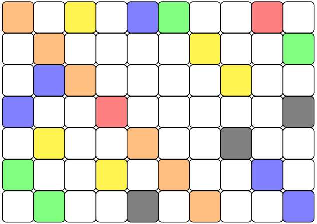
Weights of partial Latin rectangles with specified symmetry groups

Rebecca J. Stones (Nankai University, China);
with Raúl M. Falcón (University of Seville, Spain).

August 31, 2015



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If you find yourself thinking "why don't you just do [blah]?", it may simply be because I hadn't thought of it.

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Rows are labeled $\{1, 2, \ldots, r\}$. Columns are labeled $\{1, 2, \ldots, s\}$.

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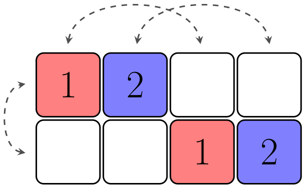
For this partial Latin rectangle



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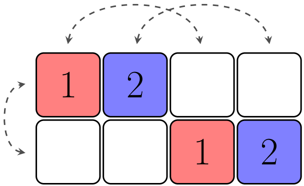

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(This why we don't want empty rows and columns, and unused symbols. E.g. if there were two empty rows, we can swap them to give an uninteresting symmetry.)

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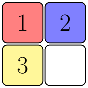

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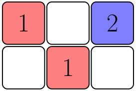

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A combination of these two types of operations is called an *paratopism*.

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- Then, we permute the coordinates of each entry in L' according to δ, i.e., if (e_1, e_2, e_3) is an entry of L', then it maps to $(e_{\delta(1)}, e_{\delta(2)}, e_{\delta(3)})$.

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E.g. it's okay to take the transpose if the number of rows equals the number of columns.

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The answer is “yes” when $H_1 = H_2$ (Phelps 1979, S. 2013).

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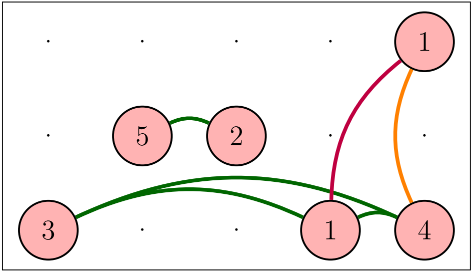
  (This is more complicated.)

Partial Latin rectangles graphs



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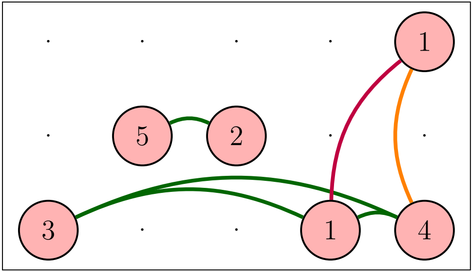

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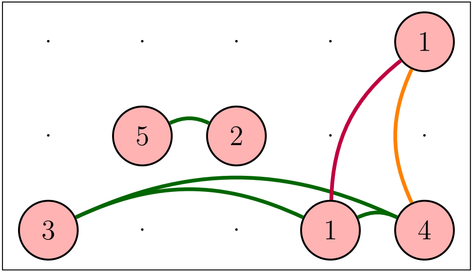

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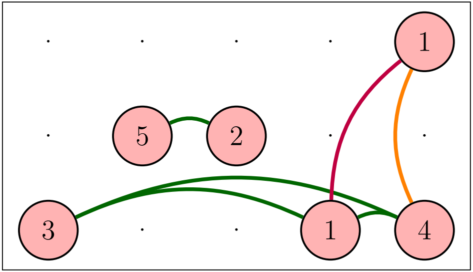

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Same row: green edge. **Same column: orange edge**. **Same symbol: purple edge**.

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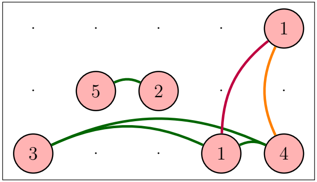

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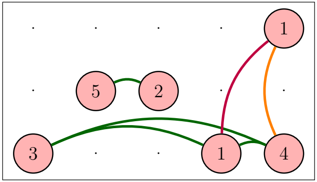


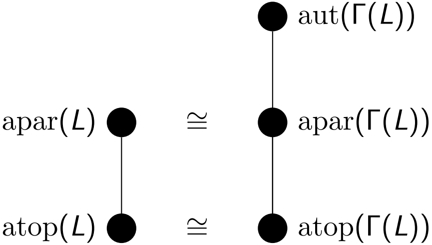
$\mathrm{atop} = \langle(\mathrm{id}, (23), (25); \mathrm{id})\rangle$ (size 2)

$\mathrm{apar} = \langle(\mathrm{id}, (23), (25); \mathrm{id}), (\mathrm{id}, (1325), (1523); (23))\rangle$ (size 4)

aut (ignoring edge colors) (size 8)

Subgroup lattice

If $\Gamma(L)$ denotes the partial Latin rectangle graph of L, then we have the (partial) subgroup lattices for $\mathrm{apar}(L)$ and $\mathrm{aut}(\Gamma(L))$:



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I'll just pretend this doesn't happen (this situation only arises in boring cases).

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  - We conclude $\mathrm{apar}(L) \cong H_1$ and $\mathrm{atop}(L) \cong H_2$.

Construction

For all odd $m \geq 5$, these partial Latin rectangles have a trivial autoparatopism group (and hence a trivial autotopism group):

$$\textit{even:} \qquad \begin{bmatrix} 2 & 3 & 4 & 5 \\ \cdot & 1 & 2 & \cdot \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ \cdot & 1 & 2 & 3 & \cdot \end{bmatrix}, \quad \cdots$$

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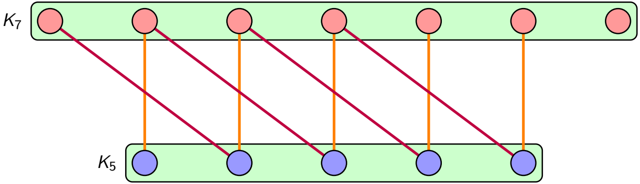
The number of entries in the first row differs from the second row, so the row permutation must be trivial. We color the vertices of the partial Latin rectangle graph red/blue according to their row.

The partial Latin rectangle graphs look like:



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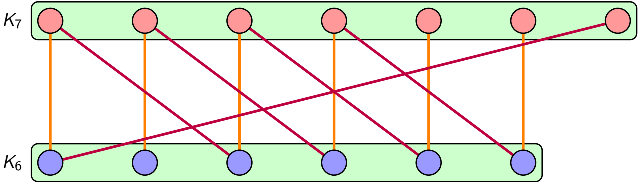

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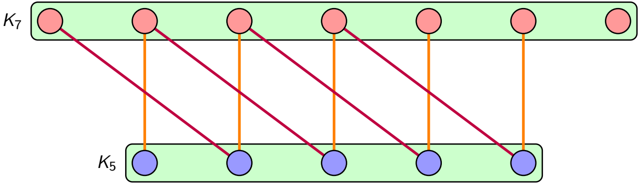

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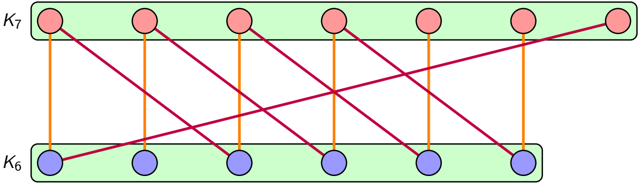

which has no automorphisms that preserve both the edge and vertex colors.
So the autoparatopism group (and hence a trivial autotopism group) of the partial Latin rectangle is trivial.

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  - One result: There exists a weight-m, connected partial Latin rectangle L with $\mathrm{apar}(L) = \mathrm{atop}(L) \cong C_k$ whenever $m \geq 2k + 2$. Open: Is this the best possible for connected PLRs?

Disconnected PLRs

Our $\mathrm{apar}(L) = \mathrm{atop}(L) \cong C_k$ construction is not, in general, the best possible if we allow disconnected cases:

$$\begin{bmatrix} 1 & 2 & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot \\ \cdot & \cdot & 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & \cdot & \cdot & \cdot \\ \cdot & 1 & 3 & \cdot & \cdot \\ 4 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 4 & \cdot & \cdot \\ 5 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 6 & 7 \end{bmatrix}.$$

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This (and other blockwise constructions) is why we focus on connected partial Latin rectangles.
(The connected PLRs also feel like the "primes", and we can glue them together blockwise to form the "composites".)

Thank you