# Extremal embedded graphs

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August 26, 2019

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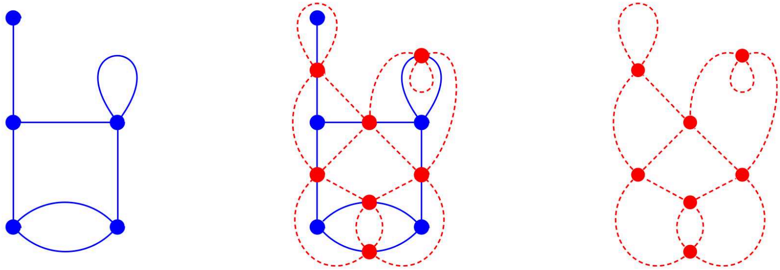

Figure: A plane graph (in blue) and its medial graph (in red).

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- The medial graph of a disconnected graph is the disjoint union of the medial graphs of each connected component.
- The medial graph of any embedded graph is a 4-regular embedded graph.

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- Let $\mu(G)$ be the number of components of a straight-ahead closed walk decomposition of $G_m$.

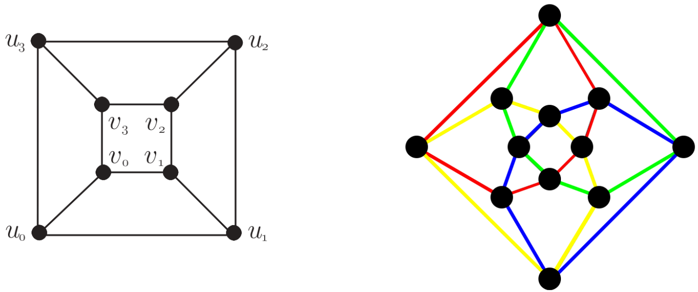


Figure: A plane graph $G$ and its medial graph with $\mu(G) = 4$.

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$$\gamma(G) = \text{Euler genus} = \begin{cases} 2g(G), & \text{if } G \text{ is orientable,} \\ g(G), & \text{if } G \text{ is non-orientable.} \end{cases}$$

[1]X. Jin, F. Dong and E. G. Tay, On graphs determining links with maximal number of components via medial construction, *Discrete Appl. Math.* **157** (2009), 3099–3110.

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**Theorem (Jin, Dong and Tay, 2009[1])**

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Let $G$ be a connected plane graph. If $G$ is extremal then $G$ is bipartite and each face of $G$ is even.

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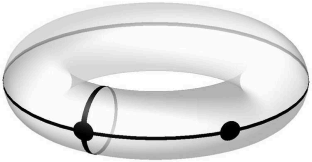
- faces are discs.



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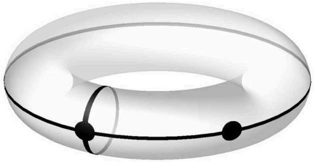

**Theorem (Huggett and Tawfik, 2015)**
Let $G$ be a graph cellularly embedded on an orientable surface of genus $g$. Then

$$1 \leq \mu(G) \leq f(G) + 2g.$$

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# Ribbon graphs

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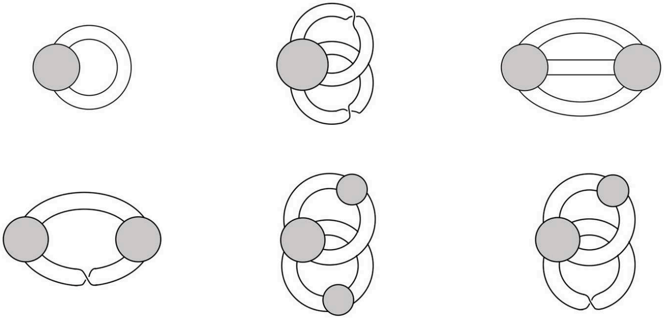

Figure: Examples of ribbon graphs.

Formally, a ribbon graph[3] $G = (V(G), E(G))$ is a surface with boundary represented as the union of two sets of discs, a set $V(G)$ of vertices, and a set $E(G)$ of edges such that

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- Every edge contains exactly two such line segments.

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- Since cellularly embedded graphs and ribbon graphs are equivalent, we can move freely between these representations, choosing whichever is most convenient at the time for our purposes.

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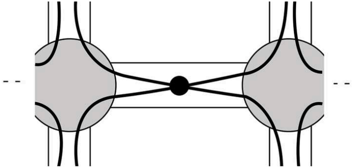

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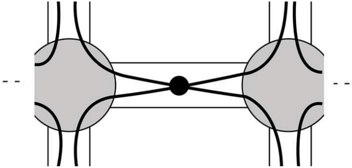
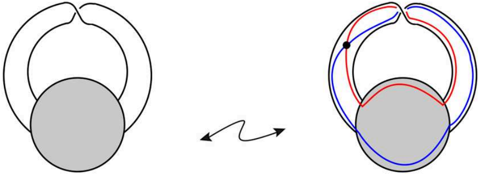


**Example**



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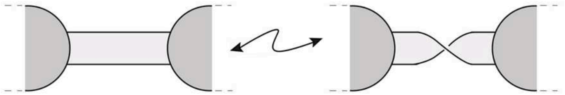
$G^{\times}$ is simply the result of giving a half-twist to all of the edges as shown below.



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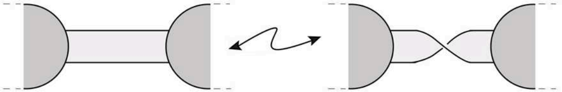
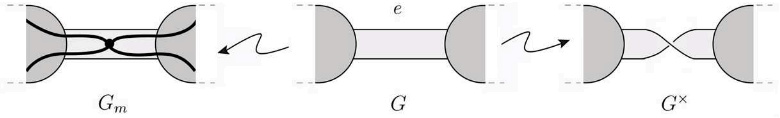


**Lemma**

Let $G$ be a ribbon graph. Then $\mu(G) = f(G^\times)$.



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**Lemma**

Let $G + e$ be the ribbon graph obtained from a ribbon graph $G$ by adding a new edge $e$ connecting two vertices of $G$. Then

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**Theorem**

Let $G$ be a ribbon graph. Then

$$k(G) \leq \mu(G) \leq f(G)+\gamma(G).$$

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**Theorem**

A ribbon graph $G$ is extremal if and only if $\gamma(G^{\times}) = 0$, i.e. $G^{\times}$ is plane.

# Minor

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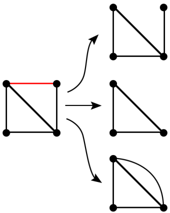
- edge (vertex) deletion
- edge contraction

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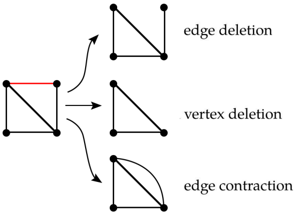


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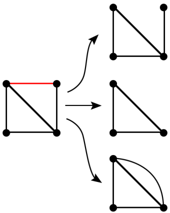


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To contract $e = (u, v)$:

- attach a disc to each $\partial$-cpt. of $v \cup e \cup u$
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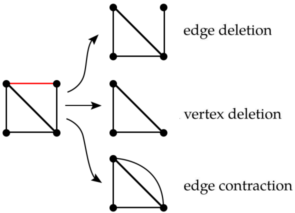


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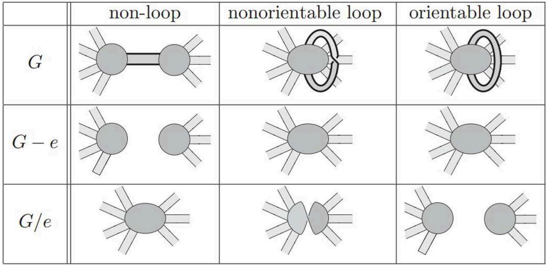


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**Lemma**

Let $G$ be a ribbon graph and $e$ be a bridge of $G$. Then $\mu(G) = \mu(G/e)$.


Figure: The medial graphs of $G$ and $G/e$.

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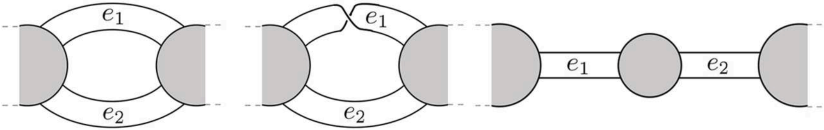


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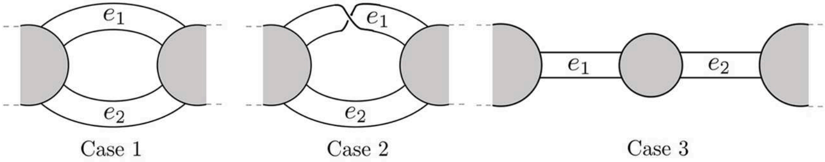

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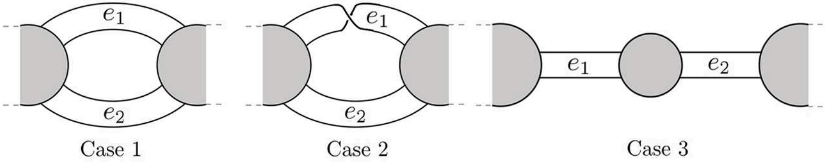

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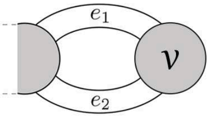


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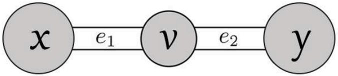

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4. Let $v$ be a vertex of degree 2 with two different adjacent vertices $x$ and $y$. Then $G/\{v, x\}/\{v, y\}$ is extremal if and only if $G$ is extremal.

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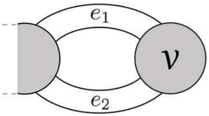


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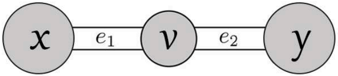

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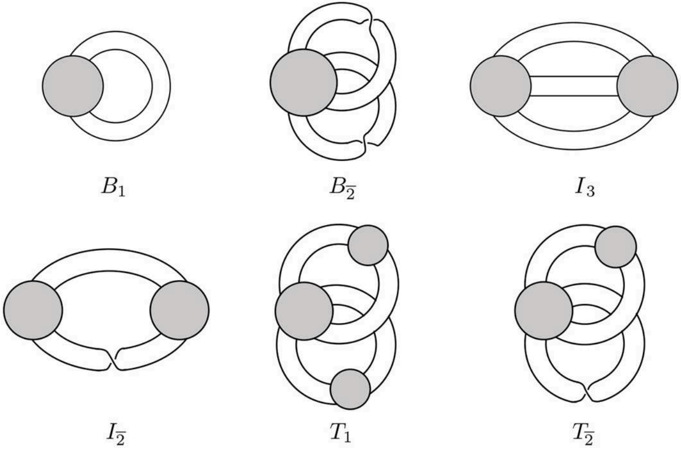
Let $G$ be a ribbon graph. We say that a ribbon graph $H$ is an extremal minor of $G$, if there is a sequence of ribbon graphs

$$G = G_0, G_1, \cdots, G_t = H$$

where for each $i$, $G_{i+1}$ is obtained from $G_i$ by either an admissible deletion or an admissible contraction.

**Theorem**

Let $G$ be a ribbon graph. Then $G$ is extremal $\Leftrightarrow$ it contains no extremal minor equivalent to $B_1, B_{\overline{2}}, I_3, I_{\overline{2}}, T_1$ or $T_{\overline{2}}$.



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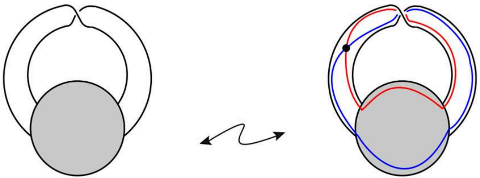

Thank you for your attention!