Structures with Features

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For Ben Nebres: more than brother, friend and priest.

Abstract

What do we look at first when checking a proof or a student's exercise? What is important in recognizing someone else's face? What are the important things that make us realize we are looking at a (mathematical) group? How do we look at a building? How do pilots navigate safely with so many instruments to look at?

All of these are examples involving *structures with features*. The concept seems to arise in so many different contexts but not to have been formalized. This is true despite the use of *Feature Set Theory* by cognitive psychologists (see [18]) and related work by Christopher Alexander (see [2] and [3]) which both date back to the mid-seventies.

We use the examples to suggest how features are important in the process of abstraction from known structures, and how and when formal descriptions correspond to the world.

We consider the operations of modifying, adding and deleting features in a structure and the idea of isomorphisms and other similarity relations between structures with features. In particular we analyse what it means for classes of structures with features to be equivalent.

1 Introduction

> *A universe comes into being when a space is severed into two. A unity is defined. The description, invention and manipulation of unities is at the base of all scientific enquiry.* (H.B. Maturana, *Autopoiesis*, [14], p. 73.)

Thus begins the introduction to Maturana's seminal work. His speciality was originally vision so it is not surprising that visual examples are particularly appro-

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[*] Special thanks to Queena Lee who made me aware of the psychologists' *Feature set theory* and provided references.

priate for giving examples of unities.[1] We shall use the word "feature" rather than "unity" since it is better for our purposes.

The beginning of the book of Genesis in the Bible also notes this dramatic separation:

> *And the earth was without form, and void; and darkness was upon the face of the deep. And the Spirit of God moved upon the face of the waters. And God said, Let there be light: and there was light. And God saw the light, that it was good: and God divided [separated] the light from the darkness.* *Genesis* I, vv. 2-4.

Suppose one walks into a sparsely furnished room with a dark carpeted floor. One of the first things that springs to one's attention is any white speck on that nice clean carpet. The eye responds to the sudden change from dark to light. The spot is one of Maturana's unities, for the speck and the surrounding darkness are the two parts into which the space[2] is separated.

In heavy traffic a flashing light commands attention because of its sudden changing. This is another feature or unity.

An experienced programmer looking at a program immediately notices those `GOTO`s that should not be there; a mathematician looking at a proof immediately notices the inductions, even though he or she may not immediately notice what kind of induction it is; and a logician looking at a formal proof written in a natural deduction (tree) form will immediately spot the "interesting" nodes.

These are all examples of people noticing features.

The idea of separation (as proposed by Maturana) is obviously a fundamental one and occurs in the very basic development of the number concept, see Seidenberg [16] and my [6], chapter I.

On the other hand, it is important to remember that the features occur in, to use Maturana's terminology, *a space*, or, as we shall put it, in a *structure*.

The Secretary General of the United Nations, Kofi Annan, recounted the following tale of his childhood.[3]

> *[I was] in boarding school, in high school, and the headmaster came in one day and put a broad white sheet with a dot in the right hand corner on the board – and we were 45 of us – and said,*
> *"Boys, what do you see?"*
> *And in unison we all put up our hands and said,*
> *"Black dot."*
> *He stood back, and said,*

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[1] Maturana says in a note on page 6 of [14] that he would have used the word "unit" if his English had been better at the time.

[2] I.e. the floor of the room.

[3] In response to a question after his speech at the National Press Club in Canberra, Australia, 23 February 2000.

> *"Not a single one of you saw the white sheet! You all saw only the black dot. Don't go through life forgetting the broader picture ...."*
> *That is something I have carried with me all my life.*

Here the white sheet is the structure containing the featured black dot.

Features therefore represent, in some sense, the "essence" of something, of some structure. Thus in recognising fellow human beings we will dwell on the eyes and other features of a face, even to the extent of neglecting lesser features. In considering homomorphisms in mathematics certain items have to be preserved, for example, the unit element of a group whereas other elements may map to the same element as some other element: the distinctness of the elements may not be preserved but we shall still have a group. In heraldry we can give a precise description of the arms because the codified description, e.g. *or, a cross gules*, yields a uniquely identifiable design: the description captures all the relevant features of the arms.

Clearly it is unlikely that a precise definition of the natural language word "feature" can be found. Certainly we do not make such a claim. Indeed, our use of the word "feature" will take into account the varying subjective interpretations that different individuals may make of "feature".

Dr Vicki Bruce, who has long worked on features (see, *inter alia*, [4]), responded to the author's question

> *What is a 'feature'?*
>
> *Here is my answer - A 'feature' is any aspect of a face (or other object) that plays some representational role in perception/cognition. Clearly the "features" of eyes, nose and mouth are features in this sense, since we have verbal labels for them, and they therefore play a role in our discourse. This does not necessarily mean that these are features for the visual system, though they do possess some properties which would lead them to be segregated as distinct entities (e.g. curvature at the bridge and base of the nose; contrast differences at the edge of the lips). Other kinds of features that may be important for the visual system include more holistic ones - we have experimented with eigenface decompositions, for example* [see [9]] *....*[4]

Feature Set Theory originated with the 1974 paper of Smith *et al.* [18]. Similar or related ideas are also to be found in Christopher Alexander's fascinating books on architecture and building. (See for example [2] and [3].) The interest for psychologists came from their interest in memory and how we remember things because of their meaning (semantics). This has now been accepted as one way of tackling the problem, see text books such as [8]. However the psychologists distinguish between *defining* features and *characteristic* features.

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[4] Private communication, 14 September 1999.

> *Defining features are central to the meaning of a concept; to be called a particular type means that the object has the defining features.*
>
> Smith *et al.* [8], p. 181.
>
> *The second type of feature is characteristic of the object, but not necessary to its definition. For example, we characteristically associate pipe smoking with college professors, but we also know that pipe smoking is not a defining feature of college professors.* *Ibid.* p. 182.

However the distinction between defining and characteristic features has not been satisfactorily explained and so feature set theory has been not so successful.

The present paper presents an introduction to a new language of structures and features which we believe makes for better understanding and simpler expression in a number of complicated situations. Moreover we see how mathematicians do not have the same difficulties as psychologists (see e.g. [8], p. 184) with the notions of defining and characteristic features.

2 Examples

Examples may be found in many areas. Here are a few basic ones. The list is not meant to be exhaustive and we shall even expand on some of these examples later.

1. *Faces*

   The structure of a human face is fairly easily delineated. The features are any or all of the following list (which is taken from Bruce & Young, [4], Subject Index, p. 276): ears; eyebrows; eyes; forehead; hair; jaws; lips; nose; mouth; skin; teeth. Faces are clearly recognizable from a few of these features, for example: :-)

2. *Architecture*

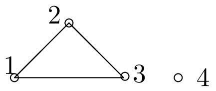
   There are two distinct ways of treating this area.

   (a) *Buildings*

   The structure is obvious. The features may be specific decorative items with or without specified locations. On the other hand they may be the sizes of the rooms. For further ideas in this area see Alexander [2].

   (b) *Design development*

   The structure is the design. The features are the contents of the design decisions about what is added or subtracted in the process of making the final design.



	1	2	3	4
1	1	1	1	0
2	1	1	1	0
3	1	1	1	0
4	0	0	0	1

Figure 1: A simple sample graph.

3. *Groups*

   A group is an algebraic structure that may be presented[5] as $\langle G, \circ, ^{-1}, 1\rangle$ where G is a set and 1 is an element of G. The set G must be closed under the binary operation $\circ$ which has an inverse $^{-1}$. A feature of [this kind of presentation of a group] is the unit, 1.

   On the other hand the group may be given by its multiplication table, e.g.

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

   and here we have to work out that the unit is a. However we may also know that we have a group because no character is repeated in any single row or column (which guarantees that we have inverses). Thus this gives a different presentation with different features. Later (in Section 7) we consider possible definitions of equivalence for classes of structures with features.

4. *Graphs*

   Like groups graphs can be represented in several different ways. The graph on the left in Fig. 1 can also be represented by the adjacency matrix on the right. In the picture the features are the lines and the nodes and the relations between them, and in the matrix the features are the zeros and ones and their locations.

   (Since the matrix is symmetric we can regard this as an undirected or a bi-directed graph.)

5. *Logical (and mathematical) proofs*

   A proof is built, starting from (zero or more) axioms and/or hypotheses using rules of inference. The features include the axioms and/or hypotheses and the

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[5] Groups may be presented in other ways. These always seem to include the binary operation but may exclude either the unit or the inverse operation (or both).

[applications of the] rules of inference.[6]

6. *Greek ship*

   There is a famous philosophical puzzle about a Greek ship or indeed any ship. Over the course of time every plank, every nail and screw, every rope is replaced. Do we still have the same ship? Using our terminology of structures and features we can say that the *physical* structure has changed but the features remain the same over time. (We can also say that the abstract structure has remained the same. So now we are talking here about different, but related, structures.)

7. *The instrument panel in aircraft*

   A pilot in an aircraft, looking at his instruments sees a structure with features. It is important for our safety that (s)he do so. Indeed the design of the instrument panel has a big influence on what features (s)he sees. The most important features are those meters and gauges which can indicate immanent disaster. this motivates the study of situational awareness:

   > *''Situation awareness" refers to a mental state, namely the internal model or mental picture that each person continuously has of the present situation. This model includes some of the circumstances that have led to the current situation and some of the expected developments from it. The model also includes several broader aspects of the situation, such as the objectives of a flight, mission or voyage, and the rules that apply to its conduct. Situation awareness, being a mental state, is personal to each individual, who is also part of it.*
   >
   > Hopkin [11], p. 8.

   So here, being aware of the (important) features can be, quite literally, a matter of life and death. having a clear picture of the important features in the structure (which is the whole display) is the key to safety. Designing the display well makes for better aircraft safety.

3 Structures and Features

Ellis and Hunt in their textbook [8], p. 37, say

> *All patterns consist of a configuration of elements, and theoretically any pattern caqn be broken down into thee basic elements. The basic elements or parts of a pattern are known as* features *of the pattern. For example, the letter A consists of the three features /, \ , and —.*

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[6] This list of features may have to be expanded to include, for example, specifying the variables which become bound and/or the replacements made.

Note, however that this definition does not specify the relationships between the components so we cannot necessarily reconstruct the structure. For example we may get \ /– instead of A.

A feature may be defined as a function of data whose value is 'interesting'. Alternatively features may be regarded as equivalence classes of structures.[7]

The first definition obviously depends on the context for a tighter specification of 'interesting'. All of us tend to use the eyes as the most distinctive feature of the face. Faces with the eyes covered up are much harder to recognise. In the case of a young child Johnson *et al.* (cited in [4], p. 251) showed that it appears that very young children can recognise something as a face if it has eye and mouth/nose markings.[8] It therefore seems appropriate to classify eyes and mouth/nose as features.

Thus in this case we can say that the "interesting" values are those of colour, luminosity, liveliness, etc.

In a more abstract way the second definition was used long ago (in 1927) by Tarski for classification. We consider spheres. There are two obvious features: 1. the centre and 2. the radius (or diameter). In his second paper in the collection [19], *Foundations of the Geometry of Solids*, Tarski defined a point as the class of all spheres centred at that point ([19], p. 27, Definition 7). In this way the centre (point) was interpreted as a *class* of spheres and the notion of point was an abstraction from the (basic) notion of sphere. Thus the centre of a sphere can be regarded as a feature. Two spheres can be called "equivalent" if they have the same centre. Likewise the diameter can be regarded as a feature. In this case two spheres are equivalent if they have the same diameter. We also know that they are identical (as abstract spheres) if they have both the features in common – same centre and same diameter. Notice that the list of "essential features" is obviously not unique: radius and centre will do just as well as diameter and centre. What is very important here is that there is some notion of invariance in a feature.

Now features in a human face change over time in terms of their dimensions, colour, etc. Nevertheless the eyes remain features.[9] Thus features are not entirely invariant. Eyes have size and colour, the latter of which many would regard as a feature. Thus features can themselves be structures that have features (*cf.* below, section 4). In this case the feature *qua* structure may change while still remaining a feature.

When we say "structure" or "feature" then we have to select. As humans we cannot cope with all the data there may be so we normally have a small list of features in which we are interested (see e.g. example 1 above).

Definition 3.1 (Structures with features) *A* structure *is an object, A, in a cat-*

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[7] These ways of giving a definition of "feature" are due to Chris Wallace, personal communication, 30 August 1999.

[8] Two dark marks were used for the two eyes and a single dark mark for the mouth/nose, hence we use this awkward phrase 'mouth/nose'.

[9] Except perhaps in the case of an accident to or surgery on the eyes.

egory $\mathcal{S}$(see [13]). A feature, F, *for such a category $\mathcal{S}$ is a functor from $\mathcal{S}$ into a category of values, $\mathcal{V}$, which has a distinguished terminal object* 0.[10] *A feature is said to be* common *to all objects in $\mathcal{S}$ if* $\mathrm{F}(A) \neq 0$ *for all A in $\mathcal{S}$.*

Regarding this definition we should point out that the objects in the category of values will normally be quite complicated.

In general we shall be interested in classes of structures which are illustrated by some of our examples.

1 *Faces*

In this example the category of structures is the category of representations of human faces. To each feature H in the list: H_{ears} for ears; $\mathrm{H}_{eyebrows}$ for eyebrows; H_{eyes} for eyes; and so on for forehead; hair; jaws; lips; nose; mouth; skin and teeth we assign a value in a category $\mathcal{V}_{ears}$ etc. Here each $\mathcal{V}$ has a default value 0 signifying that the feature is absent. The objects in the range of H_{eyes} will perhaps have a simple signature consisting of strings if we are only interested in the name of the colour of the eyes.

We could put all the values for all these facial features into one all-embracing category $\mathcal{V}_{all}$ or $\mathcal{V}$, say, if we so desired, but one of the most important things about features is that their choice is left to the user. Of course they must be specified accurately.

2 *Architecture*

2a For *Buildings* the category of structures can be taken to be the physical buildings.[11] As in the first part of example 3 the features can be regarded as individual functors for each specific decorative item or room [its dimensions], etc., with or without specified locations.

3 *Groups*

The category of structures is a category $\mathcal{G}$ of groups. If we have represented each group as $\langle G, \circ, ^{-1}, 1\rangle$ then the feature may be regarded as the functor U that takes each group G to its unit, 1_G, and the value category as the set of all such units with trivial maps between them or else as the one object category $\mathbf{1}$.

However it is more normal to regard the operation, inverse *and* identity as all being important features. In this case we would want the *representations* (*implementations*) of these to be the features.

Here we we may wish to distinguish between different representations of the same group.

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[10] This object is intended to be the default value (zero) indicating that the object does not possess the feature.

[11] In general there may be no maps between them except in the case of e.g. houses built to a common plan on a housing estate.

5 *Logical (and mathematical) proofs*

When the proof is built, starting from [zero or more] axioms and/or hypotheses using rules of inference the most commonly noted important aspects are what formulae one starts from and what rules of inference are used where.[12,13] Another way to represent such a proof rather than writing out all the formulae in every line is to write out the initial formulae (axioms and/or hypotheses) and thereafter to annotate the rules of inference employed by pointing to where they came from and what syntactic changes have been made.

Thus the category, $\mathcal{V}$, of values for features can here be taken as a category of a particular variety of labelled trees.

6 *Greek ship*

If we regard the structure of the ship as being parametrized by time, so that all the objects in the structure category are of the form $Ship(t)$, then we may take the value of a typical feature as being the location or specification of a particular plank or nail. Such a feature always has a non-zero value which does *not* change even though the physical part of $Ship(t)$ does.

4 Layering

Maturana is quick to point out that once we have identified a feature (or unity) then that itself may have features within it. For example, continuing the example of the speck on the carpet. We approach the speck and find it is in fact a folded piece of paper which, when unfolded, reveals an inscription. The space is now the piece of paper and the feature the inscription. If the inscription is letters or writing, then we may be able to distinguish other features. For example we may find there are certain words or, at a higher level, that it is written in French or at an even higher level, that it is written in Roman (as opposed to Cyrillic or Chinese or Arabic) characters.

Which of these levels of feature impinges on our attention will, of course, depend on the individual. To someone who cannot read it may just appear as writing whereas someone who knows French in addition to some other language may be surprised to find it written in French. Note that in this case the person might not even be able to answer the question "What does it say?" because he, or she, has not yet paid attention to that level of the features of the example.

There is a further problem here. If we look at an image being reflected back and forth between two parallel mirrors then there is no end to the features, the images. In order to accommodate this kind of infinite regression we mention that we use a

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[12] We are using "aspect" rather than "feature" here because we shall reserve the latter for the abstraction from the whole proof.

[13] This list of aspects may be expanded to include, for example, specifying the [occurrences of the] variables which become bound and/or the replacements made.

category-theoretic version of Aczel's non-well-founded sets, [1]. We shall discuss this in planned future work.

5 Abstraction and concretization

When we find a collection[14] interesting and we wish to share that interest with others, then our normal practice is to talk about it.[15] Since we cannot give exhaustive descriptions in most cases, we give an abbreviated description. Such a description will concentrate on certain aspects. For example we may single out the eyes in describing a face, we may distinguish mountains in a landscape, we may concentrate on the rules of inference employed in a proof. The ideal situation seems to be when we can give a nice succinct description according to a general prescription. Consider for example the descriptions given by police of suspects. Height, colour of hair, eyes, etc. are filled in on a standard form. Unfortunately, in this case the description may not pin down the suspect to one individual.[16]

In the case of groups we may start, as people did in the nineteenth century (see [5]), by noticing properties in common between various abstract collections, for example, the rotations of a disc or the reflections that preserve certain patterns. These properties were singled out and in this case it was possible to give a simple axiomatic description (in the language of first order logic). Once these had been codified then a decision was made to call *anything* that satisfied these axioms a "group". There is then a complete circle[17] going from the collection of groups (in the world[18]) to the formal description (by the axioms) and back to all the exemplars (or models) of those axioms.

This case is especially nice because we know that we have exactly captured the notion of 'group'. In the case of faces life is nothing like so simple. Here it is very hard to decide what is sufficient to characterize a face. If we take an extreme example we might say that a face is a structure with a mouth. In this case an earthworm would satisfy the description but few of us would want to say that an earthworm had a face.

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[14] We use the word "collection" instead of either of the over-worked words "set" and "class" to remind the reader that we shall be interested in what he or she finds interesting.

[15] There appear to be some cases where the use of (natural) language is unnecessary, for example, in the context of painting a picture, or the interior of a room, we may just point to some colour or colours as being the ones we wish to choose or choose between. Also it seems plausible that members of the human race should have acted in a similar way before human language was developed, for example in selecting food.

[16] It is interesting that identikit photographs have not been very successful although one would expect that the additional information they contain (over and above the verbal information in the description) would be helpful.

[17] But only because of the formal decision that was made.

[18] Here we include the abstract world of mathematics as well as the physical world of discs and Rubik's cube, of course.

In the case of structures whose descriptions can be formalized in formal logic we talk of, first the axiomatization of the structures and then the models of those axioms. If all our models are (examples of the) original structures, then we say that we have a *complete* axiomatization. One of the most notorious examples of failing to have a complete axiomatization occurs with the natural numbers $0, 1, 2, \ldots$. The Peano axioms,[19] when interpreted as being in *second* order logic, do have a model which is unique up to isomorphism However, when we write down those axioms in *first* order logic then we get unintended, or non-standard, models.[20] That is to say, there are models which are essentially different from the original structure.

In all of these examples we see, first of all a process of abstraction from existing structures, and then the giving of a description in some (possibly formal) language, followed by an examination of the models of that formal description. We can claim to have exactly captured the original structures if *all* of the models are amongst the original structures. In this process of giving the formal description we seem to use the features as the key concepts.

This process may be iterated. Consider the simple (totally abstract) function that doubles a (natural) number. Mathematicians may define such a function as a set of ordered pairs. (We shall leave aside the exact way that ordered pairs are treated: similar considerations apply to that notion too.) So we use some kind of (semi-)formal language to describe our function. Next we can move from this to an algorithm which, given a number, returns its double. Finally we can write a computer program (in some formal computer language) to take a (particular style of) representation of the number and return the required double value. Then we can look at models of the program and we shall be pleased and satisfied if the program actually does what it was supposed to do! That is to say, we want and expect the abstract value of the answer to be double the (abstract value of the) number put in to the program.

At each of the stages in this cycle what we really need to do is to preserve the feature that what we have will reflect the doubling. Thus in all of the stages there should (at least) be *one* feature that is (abstractly speaking) common to all of the stages. If it is not preserved, then our main aim, of doing the doubling, will be lost.

6 Operations on features

The basic operations we have on features are: adding, deleting, retaining and modifying. In addition we have building and simplification of sequences of applications of features.

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[19] These are actually due to Dedekind as Peano acknowledges! (See my *Emergence*

[20] This was first observed by Skolem, [17] and later the idea was heavily used by Abraham Robinson in his non-standard analysis, [15].

6.1 Adding a feature

Since features are functors, adding a feature either means adding a new functor to a new category of values or extending the given feature functor. The second is described by the following commutative diagram:

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "="] \arrow[d, "\mathrm{F}_1"] & \mathcal{S} \arrow[d, "\mathrm{F}_2"] & \text{where } \mathrm{F}_1 \subseteq \mathrm{F}_2. \\
\mathcal{V}_1 \arrow[r, "\subseteq"] & \mathcal{V}_2 &
\end{tikzcd}$$

6.2 Deleting a feature

This is simply the inverse of the previous operation of addition.

6.3 Retaining a feature

This is a trivial operation which is useful when we have sequences of operations involving features.

6.4 Modifying a feature

Here we consider the diagram

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "="] \arrow[d, "\mathrm{F}_1"] & \mathcal{S} \arrow[d, "\mathrm{F}_2"] \\
\mathcal{V}_1 \arrow[r, "\mathrm{C}"] & \mathcal{V}_2
\end{tikzcd}$$

where we change the value category by a functor C.
Sequences of operations of these kinds take us from one structure to another.
Think of building up a painting – or erecting a building!

6.5 Building and simplifying features

When we take proofs (example 5) then there are uniform operations on objects in a category, $\mathcal{S}$, which are (partial) functors. These automatically induce partial functors in the value category $\mathcal{V}$.

Suppose we have a proof P and then we apply a logical rule, say, or-introduction on the left ($\vee_1 - I$) using the formula α. If P has feature $\mathrm{F}(P)$, then the new proof will have a feature which is the tree $\mathrm{F}(P)$ with an extra node at the bottom labelled with the pair $(\vee_1 - I, \alpha)$ or, if α is an assumption, there may simply be a pointer p_α to the assumption α.

This particular functor F is in fact total. In the case of, say, and-elimination ($\& - E$) the rule can only be applied if the formula at the end of the proof is of the form $(\alpha \& \beta)$. In this case the modified feature can be written as the original tree with a label $\&_1$ or $\&_2$, depending on whether the left or right-hand conjunct is removed.

We can therefore start off with some proof and successively extend it or, in our present terminology, add features to it or build up features. In this case and also in the case of developing an architectural design we shall sometimes be able to achieve simplifications.

Thus if we introduce "and" in a logical proof and then immediately remove it again in a proof figure such as:

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A\&B}\;(\&\text{-Intro})}{A}\;(\&\text{-Elim})$$

then we can omit this section of the proof leaving just the original first part:

$$\begin{array}{c}\vdots \\ A\end{array}$$

In this way we have simplified the successive application of the features (which we also denote by) (&-Intro) and then (&-Elim) to the (trivial) process of simply retaining the earlier features, i.e doing nothing to the proof of A but discarding the proof of B.

7 Equivalence

If we take the two styles of presentation of groups mentioned in section 3 then we can construct one representation from the other. Thus we may define classes $\mathcal{S}_1, \mathcal{S}_2$ of structures to be *equivalent*$_1$ if there are functors $F_1 : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ and $F_2 : \mathcal{S}_2 \rightarrow \mathcal{S}_1$ which are inverses.

In the case of groups (example 3) if $\mathcal{S}_1$ is given in terms of a group multiplication, etc. then it is easy to construct (to write out) the multiplication table. This yields F_1. Conversely, if we are given the multiplication table then we can define the group multiplication (it is essentially the same) and then find the unit of the group and the inverse of each element by looking at the table. This procedure yields F_2.

A similar process shows the two definitions of graph are *equivalent*$_1$.

Given the picture of the graph we can construct the incidence matrix and, conversely, given the incidence matrix we can draw [a picture of] the graph.

Another definition of equivalence is the following:

Given classes $\mathcal{S}_1, \mathcal{S}_2$ of structures, then $\mathcal{S}_1$ is *equivalent*$_2$ to $\mathcal{S}_2$ if we can uniformly add features to each structure in $\mathcal{S}_1$ and in $\mathcal{S}_2$ to get a single third class $\mathcal{S}_3$.

The classes of each of a) groups and b) graphs are again equivalent$_2$ because we can compute the features to add from the information we have already been given.

Note, however, that in this case we may not be changing the underlying structure as we might do if we are using a functor as in *equivalence*$_1$.

It follows at once that *equivalence*$_2$ is a stronger form of equivalence than *equivalence*$_1$.

8 Defining versus characteristic features

In the mathematical context we can now distinguish between defining and characteristic features. A collection of features $\mathcal{F}$ for a class of structures is a collection of defining features if there is only one such class of structures. A feature is a characteristic feature is any feature which is a logical consequence of (some of) the defining features.

It follows at once that there may be many collections of defining features but that they must all be equivalent.

This therefore resolves the conflict between *defining* and *characteristic* features – but only in the mathematical context.

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