Well-quasi-ordering Binary Matroids

Jim Geelen, Bert Gerards, and Geoff Whittle

What is a binary matroid?

A binary matroid is defined by a set of vectors over the 2-element field. For example

$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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- The *independent* sets of M label linearly independent vectors.
- Linear independence is not affected by row operations, so row operations do not change the matroid.

We can delete elements from a matroid. For example, deleting f gives,

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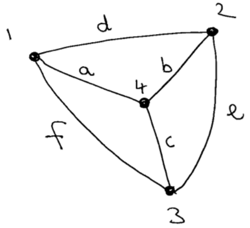
A minor is obtained by a sequence of deletions and contractions.

Minors of Graphs

Recall that for a graph G we can

- Delete an edge.
- Contract an edge.
- Obtain a minor by a sequence of deletions and contractions.

Binary matroids generalise graphs



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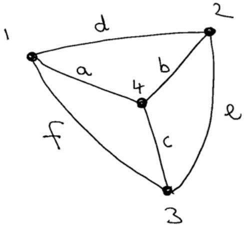

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
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- Graph G, cycle matroid $M(G)$. Will be relaxed about the distinction.

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- An antichain in a quasi-order is a set of pairwise incomparable elements.
- A well-quasi-order has no infinite antichains.

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- Do we have a well-quasi-order?
- No. There are infinitely many primes.

Graphs and Subgraphs

$H \preceq G$ if H is a subgraph of G.

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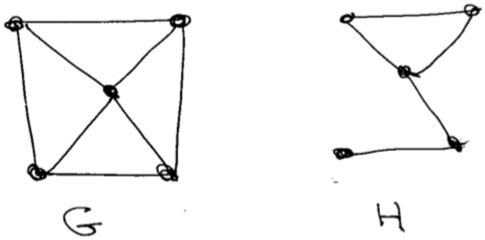

Figure: H is a subgraph of G

- Is this a well-quasi-order?

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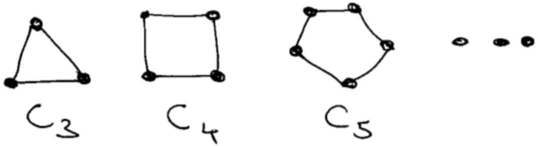

Figure: An antichain in the subgraph order

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- The cycles look more like a chain than an antichain!
- In fact C_n can be obtained from C_{n+1} by contracting an edge.
- In the minor order on graphs, $H \preceq G$ if H can be obtained from G by a sequence of deletions and contractions.
- Wagner's Conjecture: Graphs are well-quasi-ordered with respect to the minor order.

Two famous theorems

Theorem (Robertson and Seymour)

Graphs are well-quasi-ordered under the minor order.

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Theorem
Any minor-closed property of graphs can be recognised in polynomial time.

The Work Horse
The Graph Minors Structure Theorem of Robertson and Seymour describe the qualitative structure of members of proper minor-closed classes of graphs. This is where most of the work is.

Theorem (Geelen, Gerards, W)

Binary matroids are well-quasi-ordered under the minor order.

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Apologies for the sales pitch

- A rank-n graphic matroid has at most $\binom{n}{2}$ elements.

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- A rank-n graphic matroid has at most $\binom{n}{2}$ elements.
- A rank-n binary matroid can have $2^n - 1$ elements.
- So almost all binary matroids are not graphic. Graphs to binary matroids is a massive step.
- Arbitrary matroids are not well-quasi-ordered.

It's all about connectivity

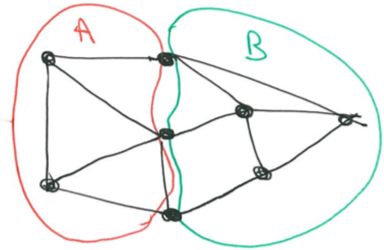
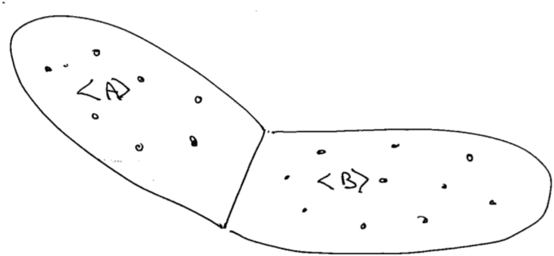
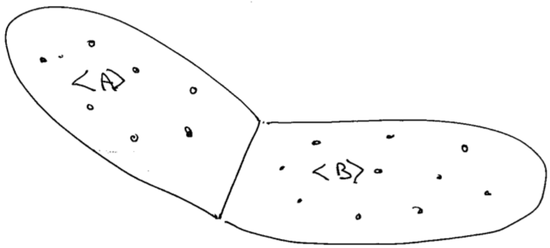


Figure: (A, B) defines a 3-separation in the graph



- (A, B) a partition of M.
- If $\langle A \rangle$ meets $\langle B \rangle$ in rank k, then (A, B) defines a $(k+1)$-separation in M.



- (A, B) a partition of M.
- If $\langle A \rangle$ meets $\langle B \rangle$ in rank k, then (A, B) defines a $(k+1)$-separation in M.
- The $+1$ makes graph connectivity and matroid connectivity coincide when M is the matroid of a graph.

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- Abundant low connectivity controls complexity in graphs or binary matroids.

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Quasitheorem

Various types of decorated trees are well-quasi-ordered.

Bounded Tree Width

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Figure: Tree width about 4 Note the tiled floor

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Theorem (GGW)

Any class of binary matroids of bounded tree width is well-quasi-ordered.

The Strategy

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3. Invoke usual minimal bad sequence argument.

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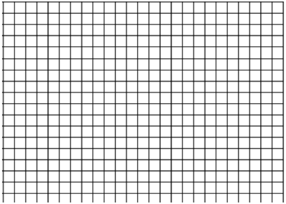
- In fact, for any k we like we can assume that all members of $\mathcal{S}$ have tree width at least k.
- But high tree width must be good for something. Otherwise we have not made progress.

Grids



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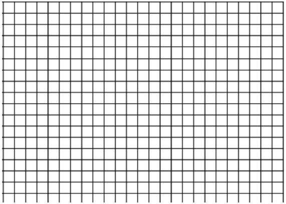

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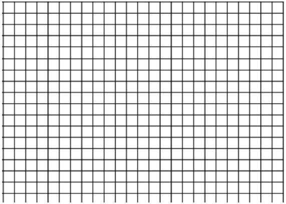

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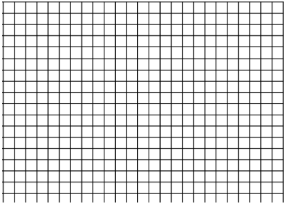

- Sufficiently large grids have arbitrarily high tree width.
- Any planar graph is a minor of a sufficiently large grid.
- There is a function $f : \mathbb{N} \to \mathbb{N}$ such that, if G is a planar graph with n vertices, then G is a minor of an $f(n) \times f(n)$ grid graph.

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- In fact a much more general result is true.

Proof of the grid theorem

- Several proofs of the grid theorem for graphs.
- None of them extend to matroids.
- Grid theorem for matroids was three years hard work.
- Current proof is *not* intuitive.

High tree width gives big grids, so that is something. But we have learnt more. Recall our antichain

$$S_1, S_2, S_3, \ldots, S_n, \ldots$$

- We know that

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  all belong to the class of structures that do not have S_1 as a minor.

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- For example, what if S_1 is a planar graph? What happens when we exclude a planar graph?

Excluding a Planar Graph

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- Voila!

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- High tree width does not give high connectivity as such.
- It gives high order tangles.

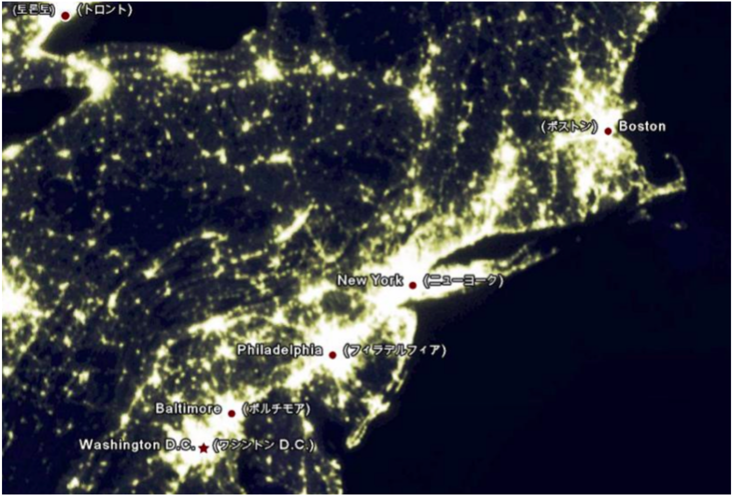

Figure: Boswash: A graph with several high order tangles

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- We'll slip over issues due to tangles.

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The graph minors structure theorem gives us a qualitative structural description of such a graph.

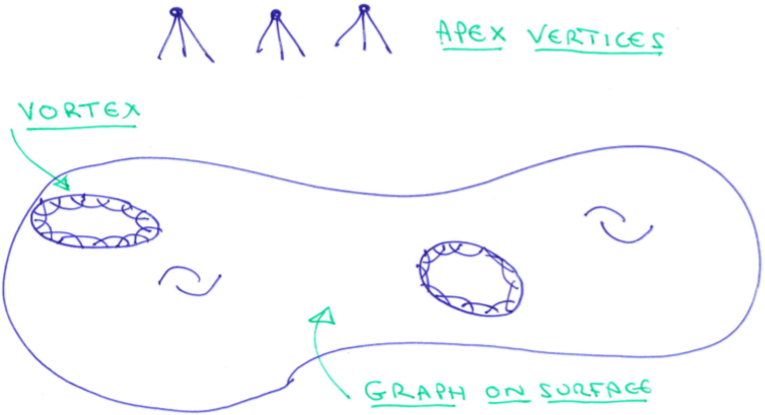

Figure: The Graph Minors Structure Theorem

The Graph Minors Structure Theorem

Theorem

For any non-planar graph H, there exists a positive integer k such that every H-free graph can be obtained as follows:

1. *We start with a graph that embeds on a surface on which H does not embed.*
2. *We add at most k vortices, where each vortex has depth at most k.*
3. *we add at most k new vertices and add any number of edges, each having at least one of its endpoints among the new vertices.*
4. *Finally, we join via k-clique-sums graphs of the above type.*

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- How much help is that?

Beyond Graphs and Cographs

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What if the members of $\mathcal{M}$ are neither matroids of graphs, nor the duals of graphs?

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What if the members of $\mathcal{M}$ are neither matroids of graphs, nor the duals of graphs?

Theorem (GGW)

Every binary matroid with no M_1 minor admits a tree decomposition into pieces that are either essentially graphic or essentially cographic.

Essentially Graphic Matroids

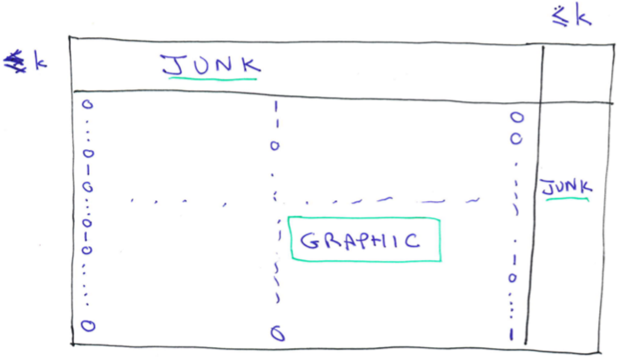

Figure: An Essentially Graphic Matroid

- Columns in B are vectors labelling edges. We have group labelled edges.

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- We almost have a doubly group labelled graph.

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2. Describe binary matroids as tree-like object built up from doubly group labelled graphs.
3. That is, describe binary matroids as certain decorated trees.

Future Work

- Extend the result to other finite fields. Many extra difficulties, but we believe we will do it.

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- Prove Rota's Conjecture. For any finite field $\mathbb{F}$ there is a finite number of forbidden minors for $\mathbb{F}$-representability.