A Latin square autotopism secret sharing scheme

Talk by Rebecca J. Stones

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- if the participants cooperate, their collective shares can be used to recover a secret message, and
- if too few participants cooperate, then the secret cannot be recovered.

A toy example...

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$$
\overbrace{\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
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0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}}^{\text{share 1}}
+
\overbrace{\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
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=
\overbrace{\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
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0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
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We can't find the secret without both shares.

We can choose share 1 uniformly at random. And choose share 2 to so that "share 1 + share 2" reveals the secret.

Shamir's Secret Sharing Scheme

Adi Shamir (of RSA fame) developed a secret sharing scheme.
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Given any t points, we can use Lagrange Interpolation to recover f, and find the secret $f(0)$.

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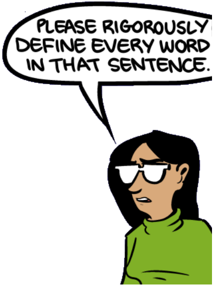
> *Secret sharing was invented independently by Adi Shamir and George Blakley in 1979. — Wikipedia.*

Latin squares (intro)



(Image source: SMBC)

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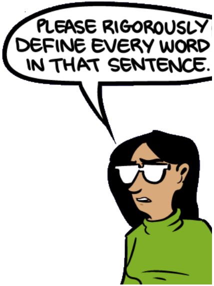

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A *Latin square* of order $n = 3$:

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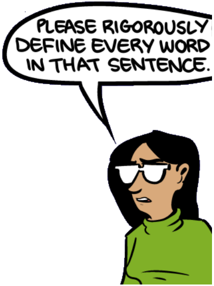

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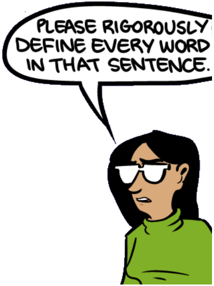

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$$\big(\overbrace{(0, 1, 2)}^{\text{row perm}}, \overbrace{(0, 1, 2)}^{\text{col perm}}, \overbrace{(0, 2, 1)}^{\text{sym perm}} \big).$$

Reconstruction from partial Latin squares

A Latin square of order 4 and a critical set:

0	1	2	3
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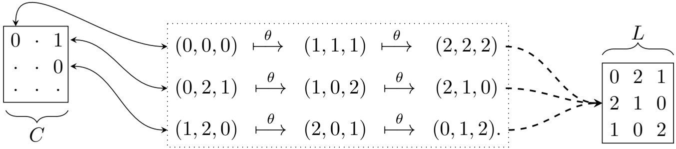
This scheme has been (harshly) criticized in the literature as impractical. (More about this later...)

Reconstruction from contours

We can reconstruct a Latin square L from knowledge of a *contour* C and an autotopism θ.



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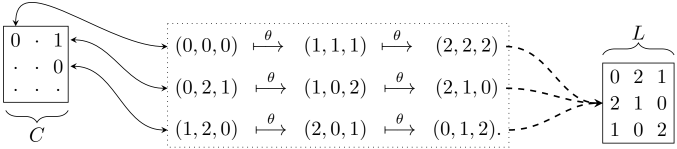

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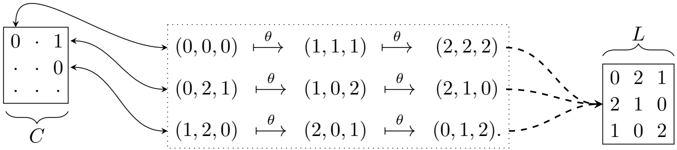

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Ganfornina (2006) proposed having a secret Latin square, and splitting contours among participants. This was not carefully analyzed in his work (it felt more like he was proposing a potential application).

Criticisms

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Initialization and reconstruction complexity Typically, it is difficult to find a critical set C, and given a critical set C, it is difficult to find the completion of C (determining if a partial Latin square admits a completion is NP-complete; Colbourn 1984).

More criticisms

Partial information The shares reveal partial information about the secret Latin square to the participants.

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Multi-level scheme It is impractical to extend these schemes to multi-level schemes (where certain subsets of the participants can combine to find the secret).

The proposed scheme

The method we propose differs in two key aspects:

- Instead of having a secret Latin square that admits an autotopism, we have a secret autotopism (and we use the Latin square for verification).

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- Instead of having a secret Latin square that admits an autotopism, we have a secret autotopism (and we use the Latin square for verification).
- We enforce particular cycle structures for the autotopism; this allows a concrete theoretical analysis.

We call an isotopism $\theta = (\alpha, \beta, \gamma)$ *suitable* if α, β, and γ all decompose into 2 disjoint $(n/2)$-cycles.

Generating the "prior" contour

We generate a random contour for the autotopism $\zeta = (\tau, \tau, \tau)$ where $\tau := (0, 1, \ldots, n/2 - 1)(n/2, n/2 + 1, \ldots, n - 1)$ by sticking 0's and $n/2$'s along the diagonals indicated below:

$$D = \left[\begin{array}{ccc|ccc} \cdot & \cdot & 0 & \cdot & \cdot & 3 \\ \cdot & 3 & \cdot & \cdot & 0 & \cdot \\ 0 & \cdot & \cdot & 3 & \cdot & \cdot \\ \hline \cdot & \cdot & 3 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & 3 & \cdot \\ 3 & \cdot & \cdot & 0 & \cdot & \cdot \end{array}\right] \xrightarrow{\text{contour}} L_{\text{prior}} = \left[\begin{array}{ccc|ccc} 5 & 1 & 0 & 2 & 4 & 3 \\ 1 & 3 & 2 & 4 & 0 & 5 \\ 0 & 2 & 4 & 3 & 5 & 1 \\ \hline 2 & 4 & 3 & 5 & 1 & 0 \\ 4 & 0 & 5 & 1 & 3 & 2 \\ 3 & 5 & 1 & 0 & 2 & 4 \end{array}\right]$$

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(for this to work we need, and hence assume $n \equiv 0 \pmod 4$).

Generating the "prior" contour

We generate a random contour for the autotopism $\zeta = (\tau, \tau, \tau)$ where $\tau := (0, 1, \ldots, n/2 - 1)(n/2, n/2 + 1, \ldots, n - 1)$ by sticking 0's and $n/2$'s along the diagonals indicated below:

$$D = \left[\begin{array}{ccc|ccc} \cdot & \cdot & 0 & \cdot & \cdot & 3 \\ \cdot & 3 & \cdot & \cdot & 0 & \cdot \\ 0 & \cdot & \cdot & 3 & \cdot & \cdot \\ \hline \cdot & \cdot & 3 & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & 3 & \cdot \\ 3 & \cdot & \cdot & 0 & \cdot & \cdot \end{array}\right] \xrightarrow{\text{contour}} L_{\text{prior}} = \left[\begin{array}{ccc|ccc} 5 & 1 & 0 & 2 & 4 & 3 \\ 1 & 3 & 2 & 4 & 0 & 5 \\ 0 & 2 & 4 & 3 & 5 & 1 \\ \hline 2 & 4 & 3 & 5 & 1 & 0 \\ 4 & 0 & 5 & 1 & 3 & 2 \\ 3 & 5 & 1 & 0 & 2 & 4 \end{array}\right]$$

(for this to work we need, and hence assume $n \equiv 0 \pmod 4$).

Instead of the original contour for D, we retain a random contour C_{prior} by replacing each entry $(i, j, d_{i,j})$ in the contour with $\zeta^t(i, j, d_{i,j})$ for $t \in \{0, 1, \ldots, n/2 - 1\}$ randomly chosen for each entry.

$$C_{\text{prior}} = \left[\begin{array}{ccc|ccc} 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 4 & 0 & \cdot \\ 0 & \cdot & \cdot & 3 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 5 & \cdot & \cdot & 2 \\ \cdot & 5 & \cdot & \cdot & 2 & 4 \end{array}\right]$$

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If we apply the random isotopism

$$\varphi = \big((0,4,1,3,5,2),(1,2,4),(1,3,2,5)\big)$$

to the earlier example, we obtain the Latin square

$L = \varphi(L_{\text{prior}}) =$

0	1	5	2	4	3
4	2	0	3	1	5
2	5	1	0	3	4
3	0	2	4	5	1
1	4	3	5	0	2
5	3	4	1	2	0

which admits the autotopism

$$\begin{aligned}\theta &= \varphi\zeta\varphi^{-1} \\ &= \big((0,4,3)(1,2,5),(0,2,4)(1,5,3),(0,3,5)(1,2,4)\big).\end{aligned}$$

Randomizing the contour (cont.)

Further, it is generated by the contour

$$C = \varphi(C_{\text{prior}}) = \begin{array}{|cccccc|} \hline 0 & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & 1 & 5 \\ \cdot & 5 & 1 & \cdot & \cdot & 4 \\ 3 & 0 & \cdot & 4 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array}$$

and the autotopism θ.

Splitting the autotopism

If we have e.g. 4 participants, we split the autotopism θ into 3 random isotopisms $\sigma_1, \sigma_2, \sigma_3$, and we choose σ_4 such that $\theta = \sigma_1\sigma_2\sigma_3\sigma_4$.

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$$\begin{aligned}
\sigma_1 &= \big((0,4)(1,5),(0,4,5,3,1),(0,5,1)(2,4,3)\big)\\
\sigma_2 &= \big((0,4)(1,3,5,2),(0,2,5),(0,1,3,4,5,2)\big)\\
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\end{aligned}$$

These are our shares and we distribute one to each participant.

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which we make public. When the shares are returned to reveal the secret, we use this to verify that the shares combine correctly.

Review

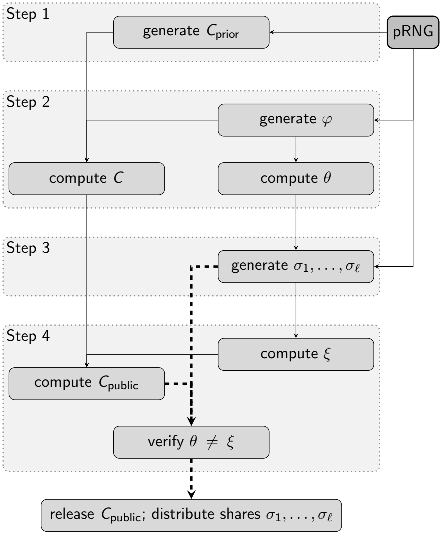

Figure : Flow chart of the proposed secret sharing scheme: initialization phase. (We also check $\theta = \xi$, restarting if this happens.)

Recovery

When all participants decide to cooperate, the participants securely send the shares $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_\ell$ to a *combiner* (possibly incorrectly—if share i is correctly sent, we have $\tilde{\sigma}_i = \sigma_i$).

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Collusion Each σ_i is a random isotopism (distributed uniformly at random from $S_n \times S_n \times S_n$); knowledge of fewer than all ℓ shares σ_i is of no more use in recovering θ or C than is a random suitable isotopism.

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Brute-force attack Search spaces are too large:

n	nr LS with autotop. ζ	nr suitable isotop.	$\mathrm{is}(L)$ lower bound
6	648	6×10^4	2×10^5
10	20820000	3×10^{14}	4×10^{14}
14	?	7×10^{26}	1×10^{27}
18	?	6×10^{40}	7×10^{39}

Security analysis (cont.)

Attack by finding a completion of C_{public} If an attacker managed to find L, they could compute its autotopism group, and find the secret θ.

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Obstacle 2: Let p denote the probability of $\theta_{\mathsf{cand}} \neq \theta$ returned assuming Obstacle 1 is overcome. This is tested experimentally:

n	experimentally $p \leq$		theoretically $p \geq$
6	4.5×10^{-5}	(99.995% confidence)	3.13×10^{-5}
10	2×10^{-11}	(99.995% confidence)	1.04×10^{-14}

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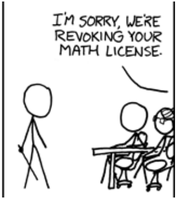
2. We can easily extend to a multi-level scheme on-the-fly.
3. We can eliminate working with Latin squares altogether (they're "behind the scenes"); this saves on space and time complexity.

Thank you!



(Image source: xkcd)

Probability $(C, \theta_{\text{cand}})$ generates a Latin square, when θ_{cand} is random We have

$$\begin{aligned} p &:= \Pr[(C, \theta_{\text{cand}}) \text{ generates a Latin square}] \\ &= \Pr[(\varphi^{-1}(C), \varphi^{-1}\theta_{\text{cand}}\varphi) \text{ generates a Latin square}] \\ &= \Pr[(C_{\text{prior}}, \varphi^{-1}\theta_{\text{cand}}\varphi) \text{ generates a Latin square}] \\ &= \Pr[(C_{\text{prior}}, \theta_{\text{cand}}) \text{ generates a Latin square}] \end{aligned}$$

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For $n = 6$, we generate 10^9 pairs $(C_{\mathsf{prior}}, \beta)$, for random suitable autotopism β, and find 43409 generate a Latin square. The upper bound on the Wald confidence interval is 4.5×10^{-5} with 99.995% confidence. For $n = 10$, we made $N := 3.6 \times 10^{11}$ samples, and no Latin square was generated this way. Using a modified "rule of three", we can be 99.995% confident that $p \leq 7.6/N \approx 2 \times 10^{-11}$.