrightarrow$ every pair of vertices of C spans edge
- Surface types admitting tight triangulations: most orientable and non-orientable surfaces S with Euler characteristic
$$\chi(S) = -\frac{1}{6}k(k-7),$$
$k \in \mathbb{Z}$, $k > 3$.
- Manifolds known to admit tight triangulations in $\dim > 2$: $K3$ surface, $\mathbb{C}P^2$, $SU(3)/SO(3)$, infinitely many sphere bundles
- Focus of this talk is $\dim(C) = 3$: Equality in combinatorial Morse relation if and only if C is tight.

[5] which we will show is a purely combinatorial condition

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- Result: "Tight triangulations of 3-manifolds are strongly minimal, and must have stacked 2-spheres as vertex links"[5]

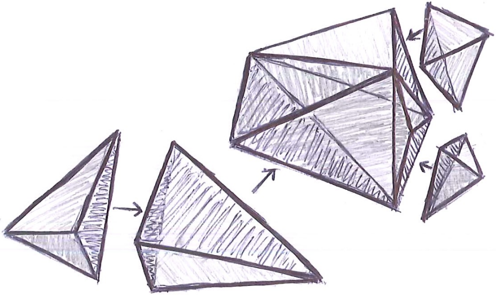
---

[5] which we will show is a purely combinatorial condition

Stacked spheres

A stacked $(d+1)$-ball is defined recursively:

- A $(d+1)$-simplex is a stacked $(d+1)$-ball.
- A simplicial complex obtained from a stacked $(d+1)$-ball B by gluing a $(d+1)$-simplex along a d-boundary face of B is a stacked $(d+1)$-ball.



Stacked $(d-1)$-sphere: boundary complex of a stacked d-ball.

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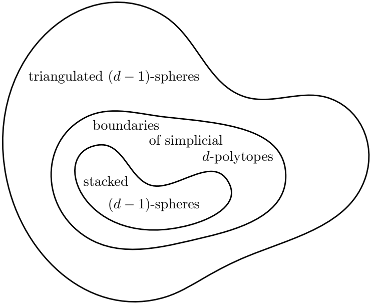

Stacked $(d-1)$-sphere: boundary complex of a stacked d-ball.

Stacked spheres

- We know: μ is an upper bound for the first Betti number
- Question: given a simplicial complex C with n vertices, how large can $\mu(C)$ be?
- $\rightarrow$ Maximise connected components of subsets of links (i.e., maximise $\sigma(\mathrm{lk}_C(v))$)
- Intuition: maximum is attained when number of edges is small

Theorem (Kalai 1987)

Let S be a triangulated d-sphere, $d \geq 3$. Then S has at least as many edges as a stacked d-sphere with equality iff S is stacked.

Stacked spheres

Vertex links $\mathrm{lk}_C(v)$ are triangulations of the 2-sphere with n vertices, e edges and t triangles.

- $2e = 3t$ (every edge is contained in exactly two triangles)
- $n - e + t = 2$ (Euler characteristic)
- $\Rightarrow f(S) = (n, 3n - 6, 2n - 4)$
- number of edges is always the same

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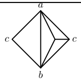
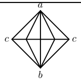
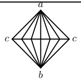
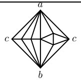
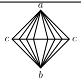
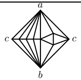
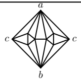
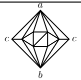
Triangulation S	$\sigma(S)$	Triangulation S	$\sigma(S)$
	0		1/5
	27/35		5/7
	11/9		71/63
	23/21		22/21

Stacked spheres

Theorem (Burton, Datta, Singh, S. 2014)
Let S be an n-vertex triangulated 2-sphere. Then

$$\sigma(S) \leq \frac{(n-8)(n+1)}{20},$$

where equality occurs if and only if S is a stacked sphere.

Bound on σ

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Bound on σ → Bound on μ → Bound on $\beta_1(M, \mathbb{Z}_2)$

Results

Corollary (See also Lutz, Sulanke, Swartz, 2008)

Let C be a triangulation of a 3-manifold M with n vertices. Then

$$n \geq \frac{1}{2}\left(9 + \sqrt{1 + 80\beta_1(M,\mathbb{Z}_2)}\right)$$

and C is tight if equality is attained.

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Let C be a tight triangulation of a 3-manifold with n vertices. Then all of its vertex links are $(n-1)$-vertex stacked 2-spheres.

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- Tightness for 3-manifold is purely combinatorial
- Tight triangulations of 3-manifolds are minimal, and their first Betti number is given by their number of vertices

Thank you

B. Burton, B. Datta, N. Singh, J. Spreer. Separation index of graphs and stacked 2-spheres. *J. Combin. Theory (A)*, 136:184-197, 2015.

B. Bagchi, B. Datta, J. Spreer. *A characterization of tightly triangulated 3-manifolds*, `arXiv:1601.00065`