Weakly linked embeddings of complete graphs

Christopher Tuffley
With Erica Flapan (Pomona) and Ramin Naimi (Occidental)

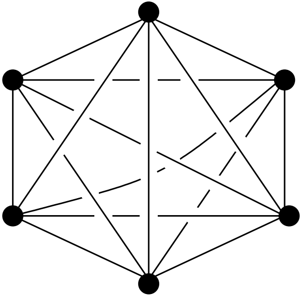
School of Fundamental Sciences
Massey University, New Zealand

November 2019

Intrinsic linking

Theorem (Conway and Gordon, 1983; Sachs, 1983)

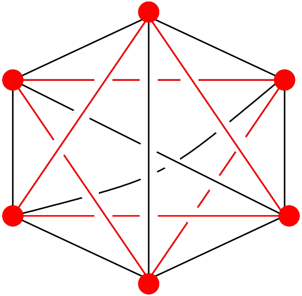
Every embedding of the complete graph K_6 in $\mathbb{R}^3$ contains a nontrivial link.



Intrinsic linking

Theorem (Conway and Gordon, 1983; Sachs, 1983)

Every embedding of the complete graph K_6 in $\mathbb{R}^3$ contains a nontrivial link.



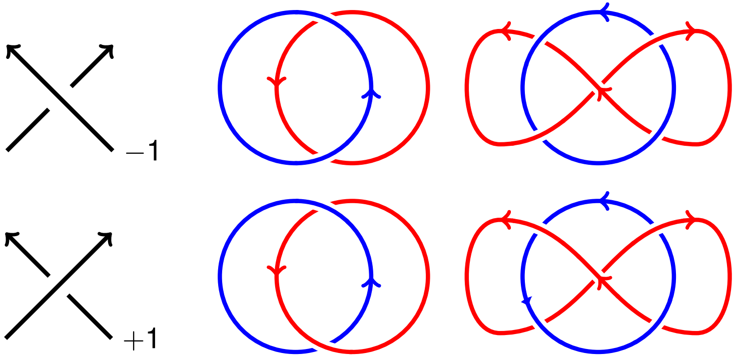
We say that K_6 is *intrinsically linked*.

Linking number

Definition

Let C, D be oriented disjoint simple closed curves in $\mathbb{R}^3$. The *linking number* of C and D, $link(C, D)$, is the signed count of crossings where C crosses *over* D.

Linking number is symmetric: $link(C, D) = link(D, C)$

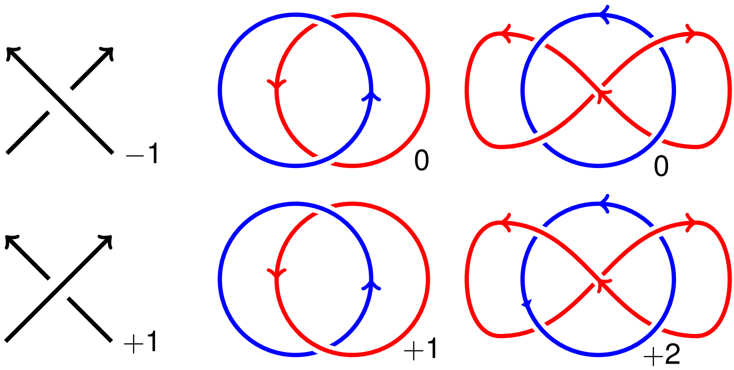


Linking number

Definition

Let C, D be oriented disjoint simple closed curves in $\mathbb{R}^3$. The *linking number* of C and D, $link(C, D)$, is the signed count of crossings where C crosses *over* D.

Linking number is symmetric: $link(C, D) = link(D, C)$



Proof K_6 is intrinsically linked

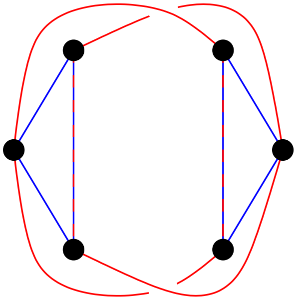
1. Define

$$\lambda = \sum_{\{L,J\}} link(L,J) \bmod 2,$$

summing over all $\frac{1}{2}\binom{6}{3} = 10$ pairs of disjoint triangles in K_6.

2. λ is unchanged by ambient isotopies and crossing changes, which suffice to take any embedding to any other.

3. λ evaluates to 1 on a specific embedding.



$\Rightarrow$ Every embedding contains an odd number of links of odd linking number, hence at least one.

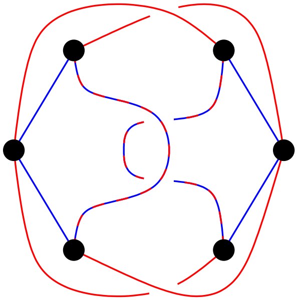
Proof K_6 is intrinsically linked

1. Define

$$\lambda = \sum_{\{L,J\}} link(L, J) \bmod 2,$$

summing over all $\frac{1}{2}\binom{6}{3} = 10$ pairs of disjoint triangles in K_6.

2. λ is unchanged by ambient isotopies and crossing changes, which suffice to take any embedding to any other.
3. λ evaluates to 1 on a specific embedding.



link $\equiv 1$,   *link* $\equiv 0$

$\Rightarrow$ Every embedding contains an odd number of links of odd linking number, hence at least one.

Proof K_6 is intrinsically linked

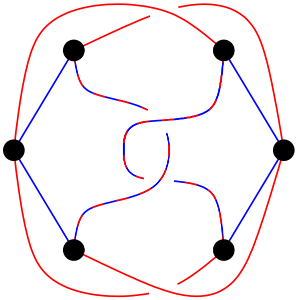
1. Define

$$\lambda = \sum_{\{L,J\}} link(L, J) \bmod 2,$$

summing over all $\frac{1}{2}\binom{6}{3} = 10$ pairs of disjoint triangles in K_6.

2. λ is unchanged by ambient isotopies and crossing changes, which suffice to take any embedding to any other.

3. λ evaluates to 1 on a specific embedding.



link $\equiv 0$, *link* $\equiv 1$

$\Rightarrow$ Every embedding contains an odd number of links of odd linking number, hence at least one.

Proof K_6 is intrinsically linked

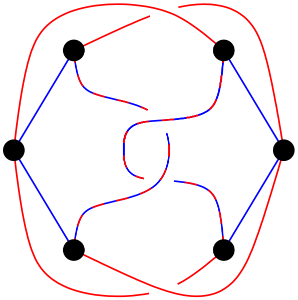
1. Define

$$\lambda = \sum_{\{L,J\}} link(L, J) \bmod 2,$$

summing over all $\frac{1}{2}\binom{6}{3} = 10$ pairs of disjoint triangles in K_6.

2. λ is unchanged by ambient isotopies and crossing changes, which suffice to take any embedding to any other.

3. λ evaluates to 1 on a specific embedding.



$link \equiv 0$, $link \equiv 1$

$\Rightarrow$ Every embedding contains an odd number of links of odd linking number, hence at least one.

Proof K_6 is intrinsically linked

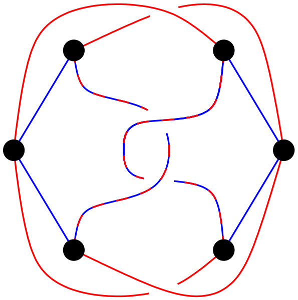
1. Define

$$\lambda = \sum_{\{L,J\}} link(L,J) \bmod 2,$$

summing over all $\frac{1}{2}\binom{6}{3} = 10$ pairs of disjoint triangles in K_6.

2. λ is unchanged by ambient isotopies and crossing changes, which suffice to take any embedding to any other.

3. λ evaluates to 1 on a specific embedding.



link $\equiv 0$, *link* $\equiv 1$

$\Rightarrow$ Every embedding contains an odd number of links of odd linking number, hence at least one.

Additional results of interest

Characterisation of linklessly embeddable graphs

The linklessly embeddable graphs are the graphs with:

- No minor among the six graphs in the *Petersen family* (Robertson, Seymour and Thomas, 1995).
- Colin de Verdière invariant $\mu \leq 4$ (Lovás and Schrijver, 1998).

Intrinsic knotting

A graph is *intrinsically knotted* if every embedding in $\mathbb{R}^3$ contains a nontrivial knot.

- K_7 is intrinsically knotted (Conway and Gordon, 1983).
- *Graph Minor Theorem* $\Rightarrow$ knotlessly embeddable graphs are characterised by a finite set of forbidden minors.
- Over 200 minor-minimal intrinsically knotted graphs are known.

Additional results of interest

Characterisation of linklessly embeddable graphs

The linklessly embeddable graphs are the graphs with:

- No minor among the six graphs in the *Petersen family* (Robertson, Seymour and Thomas, 1995).
- Colin de Verdière invariant $\mu \leq 4$ (Lovás and Schrijver, 1998).

Intrinsic knotting

A graph is *intrinsically knotted* if every embedding in $\mathbb{R}^3$ contains a nontrivial knot.

- K_7 is intrinsically knotted (Conway and Gordon, 1983).
- *Graph Minor Theorem* $\Rightarrow$ knotlessly embeddable graphs are characterised by a finite set of forbidden minors.
- Over 200 minor-minimal intrinsically knotted graphs are known.

What about larger complete graphs?

Question

Do embeddings of larger complete graphs in $\mathbb{R}^3$ necessarily exhibit more complicated linking behavior?

For example:

- Non-split links with many components?
- Two-component links with large linking number?

We say the link $C \cup D$ is a *strong link* if

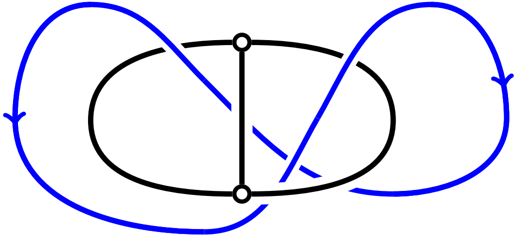
$$|link(C, D)| \geq 2.$$

Key property: additivity of linking number

Fact: For oriented simple closed curves C, D in $\mathbb{R}^3$,

$$\mathit{link}(C, D) = \text{class of } D \text{ in } H_1(\mathbb{R}^3 - C) \cong \mathbb{Z}$$

Consequence: linking number is additive.

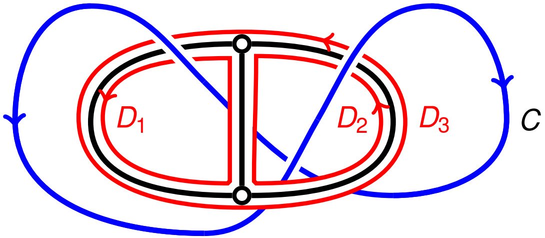


Key property: additivity of linking number

Fact: For oriented simple closed curves C, D in $\mathbb{R}^3$,

$$link(C, D) = \text{class of } D \text{ in } H_1(\mathbb{R}^3 - C) \cong \mathbb{Z}$$

Consequence: linking number is additive.



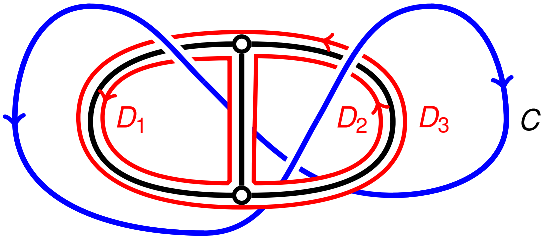
$D_3 = D_1 + D_2$ as sums of edges

Key property: additivity of linking number

Fact: For oriented simple closed curves C, D in $\mathbb{R}^3$,

$$link(C, D) = \text{class of } D \text{ in } H_1(\mathbb{R}^3 - C) \cong \mathbb{Z}$$

Consequence: linking number is additive.



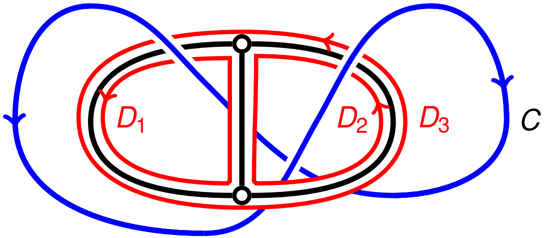
	$D_3 = D_1 + D_2$	as sums of edges
$\Rightarrow$	$[D_3] = [D_1] + [D_2]$	in $H_1(\mathbb{R}^3 - C) \cong \mathbb{Z}$

Key property: additivity of linking number

Fact: For oriented simple closed curves C, D in $\mathbb{R}^3$,

$$link(C, D) = \text{class of } D \text{ in } H_1(\mathbb{R}^3 - C) \cong \mathbb{Z}$$

Consequence: linking number is additive.



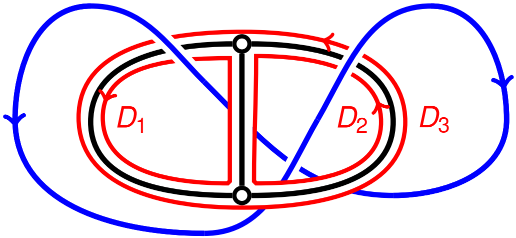
	$D_3 = D_1 + D_2$	as sums of edges
$\Rightarrow$	$[D_3] = [D_1] + [D_2]$	in $H_1(\mathbb{R}^3 - C) \cong \mathbb{Z}$
$\Rightarrow$	$link(C, D_3) = link(C, D_1) + link(C, D_2)$	

Key property: additivity of linking number

Fact: For oriented simple closed curves C, D in $\mathbb{R}^3$,

$$link(C, D) = \text{class of } D \text{ in } H_1(\mathbb{R}^3 - C) \cong \mathbb{Z}$$

Consequence: linking number is additive.



$$D_3 = D_1 + D_2 \quad \text{as sums of edges}$$

$$\Rightarrow \quad [D_3] = [D_1] + [D_2] \quad \text{in } H_1(\mathbb{R}^3 - C) \cong \mathbb{Z}$$

$$\Rightarrow \quad link(C, D_3) = link(C, D_1) + link(C, D_2)$$

here: $\quad 2 = 1 + 1$

Disjoint links implies triple link

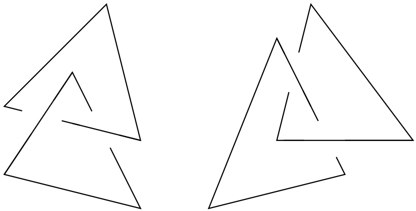
Lemma

Given a link $X_1 \cup Y_1 \cup X_2 \cup Y_2$ in K_N with

$$link(X_i, Y_i) \not\equiv 0 \bmod 2 \qquad \text{for } i = 1, 2,$$

there is a loop X in K_N with all vertices on $X_1 \cup X_2$ such that

$$link(X, Y_i) \not\equiv 0 \bmod 2 \qquad \text{for } i = 1, 2.$$



Disjoint links implies triple link

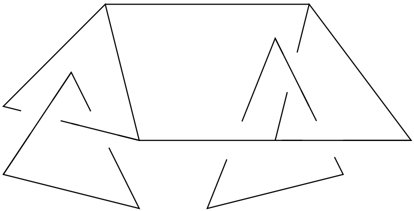
Lemma

Given a link $X_1 \cup Y_1 \cup X_2 \cup Y_2$ *in* K_N *with*

$$\mathit{link}(X_i, Y_i) \not\equiv 0 \bmod 2 \qquad \text{for } i = 1, 2,$$

there is a loop X *in* K_N *with all vertices on* $X_1 \cup X_2$ *such that*

$$\mathit{link}(X, Y_i) \not\equiv 0 \bmod 2 \qquad \text{for } i = 1, 2.$$



Disjoint links implies triple link

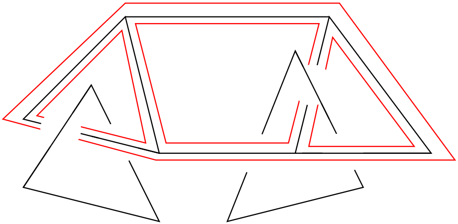
Lemma

Given a link $X_1 \cup Y_1 \cup X_2 \cup Y_2$ *in* K_N *with*

$$link(X_i, Y_i) \not\equiv 0 \bmod 2 \qquad \textit{for } i = 1, 2,$$

there is a loop X *in* K_N *with all vertices on* $X_1 \cup X_2$ *such that*

$$link(X, Y_i) \not\equiv 0 \bmod 2 \qquad \textit{for } i = 1, 2.$$



Disjoint links implies triple link

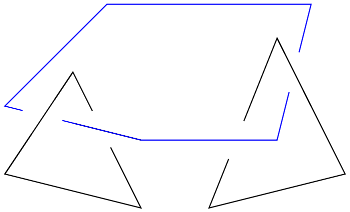
Lemma

Given a link $X_1 \cup Y_1 \cup X_2 \cup Y_2$ in K_N with

$$link(X_i, Y_i) \not\equiv 0 \bmod 2 \qquad \text{for } i = 1, 2,$$

there is a loop X in K_N with all vertices on $X_1 \cup X_2$ such that

$$link(X, Y_i) \not\equiv 0 \bmod 2 \qquad \text{for } i = 1, 2.$$



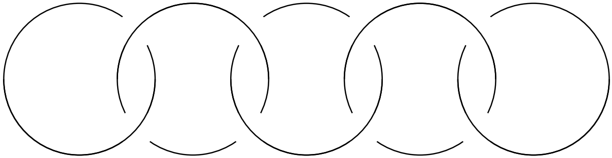
Consequence: existence of chains

Theorem (Flapan *et. al.*, 2001 (paraphrased))

Let $k \in \mathbb{N}$. For N sufficiently large, every embedding of K_N in $\mathbb{R}^3$ contains a k-component "chain": a link $L_1 \cup \cdots \cup L_k$ such that

$$link(L_i, L_{i+1}) \neq 0$$

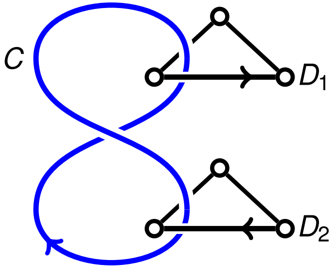
for $i = 1, \ldots, k-1$. *($N = 6(k-1)$ suffices)*



Triple link implies strong link

Lemma (Flapan 2002, special case of Lemma 1)

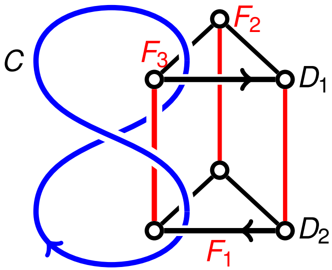
Let $C \cup D_1 \cup D_2$ be a triple link contained in an embedding of K_n in $\mathbb{R}^3$, such that $link(C, D_1) = link(C, D_2) = 1$.
Then there is a simple closed curve D in K_n, with all its vertices on $D_1 \cup D_2$, such that $link(C, D) \geq 2$.



Triple link implies strong link

Lemma (Flapan 2002, special case of Lemma 1)

Let $C \cup D_1 \cup D_2$ be a triple link contained in an embedding of K_n in $\mathbb{R}^3$, such that $link(C, D_1) = link(C, D_2) = 1$.
Then there is a simple closed curve D in K_n, with all its vertices on $D_1 \cup D_2$, such that $link(C, D) \geq 2$.

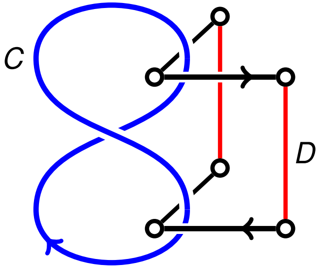


$$[F_1] + [F_2] + [F_3] = [D_1] + [D_2] = 1 + 1 = 2$$

Triple link implies strong link

Lemma (Flapan 2002, special case of Lemma 1)

Let $C \cup D_1 \cup D_2$ be a triple link contained in an embedding of K_n in $\mathbb{R}^3$, such that $link(C, D_1) = link(C, D_2) = 1$.
Then there is a simple closed curve D in K_n, with all its vertices on $D_1 \cup D_2$, such that $link(C, D) \geq 2$.



$$[F_1] + [F_2] + [F_3] = [D_1] + [D_2] = 1 + 1 = 2$$
$$[D] = [D_1 + D_2 - F_2] \geq 2$$

Consequence: existence of strong links

Theorem (Flapan, 2002)

Let $\lambda \in \mathbb{N}$. For N sufficiently large, every embedding of K_N in $\mathbb{R}^3$ contains a two component link $L \cup J$ such that

$$|link(L, J)| \geq \lambda.$$

($N = \lambda(15\lambda - 9)$ suffices)

Consequence: existence of strong links

Theorem (Flapan, 2002)

Let $\lambda \in \mathbb{N}$. For N sufficiently large, every embedding of K_N in $\mathbb{R}^3$ contains a two component link $L \cup J$ such that

$$|link(L, J)| \geq \lambda.$$

($N = \lambda(15\lambda - 9)$ suffices)

In fact:

Theorem

For all $k, \lambda \in \mathbb{N}$, for N sufficiently large every embedding of K_N in $\mathbb{R}^3$ contains a k-component link with all pairwise linking numbers

- *at least λ in absolute value (Flapan et al., 2008).*
- *a nonzero multiple of λ (T., 2019).*

Result extends to higher dimensions (T., 2019).

Motivating question

Let C, D be disjoint simple closed curves in $\mathbb{R}^3$. We say

- C and D link if $link(C, D) \neq 0$
- C and D are weakly linked if $|link(C, D)| = 1$
- C and D are strongly linked if $|link(C, D)| \geq 2$

Question

What is the least n such that K_n is intrinsically strongly linked?

That is:
What is the least n such that every embedding of K_n in $\mathbb{R}^3$ contains a strong link?

Prior results

Theorem (Flapan-Naimi-Pommersheim, 2000)

K_{10} is intrinsically triple linked, but K_9 is not.

$\Rightarrow$ K_{10} is intrinsically strongly linked

Prior results

Theorem (Flapan-Naimi-Pommersheim, 2000)

K_{10} is intrinsically triple linked, but K_9 is not.

$\Rightarrow$ K_{10} is intrinsically strongly linked

Theorem (Fleming and Mellor, 2009)

K_8 has an embedding with no strong link.

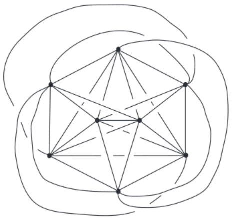


Image: Fleming and Mellor

Prior results

Theorem (Flapan-Naimi-Pommersheim, 2000)

K_{10} is intrinsically triple linked, but K_9 is not.

$\Rightarrow$ K_{10} is intrinsically strongly linked

Theorem (Fleming and Mellor, 2009)

K_8 has an embedding with no strong link.

Theorem (Naimi and Pavelescu, 2014)

Linear embeddings of K_9 are triple linked.

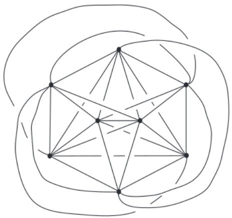


Image: Fleming and Mellor

Prior results

Theorem (Flapan-Naimi-Pommersheim, 2000)

K_{10} is intrinsically triple linked, but K_9 is not.

$\Rightarrow$ K_{10} is intrinsically strongly linked

Theorem (Fleming and Mellor, 2009)

K_8 has an embedding with no strong link.

Theorem (Naimi and Pavelescu, 2014)

Linear embeddings of K_9 are triple linked.

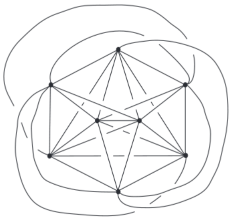


Image: Fleming and Mellor

Conjecture

K_9 is intrinsically strongly linked.

Impasse

Many partial results, including:

- A necessary and sufficient condition for an embedding of K_9 to be weakly linked.
- K_9 minus two adjacent edges has a weakly linked embedding (found by computer search).

— but so far unable to resolve the question of whether K_9 is intrinsically strongly linked.

New question: K_m–K_n embeddings

If you can't solve a problem, then there is an easier problem you can solve: find it.

— George Pólya

Problem

Algebraically characterise linked embeddings of K_m and K_n in $\mathbb{R}^3$ such that no cycle in K_m strongly links any cycle in K_n.

— now we only care about links between cycles in one graph and cycles in the other, which makes things easier.

Strategy

Characterise in turn weak linking between

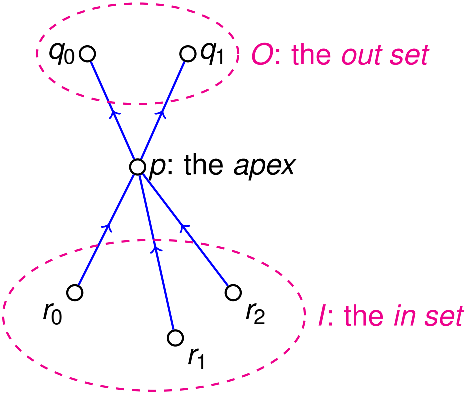
1. a simple closed curve and K_n.
2. a theta curve Θ and K_n.
3. K_4 and K_n, $n \geq 4$.
4. K_m and K_n, $m, n \geq 5$.

A common theme is that each graph gets partitioned into sets of vertices that are interchangeable with respect to linking.

Stars: definition

Definition

Let $\{\{p\}, O, I\}$ be a partition of the vertices of K_n. The *star pOI* consists of all oriented triangles of the form pqr, with $q \in O$ and $r \in I$.

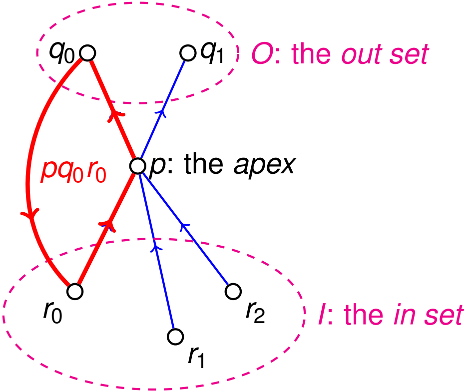


pOI is an *improper star* if $1 \in \{|O|, |I|\}$ (so apex is not unique).

Stars: definition

Definition

Let $\{\{p\}, O, I\}$ be a partition of the vertices of K_n. The *star pOI* consists of all oriented triangles of the form pqr, with $q \in O$ and $r \in I$.



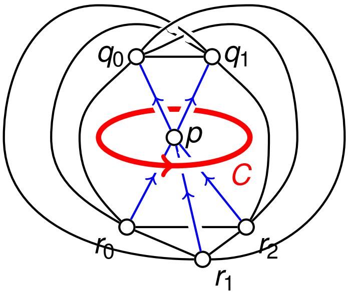
pOI is an *improper star* if $1 \in \{|O|, |I|\}$ (so apex is not unique).

Stars: linking

Definition

Let C be an oriented simple closed curve disjoint from K_n. Then C links K_n in the star pOI if it links precisely the triangles in pOI: if for all oriented triangles T in K_n,

$$link(C, T) = \begin{cases} +1 & \text{if } T \in pOI, \\ -1 & \text{if } -T \in pOI, \\ 0 & \text{else.} \end{cases}$$

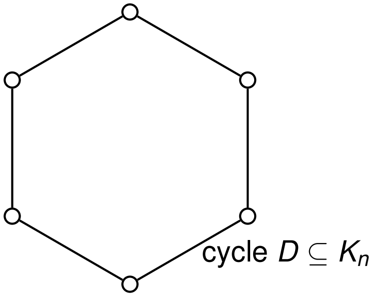


Stars and strong linking

Let C be an oriented simple closed curve disjoint from K_n.

Lemma

If C links K_n in the star pOI, then C does not strongly link K_n.

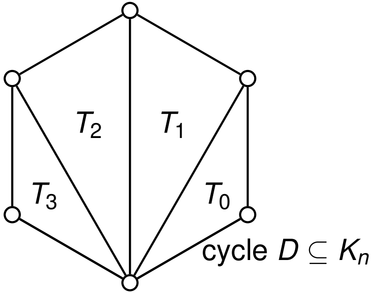


Stars and strong linking

Let C be an oriented simple closed curve disjoint from K_n.

Lemma

If C links K_n in the star pOI, then C does not strongly link K_n.



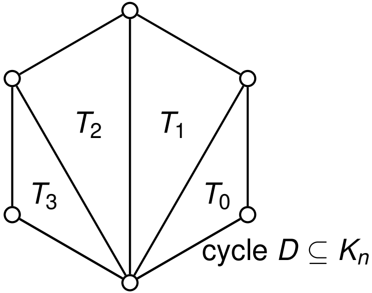
$$[D] = \sum_i [T_i]$$

Stars and strong linking

Let C be an oriented simple closed curve disjoint from K_n.

Lemma

If C links K_n in the star pOI, then C does not strongly link K_n.



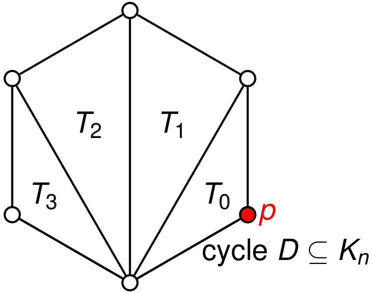
$$[D] = \sum_i [T_i] = \begin{cases} 0 & \text{if } p \notin D \\ & \end{cases}$$

Stars and strong linking

Let C be an oriented simple closed curve disjoint from K_n.

Lemma

If C links K_n in the star pOI, then C does not strongly link K_n.



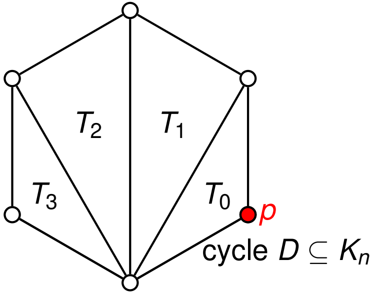
$$[D] = \sum_i [T_i] = \begin{cases} 0 & \text{if } p \notin D \\ [T_0] & \text{if } p \text{ as shown} \end{cases}$$

Stars and strong linking

Let C be an oriented simple closed curve disjoint from K_n.

Lemma

If C links K_n in the star pOI, then C does not strongly link K_n.



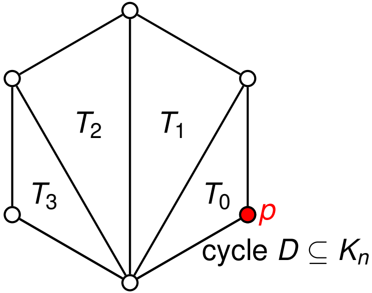
$$[D] = \sum_i [T_i] = \begin{cases} 0 & \text{if } p \notin D \\ [T_0] & \text{if } p \text{ as shown} \end{cases} \in \{0, \pm 1\}$$

Stars and strong linking

Let C be an oriented simple closed curve disjoint from K_n.

Lemma

If C links K_n in the star pOI, then C does not strongly link K_n.



$$[D] = \sum_i [T_i] = \begin{cases} 0 & \text{if } p \notin D \\ [T_0] & \text{if } p \text{ as shown} \end{cases} \in \{0, \pm 1\}$$

Lemma

Conversely, if C links but does not strongly link K_n, then it links K_n in a star.

Proof for $n = 4$

In $H_1(\mathbb{R}^3 - C)$:

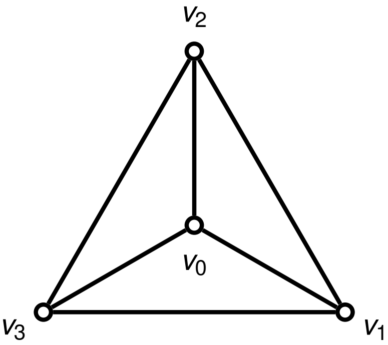
$$[T_0] + [T_1] + [T_2] + [T_3] = 0$$

If no cycle links C strongly
then (up to relabelling)

$$[T_0] = [T_2] = 0, \quad [T_1] = -[T_3] = 1$$

and C links the star

$$v_0\{v_2\}\{v_1, v_3\}.$$



Proof for $n = 4$

In $H_1(\mathbb{R}^3 - C)$:

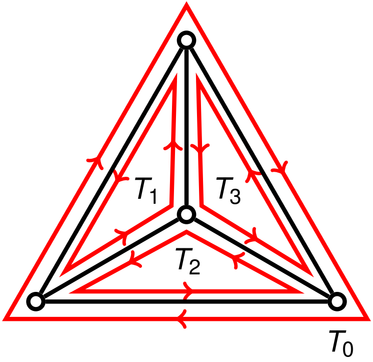
$$[T_0] + [T_1] + [T_2] + [T_3] = 0$$

If no cycle links C strongly
then (up to relabelling)

$$[T_0] = [T_2] = 0, \quad [T_1] = -[T_3] = 1$$

and C links the star

$$v_0\{v_2\}\{v_1, v_3\}.$$



Proof for $n = 4$

In $H_1(\mathbb{R}^3 - C)$:

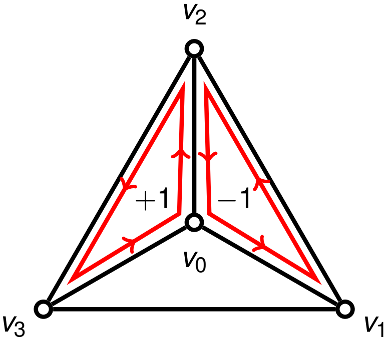
$$[T_0] + [T_1] + [T_2] + [T_3] = 0$$

If no cycle links C strongly
then (up to relabelling)

$$[T_0] = [T_2] = 0, \quad [T_1] = -[T_3] = 1$$

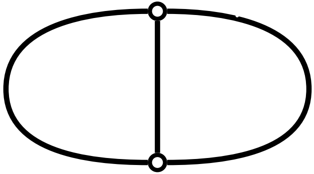
and C links the star

$$v_0\{v_2\}\{v_1, v_3\}.$$



Theta curves I

A *theta curve* is the following graph:



With respect to any simple closed curve D we have

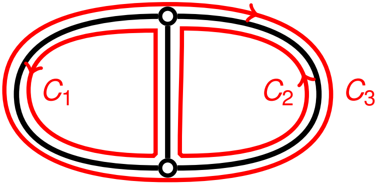
$$[C_1] + [C_2] + [C_3] = 0,$$

so if D links Θ weakly then

$$\{[C_1], [C_2], [C_3]\} = \{-1, 0, 1\}.$$

Theta curves I

A *theta curve* is the following graph:



With respect to any simple closed curve D we have

$$[C_1] + [C_2] + [C_3] = 0,$$

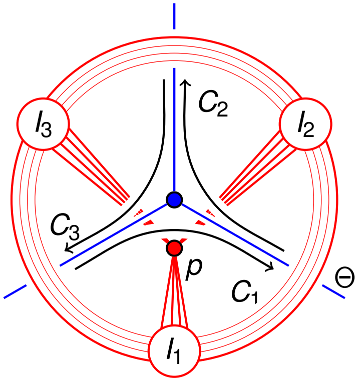
so if D links Θ weakly then

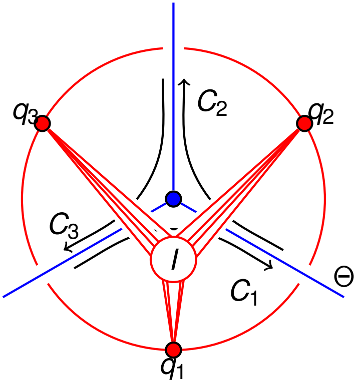
$$\{[C_1], [C_2], [C_3]\} = \{-1, 0, 1\}.$$

Theta curves II

Theorem

Let Θ be a theta curve that links but does not strongly link an embedding of K_n in $\mathbb{R}^3$. Then the linking between Θ and K_n is described by one of the pictures below.





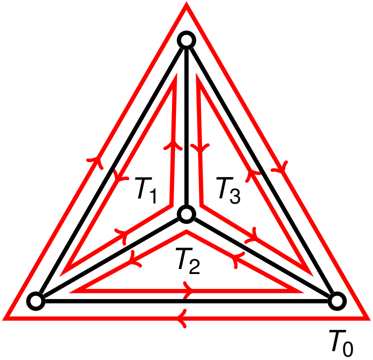
K_4

Key properties:

- Graph decomposes as a union of triangles summing to 0:

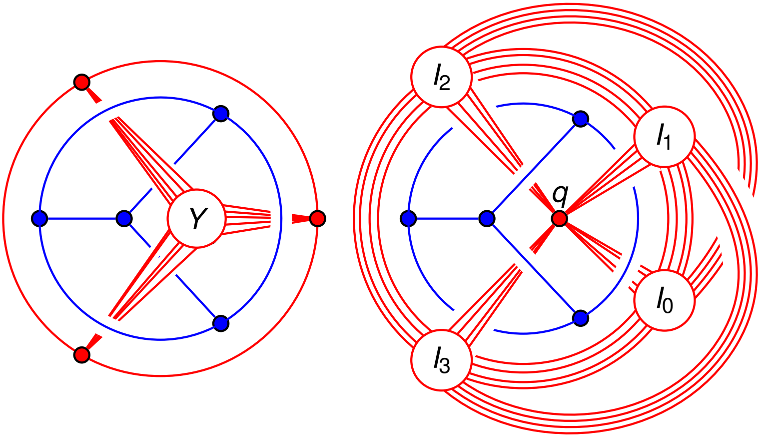
$$T_0 + T_1 + T_2 + T_3 = 0.$$

- Any two of the T_i form a theta curve.



Weakly linked K_4–K_n embeddings

Two possible pictures:



On the right $I_0 \cup I_1 \cup I_2 \cup I_3 = K_n - \{q\}$; some I_j may be empty.

$m, n \geq 5$

Key: get a "common vertex" or an "edge-incident triangle":

Key Lemma

Suppose that $G = K_m$, $H = K_n$ are weakly linked. If $m \geq 5$ then exactly one of the following occurs:

1. *There is a vertex p of G common to all triangles of G linking H.*
2. *There is a triangle T^* of G such that a triangle $T \neq T^*$ of G links H if and only if it shares an edge with T^*.*

Theorem

For $m, n \geq 5$ there are three families of weak embeddings:

1. *A common vertex in each graph.*
2. *A common vertex in one, an edge-incident triangle in the other.*
3. *An edge-incident triangle in each graph.*

$m, n \geq 5$

Key: get a "common vertex" or an "edge-incident triangle":

Key Lemma

Suppose that $G = K_m$, $H = K_n$ are weakly linked. If $m \geq 5$ then exactly one of the following occurs:

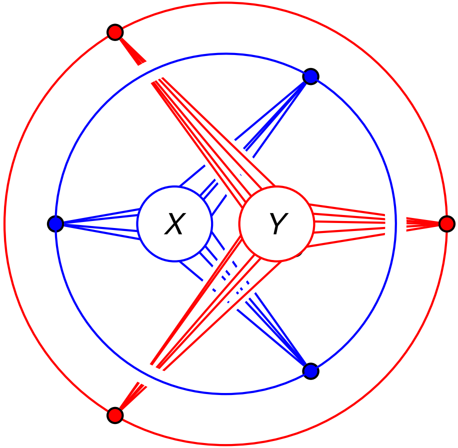
1. *There is a vertex p of G common to all triangles of G linking H.*
2. *There is a triangle T^* of G such that a triangle $T \neq T^*$ of G links H if and only if it shares an edge with T^*.*

Theorem

For $m, n \geq 5$ there are three families of weak embeddings:

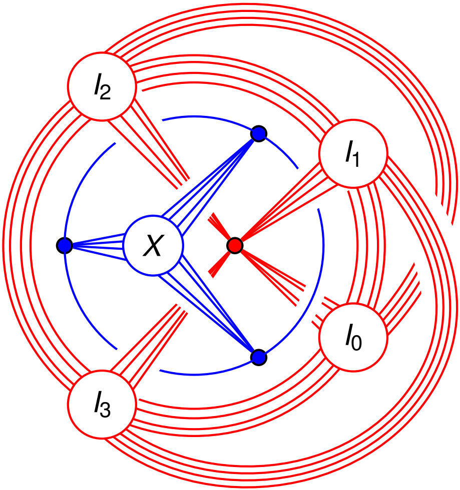
1. *A common vertex in each graph.*
2. *A common vertex in one, an edge-incident triangle in the other.*
3. *An edge-incident triangle in each graph.*

An edge-incident triangle in each graph



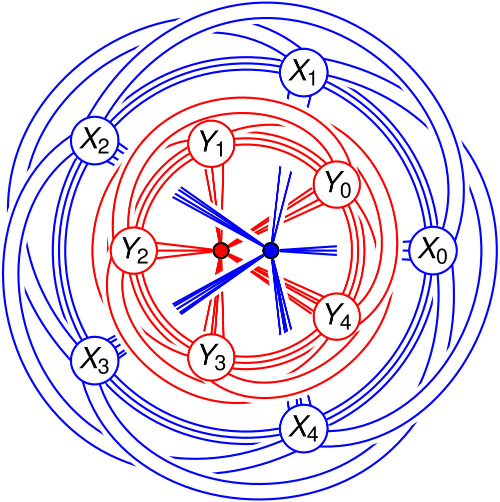
— underlying pattern is a K_4–K_4 embedding.

A common vertex with an edge-incident triangle



— underlying pattern is a K_4–K_5 embedding.

A common vertex in each graph



— underlying pattern two wheels with ℓ spokes ($\ell = 5$ shown).