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Certificates for properties of stability polynomials of graphs

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Declaration

I declare that this thesis is my own work and has not been submitted in any form for another degree or diploma at any university or other institute of tertiary education. Information derived from the work of others has been acknowledged.

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Abstract

A *stable* (or *independent*) set is a set of vertices where no two of the vertices in the set are adjacent. The stability polynomial $A(G;p)$ of a graph G is the probability that a set of randomly chosen vertices is stable where the probability of each vertex being chosen is p. Factorisation is an important algebraic property of polynomials, but there has been little research on factorisation of the stability polynomial. This research considers the s-factorisations of stability polynomials. The stability polynomial $A(G;p)$ is said to have an *s-factorisation* with *s-factors* H_1 and H_2 if $A(G;p) = A(H_1;p)A(H_2;p)$.

Stable sets and the stability polynomial are related to some other graph polynomials. There is an analogy between the chromatic polynomial and the stability polynomial. The method of studying chromatic factorisation can also be used in s-factorisation. By implementing a similar research method on stability polynomials of connected graphs with order at most 9, we found 152 different s-factorisations and used certificates, sequences of steps applying the properties of the stability polynomial, to explain these s-factorisations. Upper bounds for the length of the certificates for stability equivalence and s-factorisations are given. Moreover, we have found short certificates to explain 2 infinite families of stability equivalences and 2 infinite families of s-factorisations. These short certificates for s-factorisations were generalised to 2 certificate schemas.

1 Introduction

A *graph* $G = (V, E)$ is a set of vertices and a set of edges that are connections between pairs of vertices [21]. Graphs can be used to model objects by abstracting the elements as vertices and the connections between these elements as edges. Graphs are used to model networks including social networks [19], transport systems [9] and electric grids. People in a social network can be represented by vertices in a graph, while their relationships between each other can be represented by edges in the graph. The vertices of a graph can be also used to denote the stations of a transport system while the edges are used to denote the paths between these stations. Graphs offers us a useful tool for solving problems in many different areas. Studying the properties of graphs can produce valuable knowledge for many practical applications.

Graph polynomials are polynomials that give special information about graphs. There are many kinds of graph polynomials. Each polynomial gives information about a different graph property. The *chromatic polynomial* $P(G;\lambda)$ of a graph G gives the number of different ways colours can be assigned to the vertices of a graph such that no pair of adjacent vertices get the same colour [1]. This is a useful property for resource allocation [5]. The *stability polynomial* $A(G;p)$ of a graph G gives the probability that a set of randomly chosen vertices is stable [11]. The stability polynomial is useful for object selection. Furthermore, there are other polynomials including the flow polynomial, the matching polynomial, the vertex cover polynomial [8], the clique polynomial and the independence polynomial.

In this thesis we will consider the stability polynomial and study its factorisation properties. First we will introduce the definitions of some basic concepts. Two vertices are said to be *adjacent* or *neighbours* if there is an edge connecting them. The set of neighbours of a vertex v in graph G is donated by $N_G(v)$ or $N(v)$. The order of a graph G is the number of vertices of this graph [7]. A *stable set* (or *independent set*) is a set of vertices where no two of the vertices in the set are adjacent [7]. The independence number $\alpha(G)$ is size of a maximum stable set in graph G [20].

Let G be a graph and $p \in [0, 1]$ be the probability of each vertex being chosen. The vertex choices are made independently. The stability polynomial $A(G;p)$ of a graph G

is the probability that a set of randomly chosen vertices is stable [11]. Let H_1 and H_2 be graphs. The stability polynomial of graph G is said to have an *s-factorisation* with *s-factors* H_1 and H_2 if

$$A(G;p) = A(H_1;p)A(H_2;p). \tag{1}$$

The graph G is said to be *s-factorised* or G is *s-factorisable*. In this research we will focus on the s-factorisations of connected graphs because it is known that any disconnected graph always has an s-factorisation [10].

Some s-factorisations of graph G can be used to reduce the computational complexity of calculating $A(G;p)$ by calculating the product of the stability polynomial of two smaller graphs H_1 and H_2. We illustrate this with the following example. We can use a graph G to represent the social network of Facebook. Vertices in G represent the users on Facebook and edges between vertices show the 'friendship' connections between the users. If Facebook is planning to organise an 'All strangers party' to help users meet new friends, it will try to choose a group of users X, so that any one in X do not know each other. Thus, X is a stable set. If every user is chosen independently with probability p, then the probability that the randomly chosen users are all strangers to each other is $A(G;p)$. If the size of graph G is too big, the computational complexity of calculating $A(G;p)$ will be too high. We can search for some s-factorisatons $A(G;p) = A(H_1;p)A(H_2;p)$ to decompose the computation of $A(G;p)$.

As a polynomial is an algebraic object, the study of its algebraic properties, including factorisation, is important. But there has been little research done on the factorisation of graph polynomials. One objective of this research is to search for s-factorisations of connected graphs and find short theoretical certificates to explain these cases. We searched all the stability polynomials of connected graphs of order at most 9 and found 17,461,965 s-factorisations corresponding to 273,192 graphs. We use *certificates* to explain these s-factorisations. A *certificate* is a sequence of expressions $S_1, S_2, ..., S_k$. The first expression S_1 is the graph G. Each expression S_i is obtained from S_{i-1} by applying a property of the stability polynomial or an algebraic property.If the last expression is the graph H then we have a certificate of equivalence. If the last expression S_k is the product of the graphs H_1 and H_2 then we have a certificate of factorisation. A *certificate step* is a rule that uses some properties of the graph polynomial or algebraic property to transform one expression to another [17]. A single certificate step is usually not enough to explain the s-factorisation. In this research, short certificates of s-factorisations for all graphs of order at most 6 are given. A short certificate of one s-factorisation for a graph of order 7 was also found.

Another objective of this research is to find upper bounds of the certificate length of s-factorisations and stability equivalence. We found these upper bounds by studying the properties of stability polynomials. The upper bounds of the length of certificates of s-factorisations and stability equivalence are both exponential.

The final objective of this project is to find certificate schemas of the certificates for s-factorisations. If there is a group of graphs that have s-factorisations which can be explained by certificates that use the same sequence of steps, these certificates will be generalised to a certificate schema. We found 2 certificate schemas by analysing the patterns of short certificates of s-factorisations.

This thesis contains six sections. Section 1 gives an overview of this research and introduces some definitions and notation. Section 2 shows the relationship between the stability polynomial and some other graph polynomials and introduces the previous research on graph polynomial factorisation. Section 3 gives the research methodology of this project. Section 4 gives the computation results. Section 5 presents the theoretical research results including the upper bounds of certificates of stability equivalences, certifi-

cates and certificate schemas of s-factorisations. Section 6 summarised our work and and points some directions of future research.

1.1 Contributions

The contributions of this research are listed as following:

1. Implementing a program for calculating the stability polynomial of any given graph.
   (Section 3.1)

2. Implementing a program for searching s-factorisations from a list of stability polynomials.
   (Section 3.1)

3. Calculating stability polynomials of all graphs of order at most 9.
   (Section 4)

4. Identifying all connected graphs of order ≤ 9 that have s-factorisations.
   (Section 4)

5. Giving certificate steps for certificates of s-factorisations.
   (Section 5.1)

6. Giving an upper-bound of the length of certificates of stability equivalence and s-factorisations.
   (Section 5.2)

7. Finding short certificates for stability equivalence and identifying two infinite families of stability equivalent graphs.
   (Section 5.3)

8. Finding short certificates for s-factorisations and identifying two infinite families of graphs that have s-factorisation.
   (Section 5.4)

9. Identifying that for any graph G, there exists a connected graph G' such that G is an s-factor of G'.
   (Section 5.4)

10. Finding two certificate schemas of s-factorisations.
    (Section 5.5)

1.2 Definitions and notation

A *matroid* is a pair (E, ρ) where E is a non-empty finite set and ρ is a rank function satisfying [21]:

1. $0 \leqslant \rho(A) \leqslant |A|$

2. If $A \subseteq B \subseteq E$, then $\rho(A) \leqslant \rho(B)$

3. $\rho(A) + \rho(B) - \rho(AB) - \rho(AB) \geqslant 0$ for all $A, B \subseteq E$

A *null graph* is graph with no edges.

Some notation used in this thesis are listed in Table 1 and graphs defined in this research are listed in Table 2.

Table 1: Definitions and Notation

Notation	Explanation
$\mathbb{R}^-$	The set of negative real numbers
$G_{n,\emptyset}$	The graph of order $n \geq 0$ with no edges
C_n	The cycle of order n
K_n	The complete graph of order n
P_n	Path graph of order n with $n-1$ edges
$G \simeq G'$	Graphs G and G' are isomorphic
$G - v$	A graph obtained from G by deleting vertex v
$G \cup G'$	Disjoint union of graphs G and G'
$G + e$	A graph obtained from G by adding the edge e
$G + G'$	A graph obtained from G and G' by choosing one vertex from each graph and adding an edge between this pair of vertices.

Table 2: Graphs defined in this thesis

Notation	Explanation	Location in thesis
K_n^*	The graph of order $2n$ with $E(K_n^*) = \{v_i v_{n+i} : 0 \leq i \leq n-1\}$ $\cup\{v_j v_k : 0 \leq j, k \leq n-1, j \neq k\}$	Cor 4
$K_{n\sim n}$	The graph of order $2n$ with $E(K_{n\sim n}) = \{v_i v_j : 0 \leq i, j \leq n-1, i \neq j\}$ $\cup\{v_k v_l : n \leq k, l \leq 2n-1, k \neq l\} \cup \{v_{n-1} v_n\}$	Cor 7
Q_n	The graph of order n with $E(Q_n) = \{v_0 v_2\} \cup \{v_i v_{i+1} : 0 \leq i \leq n-2\}$	Thm 23, Cor 3 Cor 8
Y_n	The graph of order n with $E(Y_n) = \{v_0 v_2\} \cup \{v_i v_{i+1} : 0 \leq i \leq n-3\} \cup \{v_{n-1} v_1\}$	Cor 3
$\Upsilon_{m,n}$	The graph of order n with $E(T_{m,n}) = \{v_i v_{i+1} : 0 \leq i \leq n-2\}$ $\cup\{v_{m-1} v_j : 0 \leq j \leq m-3\}$ where $2 \leq m \leq n$	Thm 25

2 Background

This section will explain the relationships between the stability polynomial and some other graph polynomials, study the properties of roots of stability polynomials, discuss current research and identify gaps in the current state of knowledge. It will consist of three topics:

1. Relationships between the stability polynomial and other graph polynomials: The stability polynomial is strongly correlated with the independence polynomial. We will consider six graph polynomials and discuss the analogy between the stability polynomial, independence polynomial and chromatic polynomial. The similarities and differences between the properties of these graph polynomials will be analysed. The stability polynomial and the chromatic polynomial of graphs can be considered as specialisations of the chromatic polynomial of a matroid.

2. Roots of stability polynomials: The roots of a polynomial are one of its important algebraic properties. A common root of stability polynomials is 1. In this section, we

will discuss the strong relationship between the stability polynomial and the independence polynomial and then survey current research on the roots of independence polynomial.

3. Research of polynomial factorisation: We will study current research on chromatic factorisations and discuss if the method used in this research can applied on s-factorisations. Then the properties of stability polynomials will be analysed and be converted to certificate steps. Finally we will discuss the feasibility of stability polynomial s-factorisations research, study the theoretical properties of the stability polynomial and introduce an efficient algorithm for calculating the stability polynomial.

2.1 Relationships between the stability polynomial and other graph polynomials

The stability polynomial is closely related to the independence polynomial and other graph polynomials. In this section, we survey six graph polynomials and analyse the relationship between the stability polynomial and these polynomials. We also give a more general definition of chromatic polynomials for matroids. Stability polynomials and chromatic polynomials of graphs can be considered as specialisations of hypermatroid chromatic polynomials.

2.1.1 Properties of the stability polynomial

The stability polynomial is closely related to the independence polynomial and some other polynomials. In this section we give five interesting properties of the stability polynomial which were identified by Farr [10]. We then compare them with similar properties of some related graph polynomials.

Theorem 1. *For any vertex $v \in V(G)$ which is not incident to a loop, there is*

$$A(G;p) = (1-p)A(G-v;p) + p(1-p)^d A(G-v-N(v);p) \tag{2}$$

where d is the degree of vertex v and $N(v)$ is the neighbour set of v.

Theorem 2. *For any edge $uv \in E(G)$,*

$$A(G;p) = (1-p)A(G-u;p) + (1-p)A(G-v;p) - (1-p)^2 A(G-u-v;p).$$

Theorem 3. *Let H_1 and H_2 be two disjoint graphs. If $G = H_1 \cup H_2$, then*

$$A(G;p) = A(H_1;p)A(H_2;p). \tag{3}$$

Theorem 4. *Let G be a graph with loops, L is the set of vertices incident to loops. Then*

$$A(G;p) = (1-p)^{|L|} A(G-L;p).$$

Theorem 5. *For any edge $uv \in E(G)$*

$$A(G;p) = A(G \backslash uv;p) - p^2(1-p)^{|(N(u)\backslash\{v\})\cup(N(v)\backslash\{u\})|} A(G-u-v-N(u)-N(v);p).$$

Theorem 3 is a basic case of stability polynomial factorisation. It gives a case of factorisation when the graphs H_1 and H_2 are disjoint. In this project we will find s-factorisations of graphs that do not have disjoint components but still satisfy Equation (3). Studying s-factorisations may result in new rules for stability polynomials. A new rule is a proposition which can be explained by a certificate schema.

2.1.2 The independence polynomial

The *independence polynomial* $I(G;x)$ is a generating polynomial for the number of stable sets s_i of cardinality i [13]. It can be expressed as

$$I(G;x) = \sum_{\text{X stable}} x^{|X|} = \sum_{i=0}^{\alpha(G)} s_i x^i \tag{4}$$

where $\alpha(G)$ is the size of a maximum stable set in G. If the probability of each vertex in G being chosen is p, and X is a randomly chosen vertex subset of graph G then

$$A(G;p) = Pr(\text{X is stable}) \quad [14]. \tag{5}$$

When X is a stable set of G and the probability of X being chosen is $p^{|X|}(1-p)^{n-|X|}$, the stability polynomial can also be defined as

$$A(G;p) = \sum_{\text{X stable}} p^{|X|}(1-p)^{n-|X|} = \sum_{i=0}^{n} s_i p^i (1-p)^{n-i}, \tag{6}$$

where n is the order of G.

Comparing Equation (4) with Equation (6), it is clear that $A(G;p)$ and $I(G;x)$ are related to each other by the following transformation [10]

$$A(G;p) = (1-p)^n I\left(G; \frac{p}{1-p}\right) \tag{7}$$

and

$$I(G;x) = (1+x)^{-n} A\left(G; \frac{x}{1+x}\right) \tag{8}$$

Equations (7) and (8) show that the properties of the independence polynomial can be transferred to properties of the stability polynomial. Studies of the independence polynomial can produce information for research on the stability polynomial.

Properties of independence polynomials

The research on the properties of independence polynomials was started by Gutman and Harary [13]. A number of papers have investigated these properties and presented useful results [2][6][16][20]. Following are some properties of independence polynomials. We can compare the following theorems with the theorems of the stability polynomial presented in Section 2.1.1.

Theorem 6. *[13] For every vertex v of graph G,*

$$I(G;x) = I(G-v;x) + xI(G-v-N(v);x)$$

where $N(v)$ is the neighbourhood of v.

Comparing Theorem 6 with Theorem 1 shows that Theorem 1 can be converted to Theorem 6 by replacing $1-p$ with 1, and p with x.

Theorem 7. *[16] For any edge $uv \in E(G)$, there is*

$$I(G;x) = I(G-uv;x) - x^2 I(G - N(u) \cup N(v);x)$$

where $N(u)$ and $N(v)$ are the neighbourhoods of vertex u and vertex v respectively.

Theorem 2 can be converted to Theorem 7 by replacing $1-p$ with 1, and p with x.

Theorem 8. *[13] If G_1 and G_2 are two disjoint graphs, then*

$$I(G_1 \cup G_2; x) = I(G_1; x)I(G_2; x).$$

Theorem 8 is precisely the same as Theorem 3, which means the independence polynomial and stability polynomial have the same property on disjoint graphs. Comparing the theorems of the independence polynomial with those of the stability polynomial shows that the independence polynomial and the stability polynomial have similar properties.

2.1.3 The clique polynomial

A *clique* is set of vertices where every pair of vertices in the set are adjacent [7]. The clique polynomial of a graph was defined by Hoede and Li [15]. Let n be the order of graph G and $a_i(G)$ be the number of cliques with i vertices. The polynomial

$$C(G; x) = \sum_{i=0}^{n} a_i(G)x^i$$

is called the *clique polynomial* of graph G.

Hoede and Li [15] studied the relationship between the clique polynomial and the independence polynomial and gave the following results:

Theorem 9. *If $\overline{G}$ is the complement of graph G, then*

$$C(G; x) = I(\overline{G}; x).$$

According to Equation (8), Theorem 9 can be transformed to

$$C(G; x) = (1+x)^{-n} A\left(\overline{G}; \frac{x}{1+x}\right).$$

A vertex subset of a graph G is a stable set if and only if it is a clique in $\overline{G}$. The size of a maximum stable set of G equals the size of a maximum clique of $\overline{G}$. Thus the independence polynomial of G equals the clique polynomial of $\overline{G}$.

Theorem 10. *If G_1 and G_2 are two disjoint graphs, then*

$$C(G_1 \cup G_2; x) = C(G_1; x) + C(G_2; x) - 1$$

and

$$I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$

Theorem 11. *If G_1 and G_2 are two disjoint graphs, then*

$$I(G_1 \cup G_2; x) = I(G_1; x)I(G_2; x)$$

and

$$C(G_1 + G_2; x) = C(G_1; x)C(G_2; x).$$

Theorem 10 and 11 show that clique polynomials and independence polynomials have some similar properties. Since an independence polynomial can be transformed to a stability polynomial by applying Equation (7), it is clear that the cliques polynomial is related to the stability polynomial.

2.1.4 The vertex cover polynomial

A *vertex cover* of a graph $G = (V, E)$ is a set of vertices $V' \subseteq V$ such that for any edge $uv \in E$, there must be $u \in V'$ or $v \in V'$. Dong et al. [8] define the *vertex cover polynomial* for graphs as:

$$\Psi(G; x) = \sum_{k=0}^{n} cv(G, k)x^k$$

where n is the order of G and $cv(G, k)$ is the number of vertex covers with k vertices. They also identified the following three theorems about the vertex cover polynomial which are related to the properties of the stability polynomial.

Theorem 12. *Let G be a graph with loops and L be the set of vertices which are adjacent to loop. Then*

$$\Psi(G; x) = x^{|L|}\Psi(G - L; x).$$

This result has the same structure as Theorem 4 for $A(G; p)$.

Theorem 13. *Let G be a graph without loops. For any vertex $v \in V(G)$ there is*

$$\Psi(G; x) = x\Psi(G - v; x) + x^d\Psi(G - v - N(v); x)$$

where d is the degree of vertex v and $N(v)$ is the neighbour set of v.

This theorem can be obtained from Theorem 1 by replacing p and $(1 - p)$ with x.

Theorem 14. *Let G_1 and G_2 be two disjoint graphs. If $G = G_1 \cup G_2$, then*

$$\Psi(G; x) = \Psi(G_1; x)\Psi(G_2; x).$$

This result has the same structure as Theorem 3.

Comparing Theorems 12-14 with Theorem 1, 3 and 4 of the stability polynomials shows that there is an analogy between the stability polynomial and the vertex cover polynomial.

2.1.5 The positive matching polynomial

The matching polynomial is a generalised polynomial defined as

$$M(G; x) = \sum_{k=0}^{|E(G)|} (-1)^k m_k x^{n-2k},$$

where m_k is the number of matchings with k edges [13]. In this paper, we are more interested in *the positive matching polynomial*, which is defined as

$$M^+(G; x) = \sum_{k=0}^{|E(G)|} m_k x^k$$

Comparing this definition with the definition of the independence polynomial shows that

$$M^+(G; x) = I(L(G); x),$$

where $L(G)$ is the line graph of G [13]. This finding shows that the positive matching polynomial is closely related to independence polynomial. Thus the positive matching polynomial is related to the stability polynomial by Equation (8) as follows

$$M^+(G; x) = I(L(G); x) = (1 + x)^{-n} A\left(L(G); \frac{x}{1 + x}\right).$$

2.1.6 The chromatic polynomial

The chromatic polynomial $P_G(k)$ counts the number of proper colourings of a graph G in at most k colours [1]. A *proper colouring* is a colouring such that any two adjacent vertices were assigned different colours [21]. The chromatic number $\chi(G)$ is the minimum number of colours required to properly colour G. The chromatic polynomial has the following interesting property [12]:

Theorem 15. *The chromatic polynomial of a graph $G = (V, E)$ is*

$$P(G;\lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{n-\rho(X)},$$

where n is the number of vertices of G. $\rho(X) = |V_X| - k(V_X, X)$ is The rank of X and k is number of components of the graph $G_X = (V_X, X)$.

It follows from Theorem 15 that

$$\frac{P(G;\lambda)}{\lambda^n} = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{-\rho(X)}. \tag{9}$$

The stability polynomial can be written

$$A(G;p) = \sum_{X \subseteq E} (-1)^{|X|} p^{\upsilon(X)} \tag{10}$$

where $\upsilon(X)$ is the number of vertices which meet X [14].

When $p = \frac{1}{\lambda}$, the only difference between the right hand sides of Equation (9) and (10) is that $\rho(X)$ is the rank of the matroid of G while $\upsilon(X)$ is the rank function of a hypermatroid. Comparing Equation (9) with (10) shows the close relationship between the stability polynomial and the chromatic polynomial which we will discuss in detail in Section 2.1.7.

2.1.7 The chromatic polynomial of a matroid

Helgason [14] studied the chromatic polynomial and the vertex cover polynomial of a matroid. He gave a more general definition of the vertex cover polynomial of a hypergraph before Dong et al. [8] gave their definition on a graph.
According to Helgson [14], if $H = (V, E)$ is a hypergraph and H has no empty edges. The characteristic polynomial of the covering hypermatroid of H is

$$\chi_H(\lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{|V|-|\cup X|} \tag{3.7}$$

When a graph $G = (V, E)$, *the characteristic polynomial of the chromatic matroid* $M(G)$ is defined as

$$P(M(G);\lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{\rho(E)-\rho(X)} \tag{11}$$

where the matroid $M(G) = (E(G), \rho)$ and ρ is the rank function on E defined as:

$$\rho(X) := (\text{number of vertices adjacent to } X) - (\text{number of connected components of } X)$$

Based on this definition, we can see the relationship between $P(M(G);\lambda)$, the chromatic polynomial of graphic matroid of G, and $P(G;\lambda)$, the chromatic polynomial is

$$\begin{aligned} P(G;\lambda) &= \sum_{X\subseteq E}(-1)^{|X|}\lambda^{n-\rho(X)} \\ &= \lambda^{n-\rho(E)}\sum_{X\subseteq E}(-1)^{|X|}\lambda^{\rho(E)-\rho(X)} \\ &= \lambda^{k(G)}\sum_{X\subseteq E}(-1)^{|X|}\lambda^{\rho(E)-\rho(X)} \\ &= \lambda^{k(G)}P(M(G);\lambda). \end{aligned}$$

The results given in this section show that the vertex cover polynomial and chromatic polynomial are specialisations of the characteristic polynomial of the matroid. The structure of the characteristic polynomial of a matroid is similar to the structure of the stability polynomial.

2.2 Roots of graph polynomials

The nature and location of the roots of a graph polynomial are important algebraic properties. In this section, a common root of stability polynomials was found. By analysing the definitions of the stability and independence polynomial, we find that there is a strong correlation between the roots of stability polynomials and the roots of independence polynomials. This finding shows that studying the properties of roots of independence polynomials can provide useful knowledge for the research of roots of stability polynomials.

2.2.1 Roots of stability polynomials

When G has at least one edge there is no stable set of size n that is, $s_n = 0$ in Equation (4). Thus the stability polynomial $A(G;p)$ has a factor $(1-p)$ and so $p = 1$ is a root of the stability polynomial of any graph that has edges. This is because when $p = 1$ all the vertices are chosen and so the chosen set is not stable if the graph has at least one edge.

2.2.2 The relationship between the roots of stability and independence polynomials

In Section 2.1, we discussed the relationship between the independence polynomial and stability polynomial. Two important equations (7) and (8) showed how the independence polynomial and the stability polynomial are related. These equations show the relationship between the roots of the stability polynomial and the independence polynomial. From

$$A(G;p) = (1-p)^n I\left(G;\frac{p}{1-p}\right)$$

we see that if p_1 is a root of the stability polynomial of graph G, then $p_1 = 1$ or $\frac{p_1}{1-p_1}$ is a root of the independence polynomial of G. From

$$I(G;x) = (1+x)^{-n} A\left(G;\frac{x}{1+x}\right)$$

we see that if x_1 is a root of the independence polynomial of graph G, then $x_1 = 1$ or $\frac{x_1}{1+x_1}$ is a root of the independence polynomial of G.

Based on these findings, we can say that the roots of stability and independence polynomials are closely related. Thus we can study the nature of roots of the stability polynomial by studying the roots of the independence polynomial.

2.2.3 Roots of independence polynomials

Since there is little research on the roots of stability polynomials, and the roots of stability polynomials and the independence polynomials are strongly related, we can study the results of research on roots of independence polynomial to give us information about the roots of the stability polynomial. The nature of the roots of the independence polynomials have been investigated by several research papers. These studies give the following properties:

Theorem 16. *[2] All the real roots of an independence polynomial are negative.*

Theorem 17. *[2] The real roots of an independence polynomial are dense in* $(-\infty, 0]$ *and the complex roots are dense in* $\mathbb{C}$.

Theorem 18. *[3] The modulus of any root of* $I(G;x)$ *is not greater than* $(n/\alpha(G))^{\alpha(G)-1} + O(n^{\alpha(G)-2})$ *where* $\alpha(G)$ *is the independence number of graph* G.

Theorem 19. *[3] Let* G *be the line graph of a tree* T *and* $\alpha(G)$ *be the independence number of* G*, then the modulus of any root of* $I(G;x)$ *is not greater than* $\binom{\alpha(G)}{2}$.

Theorem 20. *[13] Let* G *be a line graph, then every root* r *of* $I(G;x)$ *satisfies* $r \in \mathbb{R}^-$

Brown and Nowakowski [4] researched the average independence polynomials. The *average independence polynomial* of simple graphs of order n is defined as

$$aip_n(x) = \frac{1}{2^{\binom{n}{2}}} \sum_{G \in \mathcal{G}_n} I(G;x)$$

where $\mathcal{G}_n$ is the set of all simple graphs of order n. They proved the following theorem:

Theorem 21. *Average independence polynomials always have real, simple roots [4].*

A *claw* is a graph $G_c = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_1v_3, v_1v_4\}$. If a graph G does not have any induced subgraph which is isomorphic to G_c, then G is a *clawfree* graph [6]. If G is a clawfree graph then the independence polynomial of G has only real roots [6]. Furthermore, Rosenfeld [20] proved that clawfree graphs are not the only graphs whose roots of independence polynomial are all real.

2.3 Graph polynomial factorisation research

In this section we will study current research on chromatic factorisations and discuss how the approach used in chromatic factorisations research can be applied on s-factorisation research.

2.3.1 Research on chromatic factorisations

The chromatic polynomial $P(G, \lambda)$ is said to have *chromatic factorisation* [17] with chromatic factors H_1 and H_2 if

$$P(G, \lambda) = \frac{P(H_1, \lambda)P(H_2, \lambda)}{P(K_r, \lambda)} \tag{12}$$

where H_1 and H_2 are not isomorphic to K_r and have $\chi(H_i) \geq r, i = 1, 2$.

A *certificate of chromatic factorisation* is a sequence of expressions $P_0, P_1, ..., P_i$ where $P_0 = P(G, \lambda)$ and $P_i = P(H_1; \lambda)P(H_2; \lambda)P(K_r, \lambda)$. For every $j, 1 \leq j \leq i$, P_j is obtained from P_{j-1} by applying an algebraic property or a property of the chromatic polynomial. The certificate is used to explain why a given graph G has a chromatic factorisation with chromatic factors H_1 and H_2.

Research on chromatic factorisation of chromatic polynomials of graphs has been done by Morgan and Farr [18]. Their research consisted of four parts.

1. Compute the chromatic polynomials of all connected graphs of order at most 10 and find all chromatic factorisations.

2. List the certificate steps according to the properties of the chromatic polynomial and algebraic operations.

3. Use certificate steps to construct certificates to explain these chromatic factorisations.

4. Generalise the chromatic factorisation certificates which use the same sequence of certificate steps to certificate schemas.

Morgan and Farr [18] found a class graphs that have chromatic factorisations and successfully generalised the certificates of these graphs to a certificate schema.

As the stability polynomial is closely related to the chromatic polynomial (see Equation (9) and (10)) it seems likely that interesting factorisations for the stability polynomials can be found in the same way.

2.3.2 Feasibility of s-factorisation research

In the research on chromatic factorisations, certification steps which were based on properties of the chromatic polynomial were used to explain chromatic factorisations. In the stability polynomial, there are also some properties which can be used to obtain certification steps. Section 2.1.1 listed five theorems about the stability polynomial.

Theorem 1, 2 and 5 can be used to construct recursive algorithms for the computation of the stability polynomial. We implemented an algorithm using Equation (2) to calculate the stability polynomial. According to Farr [12], the time complexity of this algorithm is $O(2^n poly(n))$, which is more efficient than the usual recursive algorithm for computing the chromatic polynomial. This means that the stability polynomial can be computed for graphs of larger order than the chromatic polynomial.

3 Design

In this project, we have used two different methods to search for s-factorisations. The practical method calculated the stability polynomials of all connected graphs of order at most 9 and exhaustively search for s-factorisations. The theoretical method used the properties of stability polynomials to build graphs that can be s-factorised. Details of these two methods are given in Section 3.1 and 3.2.

3.1 Practical method

Our program for computing the stability polynomial uses a recursive algorithm based on Theorem 1. The base case of this recursive algorithm is $A(G_{n,\emptyset}; p) = 1$. If G has edges, then we compute $A(G; p)$ as the sum of two recursive calculations, $A(G - v; p)$ and $A(G - v - N(v); p)$. In this project, we built a program to search for s-factorisations by going through four steps:

Step 1 : Use Theorem 1 to build a tool for calculating the stability polynomial of any given connected graphs.

Step 2 : Compute the stability polynomial of all connected graphs of order at most 9.

Step 3 . Factorise all stability polynomials of graphs of order at most 9. Store all the factorised stability polynomials in a new list 'Factorised Stability Polynomials List'.

Step 4 . Exhaustively search for s-factorisations of all stability polynomials in the 'Factorised Stability Polynomials List'.

3.2 Theoretical method

Exhaustive search enable us to find all s-factorisations of graphs of small order. Instead of exhaustively searching all stability polynomials of graphs to find s-factorisations, the theoretical method constructs s-factorisations according to the known properties of stability polynomials. This approach consists of three steps:

Step 1 : List all available certificate steps based on the known properties of stability polynomials.

Step 2 : Do theoretical investigation. Use the listed certificate steps to build graphs that can be s-factorised.

Step 3 : Try to build certificate schemas corresponding to common sequences of steps used to explain similar s-factorisations.

Computational results provide some ideas on how to construct graphs that have s-factorisations. The outcomes of theoretical method can also guide the s-factorisation search in the practical method.

After finding s-factorisations by applying the practical method, we analysed the structures of some s-factorisable graphs and the reasons why these graphs can be s-factorised. Then we built new s-factorisable graphs by modifying the s-factorisable graphs found by practical method. Some s-factorisations caused by special structure of graphs are independent to the size of input graphs. By modifying these s-factorisations we found infinite families of s-factorisable graphs.

4 Computational results

According to Theorem 8, any disconnected graph G has an s-factorisation $A(G;p) = A(H_1;p)A(H_2;p)$ where H_1 and H_2 are two disjoint components of G. In this thesis, we focus on s-factorisations of connected graphs.

We used the program designed in Section 3.1 to calculate the stability polynomials of all connected graphs of order at most 9 and searched for s-factorisations. The computational results show that there are 17,461,965 s-factorisations corresponding to 28,576 s-factorisable graphs. These s-factorisable graphs have 1,52 different stability polynomials. Detailed results are given in Table 3 and Table 4.

For each order, Table 3 gives the number of non-isomorphic graphs, the number of different stability polynomials, the number of graphs that have s-factorisation and the number of different stability polynomials that can be s-factorised. Table 4 gives the accumulated data about s-factorisations of graphs. It presents the main results of this project including the number of connected graphs that can be s-factorised (Column C), the number of different stability polynomials that can be s-factorised (Column D) and the number of different s-factorisations (Column F).

Table 3 shows that connected graphs which have s-factorisations have order ≥ 6. By analysing the ratio between the number of graphs of order n and the number of different stability polynomials of these graphs, we can see that when the order of graphs increase, the probability that different graphs have the same stability polynomials gets higher as n increase.

Table 4 shows that there are 28,576 of the 273,192 connected graphs of order 2 to 9 can be s-factorised. These s-factorisable graphs correspond to 152 different stability

Table 3: Graphs and the stability polynomials

Order	#Graphs	#Different polys	#s-factorisation graphs	#Different s-factorised polys
2	1	1	0	0
3	2	2	0	0
4	6	5	0	0
5	21	13	0	0
6	112	38	4	3
7	853	116	50	9
8	11117	391	955	42
9	261080	1438	27567	143

Table 4: s-factorisations

Order	A	B	C	D	E	F
2	1	1	0	0	0	0
2-3	3	3	0	0	0	0
2-4	9	7	0	0	0	0
2-5	30	17	0	0	0	0
2-6	142	45	4	3	18	3
2-7	995	130	54	10	741	10
2-8	12112	420	1009	44	73617	44
2-9	273192	1503	28576	152	17461965	152

(A) Number of non-isomorphic graphs, (B) Number of different stability polynomials, (C) Number of s-factorisable graphs, (D) Number of different s-factorisable stability polynomials, (E) Number of s-factorisations, (F) Number of different s-factorisations.

polynomials, and have 152 different s-factorisations. Another interesting fact given by Table 4 is that the values in Column D are the same as the values in Column F, which means the number of different s-factorisable stability polynomials is equal to the number of different s-factorisations. This finding may indicate that one the stability polynomial can be s-factorised in at most one way.

5 Theoretical results

Based on the computational results given in Section 4, we conducted theoretical research on stability equivalence and s-factorisations. The main theoretical results presented in this section are organised as follows. Section 5.1 lists the certificate steps for explaining stability equivalences and s-factorisations. Section 5.2 gives upper bounds on length of certificates. Some infinite families of stability equivalent graphs and short certificates for these stability equivalences are given in Section 5.3. Section 5.4 presents the s-factorisations that can be explained by short certificates. By analysing the similarities of these short certificates, we found certificate schemas and presented them in Section 5.4.

5.1 Certificate steps

In their research on chromatic factorisations, Morgan and Farr [18] defined certificates of chromatic factorisation and converted the properties of chromatic polynomials to certificate steps. Based on the properties of the stability polynomial which are presented in Section 2.1.1, we can also define certificates for s-factorisation and give certificate steps for explaining s-factorisations.

We can define a *certificate of s-factorisation* as a sequence of expressions $P_0, P_1, \ldots, P_i$ where $P_0 = A(G, p)$ and $P_i = A(G_1; p)A(G_2; p)$. For every $j, 1 \leq j \leq i$, P_j in the sequence is obtained from P_{j-1} by applying one of the certificate steps of stability polynomials. The certificate steps which are obtained from the properties of the stability polynomial are listed as follows.

CS1 $A(G_{n,\emptyset}; p)$ becomes 1.

CS2 1 becomes $A(G_{n,\emptyset}; p)$.

CS3 $A(G; p)$ becomes $(1-p)A(G-v; p) + p(1-p)^d A(G-v-N(v); p)$ for some vertex $v \in V(G)$ where d is the degree of vertex v.
(Theorem 1)

CS4 Let $U = \{u_1, u_2, u_3, ..., u_d\}$ be d distinct vertices in $V(G_1)$.Then $(1-p)A(G_1; p) + p(1-p)^d A(G_2; p)$ becomes $A(G; p)$ where $G = (V, E)$ is isomorphic to the graph G' with vertex set $V' = V(G_1) \cup \{v\}$, and edge set $E' = E(G_1) \cup \{vu_i : \text{where } u_i \in U\}$, and where $G_2 \cong G_1 - U - \{v\}$.
(Theorem 1)

CS5 $A(G; p)$ becomes $(1-p)A(G-u; p) + (1-p)A(G-v; p) - (1-p)^2 A(G-u-v; p)$ for some $uv \in E(G)$.
(Theorem 2)

CS6 Let $N_1 \subseteq V(G_3)$ and $N_2 \subseteq V(G_3)$. Then $(1-p)A(G_1; p) + (1-p)A(G_2; p) - (1-p)^2 A(G_3; p)$ becomes $A(G; p)$ for some $uv \in E(G)$ where the following conditions are satisfied:

(1) $G_1(V_1, E_1)$ is isomorphic to graph with $V_1' = V(G_3) \cup \{u\}$ and $E_1' = E(G_3) \cup \{ua : a \in N_1\}$,

(2) $G_2(V_2, E_2)$ is isomorphic to graph with $V_2' = V(G_3) \cup \{v\}$ and $E_2' = E(G_3) \cup \{vb : b \in N_2\}$,

(3) $G(V, E)$ is isomorphic to graph with $V' = V(G_3) \cup \{u\} \cup \{v\}$ and $E' = E(G_3) \cup \{ua : a \in N_1\} \cup \{vb : b \in N_2\}$.

(Theorem 2)

CS7 $A(G; p)$ becomes $A(G \setminus e; p) - p^2(1-p)^d A(G-u-v-N(u)-N(v); p)$ for some $uv \in E(G)$ and $d = |(N(u) \setminus \{v\}) \cup (N(v) \setminus \{u\})|$.
(Theorem 5)

CS8 $A(G_1; p) - p^2(1-p)^d A(G_2; p)$ becomes $A(G; p)$ where
$G(V, E)$ is isomorphic to graph with $V = V(G_2) \cup \{u\} \cup \{v\}$ and $E = E(G_2) \cup \{ua : a \in N_1\} \cup \{vb : b \in N_2\}$ when $N_1 \subseteq V(G_2), N_2 \subseteq V(G_2)$ and $d = |N_1 \cup N_2|$.
(Theorem 5)

CS9 $A(G; p)$ becomes $A(G_1; p)A(G_2; p)$ where G_1, G_2 are two disjoint graphs and $G_1 \cup G_2$ is isomorphic to G.
(Theorem 3)

CS10 $A(G_1;p)A(G_2;p)$ becomes $A(G;p)$ where G_1, G_2 are two disjoint graphs and G is isomorphic to $G_1 \cup G_2$.
(Theorem 3)

These certificate steps can be used to explain s-factorisations and stability equivalences. We consider how the length of the certificates depends on the order of corresponding graphs.

5.2 Upper bound of certificate length

In this section we will discuss the length of certificates for stability equivalences and s-factorisations. We use a naive approach to find an upper bound on length of certificates. In practice we found much shorter certificates. To show two graphs are stability equivalent we must transform expression $A(G;p)$ to expression $A(H;p)$. We can first express the stability polynomial of G as an expression in null graphs. Then we transform this expression into the stability polynomial of H.

To show $A(G;p)$ has an factorisation $A(H_1;p)A(H_2;p)$ we must transform expression $A(G;p)$ to expression $A(H_1;p)A(H_2;p)$. We can also first express the stability polynomial of G as an expression in null graphs and then transform this expression into product of the stability polynomial of H_1 and the stability polynomial of H_2.

First of all, we use a naive approach to express the stability polynomial of graph G as an expression in null graphs to find an upper bound for the number of certificate steps to express $A(G;p)$ in terms of stability polynomial of null graphs. For any graph G, if there exists a vertex $v \in V(G)$ with degree ≥ 1, we apply certificate step CS3 on v and transform $A(G;p)$ to $(1-p)A(G-v;p) + p(1-p)^{degv}A(G-v-N(v);p)$. If graph $G-v$ and $G-v-N(v)$ are not null graphs, we recursively apply CS3 on these graphs until all graphs in the expression are null graphs. Let $T(n)$ be the maximum certificate steps the naive approach need to express the stability polynomial of a graph G of order n in terms of stability polynomials of null graphs. Then

Lemma 1. *For any* $0 \leq a \leq b$,

$$T(a) \leq T(b).$$

Proof. Assume that $T(a) > T(b)$ in order to obtain a contradiction. According to the definition of $T(a)$, there is a graph G_a of order a that has a certificate where CS3 is applied $T(a)$ times to transform $A(G_a;p)$ to an expression in null graphs. Because $a \leq b$, there exists a graph G_b of order b such that G_a is a subgraph of G_b. To transform $A(G_b;p)$ to an expression in null graphs, we need to apply CS3 for at most $T(b)$ times.

(1)If $a = b$, by the definition of $T(a)$ and $T(b)$ we have $T(a) = T(b)$, which contradicts assumption assumption $T(a) > T(b)$.

(2)If $a < b$, we let $G_b \cong G_a \cup G_{b-a,\emptyset}$. Then the naive approach transforms $A(G_b;p)$ to an expression in null graphs by transforming $A(G_a;p)$ to an expression in null graphs. It takes $T(a)$ steps to transform $A(G_a;p)$ to an expression in null graphs, but G_b needs at most $T(b)$ steps to transform $A(G_b;p)$ to an expression in null graphs by definition of T. This contradicts our assumption $T(a) > T(b)$.

According to (1) and (2), the assumption $T(a) > T(b)$ is wrong. So $T(a) \leq T(b)$. □

Theorem 22. *For any* $n \geq 2$,

$$T(n) \leqslant \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} - 1.$$

Proof. First we will prove three properties about $T(n)$:

1. $T(1) = 0$. A graph with only one vertex is a null graph. Thus $T(1) = 0$

2. $T(2) = 1$. Let G_2 be a connected graph of order 2. Then $G_2 \cong K_2$. We can apply certificate step CS3 on K_2, then $A(K_2; p)$ can be expressed by $(1-p)A(K_1; p) + p(1-p)^{|N(v)|}A(G_{0,\emptyset}; p)$ where $v \in V(K_2)$. Since K_1 and $G_{0,\emptyset}$ are both null graphs, $T(2) = 1$.

3. For any $n \geq 3$, $T(n) \leqslant 1 + T(n-1) + T(n-2)$. Let G_n be a graph of order n,

   (a) If G_n is a null graph, the maximum steps we need for expressing $A(G_n - v; p)$ in terms of null graphs is 0 and $0 \leqslant 1 + T(n-1) + T(n-2)$.

   (b) If G_n is not a null graph, there exist a vertex $v_c \in V(G_n)$ such that $|N(v_c)| \geq 1$. Then $A(G_n; p)$ can be transformed to $(1-p)A(G_n - v_c; p) + p(1-p)^{|N(v_c)|}A(G_n - v_c - N(v_c); p)$ by applying certificate step CS3. The graph $G_n - v_c$ has order $n-1$ and $G_n - v_c - N(v_c)$ is a graph of order $\leq n-2$. According to Lemma 1, for any $a \leq n-2$, $T(a) \leq T(n-2)$. Thus the certificate steps we need for expressing $A(G_n; p)$ in terms of null graphs is $\leqslant 1 + T(n-1) + T(n-2)$

   According to (a) and (b), for any $n \geq 3$, there is

$$T(n) \leqslant 1 + T(n-1) + T(n-2). \tag{13}$$

Now Equation (13) is closely related to Fibonacci sequence. Solving this recurrence we get

$$T(n) \leqslant \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} - 1$$

□

With Theorem 22, we can get the upper bound of the length of certificates for stability equivalence and s-factorisations.

Corollary 1. *If $A(G; p) = A(H; p)$, G is a graph of order n_G and H is a graph of order n_H, then there exist a certificate of $A(G; p) = A(H; p)$ of length at most*

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_G+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_G+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_H+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_H+1}}{\sqrt{5}} + n_G + n_H + 1$$

Proof. We can analyse the upper bound of certificate length of stability equivalence by going through three stages.

(1) According to Theorem 22, $A(G; p)$ can be transformed to an expression S_k which is an expression in null graphs by applying k certificate steps, $k \leq T(n_G)$. Let this sequence of expressions be $S_0 = A(G; p)$,$S_1, S_2, S_3, \ldots, S_k$. We can do an alegibric transformation to transform S_k to S_{k+1}. Expression $S_{k+1} = \sum_{i=0}^{x} A(G_{i,\emptyset}; p)P_i(p)$, $x \leq n_G$ and $P_i(p)$ is a polynomial in p. Then we can apply certificate step CS1 for at most $n_G + 1$ times to transform S_{k+1} to an expression E_G^p which is an polynomial in p.

(2) According to Theorem 22, $A(H; p)$ can be transformed to an expression E_l which is an expression in null graphs by applying l certificate steps, $l \leq T(n_H)$. Let this sequence of expressions be $E_0 = A(H; p)$,$E_1, E_2, E_3, \ldots, E_l$. We can do an alegibric transformation to transform E_l to E_{l+1}. Expression $E_{l+1} = \sum_{j=0}^{x} A(G_{j,\emptyset}; p)P_j(p)$, $x \leq n_H$ and $P_j(p)$ is a polynomial in p. Then we can apply certificate step CS1 for at most $n_H + 1$ times to transform E_{l+1} to an expression E_H^p which is an polynomial in p.

(3) Then S_0,$S_1, S_2, S_3, \ldots, S_k, S_{k+1}, \ldots, E_G^p$,$E_H^p, \ldots, E_{l+1}, E_l, E_{l-1}, E_{l-2}, \ldots, E_2, E_1$,$E_0$ is a certificate for $A(G;p) = A(H;p)$. Length of this certificate is at most $k+1+n_G+1+l+1+n_H$ steps. According to Theorem 22, we have

$$k+l+n_G+n_H+3 \leq \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_G+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_G+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_H+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_H+1}}{\sqrt{5}} + n_G + n_H + 1$$

□

This upper bound is approximately $0.724 \times (1.618^{n_G} + 1.618^{n_H}) + n_G + n_H + 1$.

Corollary 2. *If $A(G;p) = A(H_1;p)A(H_2;p)$, G is a graph of order n_G, H_1 is a graph of order n_{H_1} and H_2 is a graph of order n_{H_2}, then there exist a certificate for $A(G;p) = A(H_1;p)A(H_2;p)$ of length at most*

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_G+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_G+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_{H_1}+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_{H_1}+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_{H_2}+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_{H_2}+1}}{\sqrt{5}} + n_G + n_{H_1} + n_{H_2} + 1$$

Proof. We can analyse the upper bound of certificate length of s-factorisation by going through four stages.

(1) According to Theorem 22, $A(G;p)$ can be transformed to an expression S_k which is an expression in null graphs by applying k certificate steps, $k \leq T(n_G)$. Let this sequence of expressions be $A(G;p)$,$S_1, S_2, S_3, \ldots, S_k$. We can do an alegibric transformation to transform S_k to S_{k+1}. Expression $S_{k+1} = \sum_{i=0}^{x} A(G_{i,\emptyset};p)P_i(p)$, $x \leq n_G$ and $P_i(p)$ is a polynomial in p. Then we can apply certificate step CS1 for at most n_G+1 times to transform S_{k+1} to an expression E_G^p which is an polynomial in p.

(2) According to Theorem 22, $A(H_1;p)$ can be transformed to an expression E_l which is an expression in null graphs by applying l certificate steps, $l \leq T(n_{H_1})$. Let this sequence of expressions be $A(H_1;p)$,$E_1, E_2, E_3, \ldots, E_l$. We can do an alegibric transformation to transform E_l to E_{l+1}. Expression $E_{l+1} = \sum_{j=0}^{x} A(G_{j,\emptyset};p)P_j(p)$, $x \leq n_{H_1}$ and $P_j(p)$ is a polynomial in p. Then we can apply certificate step CS1 for at most n_H+1 times to transform E_{l+1} to an expression $E_{H_1}^p$ which is an polynomial in p. And $A(H_2;p)$ can be transformed to an expression F_m which is an expression in null graphs by applying m certificate steps, $m \leq T(n_{H_2})$. Let this sequence of expressions be $A(H_2;p)$,$F_1, F_2, F_3, \ldots, F_m$. We can do an alegibric transformation to transform F_l to F_{m+1}. Expression $F_{m+1} = \sum_{j=0}^{x} A(G_{j,\emptyset};p)P_j(p)$, $x \leq n_{H_2}$ and $P_j(p)$ is a polynomial in p. Then we can apply certificate step CS1 for at most $n_{H_2}+1$ times to transform F_{m+1} to an expression $E_{H_2}^p$ which is an polynomial in p.

(3) According to (2), we can apply certificate step CS1 for at most $n_{H_1}+1+n_{H_2}+1$ times to transform $E_{l+1} \cdot F_{m+1}$ to an expression $E_{H_1}^p \cdot E_{H_2}^p$ which is an polynomial in p.

(4) Then $A(G;p)$,$S_1, S_2, S_3, \ldots, S_k, , S_{k+1}$,$E_G^p$,$E_{H_1}^p \cdot E_{H_2}^p$,,$E_{l+1} \cdot F_{m+1}$,$E_l \cdot F_m$,$E_{l-1} \cdot F_m$, $E_{l-2} \cdot F_m, \ldots, E_2 \cdot F_m$, $E_1 \cdot F_m$,$A(H_1;p) \cdot F_m$,$F_{m-1} \cdot A(H_1;p)$, $F_{m-2} \cdot A(H_1;p), \ldots, F_2 \cdot A(H_1;p)$, $F_1 \cdot A(H_1;p)$, $A(H_2;p) \cdot A(H_1;p)$ is a certificate for $A(G;p) = A(H_1;p)A(H_2;p)$. Length of this certificate is $k+1+n_G+1+l+1+n_{H_1}+m+1+n_{H_2}$. According to Theorem 22, we have

$$\begin{aligned} k+l+m+n_G+n_{H_1}+n_{H_2}+4 \leq & \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_G+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_G+1}}{\sqrt{5}} \\ & + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_{H_1}+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_{H_1}+1}}{\sqrt{5}} \\ & + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_{H_2}+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_{H_2}+1}}{\sqrt{5}} \\ & + n_G + n_{H_1} + n_{H_2} + 1 \end{aligned}$$

□

This upper bound is approximately $0.724 \times (1.618^{n_G} + 1.618^{n_{H_1}} + 1.618^{n_{H_2}}) + n_G + n_{H_1} + n_{H_2} + 1$.

5.3 Stability equivalence

The computational results given in Section 4 show that many non-isomorphic graphs have the same stability polynomial. In this section, short certificates for two infinite families of stability equivalent graphs are presented.

Theorem 23. *The stability polynomial $A(C_n;p) = A(Q_n;p)$ for $n \geq 3$.*

Proof. The following is a certificate of equivalence to show $A(C_n;p) = A(Q_n;p)$:

$$A(C_n;p) = (1-p)A(P_{n-1};p) + p(1-p)^2 A(P_{n-3};p) \tag{CS3}$$
$$= A(Q_n;p). \tag{CS4}$$

This certificate is illustrated in figure 1. □

In this thesis, we use the convention of representing the stability polynomial of a graph by a picture of the graph itself.

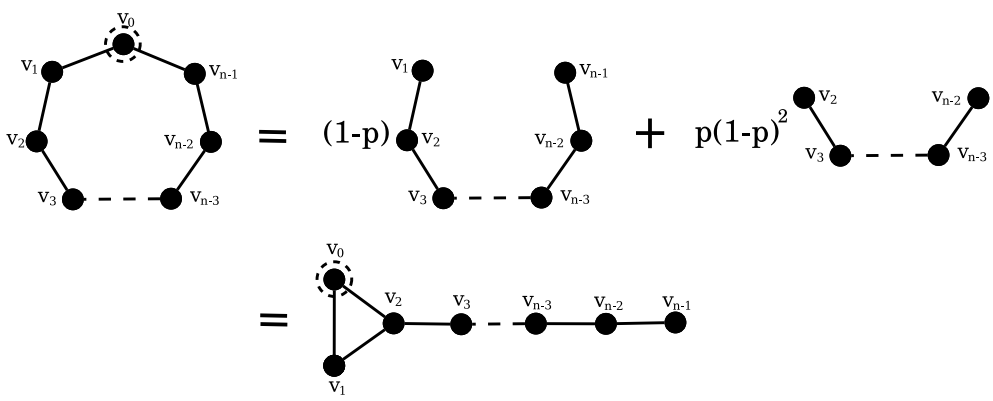

Figure 1: $A(C_n;p) = A(Q_n;p)$

Theorem 23 shows that for any $n \geq 3$, $A(C_n;p) = A(Q_n;p)$ have certificates of length $= 2$. The length of certificates for this infinite family of stability equivalences is much lower than the exponential upper bound given in Section 5.2.

Theorem 24. *For any $uv \in E(G)$, if $(N(v)\backslash\{u\}) \subseteq (N(u)\backslash\{v\})$. Then*

$$A(G;p) = A(G';p)$$

where G' is a graph with $V(G') = V(G) \cup \{w\}, E(G') = (E(G)\backslash uv) \cup \{uw\}$

Proof. The following is a certificate of the stability equivalence:

$$\begin{aligned}
A(G;p) =& (1-p)A(G-u;p) + p(1-p)^{|N(u)|}A(G-N(u);p) && \text{(CS3)}\\
=& (1-p)A(G-u;p)A(K_1;p) + p(1-p)^{|N(u)|}A(G-N(u);p)A(K_1;p) && \text{(CS2)}\\
=& (1-p)A(\{G-u\} \cup K_1;p) \\
&+ p(1-p)^{|(N(u)\backslash\{v\})\cup\{w\}|}A(\{G-N(u)\} \cup K_1;p) && \text{(CS10)}\\
=& A(G';p). && \text{(CS4)}
\end{aligned}$$

Figure 2 illustrates this certificate. □

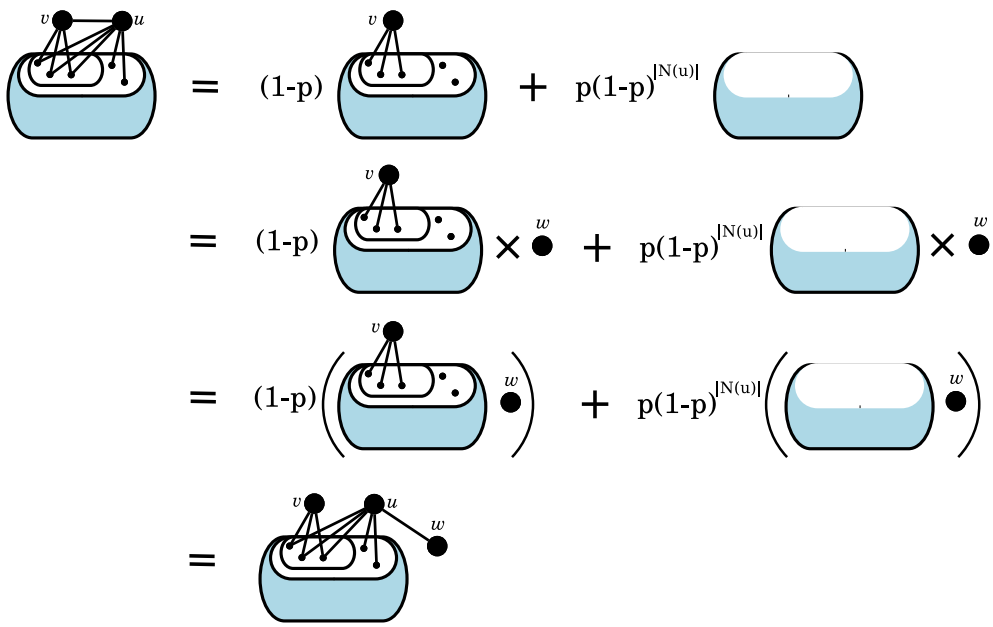

Figure 2: Edge breaking theorem

Theorem 24 shows that an infinite family of graphs have stability equivalences that can be explained by certificates of length $= 4$. The length of certificates for these stability equivalences is much lower than the exponential upper bound given in Section 5.2 and is independent of the order of the graph. Theorem 24 is a powerful theorem which can be used to show the following two infinite families of graphs are stability equivalent.

Corollary 3. *For any $n \geq 3$, there is*

$$A(Q_n;p) = A(Y_{n+1};p)$$

Proof. The certificate in Theorem 24 is a certificate of stability equivalence of this infinite family of graphs. The certificate is illustrated in Figure 3. The length of certificates provided by this corrollary is 4. □

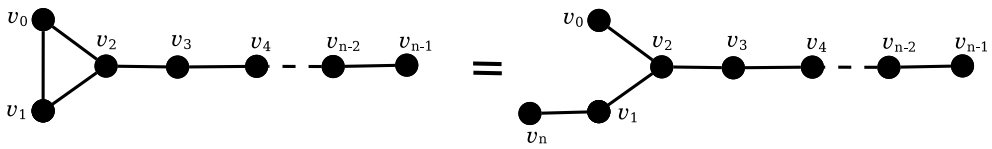

Figure 3: $A(Q_n; p) = A(Y_{n+1}; p)$

Corollary 4. *For any $n \geq 2$, there is*

$$A(K_{n+1}; p) = A(K_n^*; p)$$

Proof. Let $\{v_i : 0 \leq i \leq n-1\} \cup \{w\}$ be the vertex set of K_{n+1}. Applying Theorem 24 on all edges in $\{v_i w : 0 \leq i \leq n-1\}$ will provide certificates for this infinite family of stability equivalences. Theorem 24 is used for n times. Thus the length of certificates provided by this proof is $4n$. This length is linear, which is much shorter than the exponential upper bound given in Section 5.2. This proof is illustrated in Figure 4. □

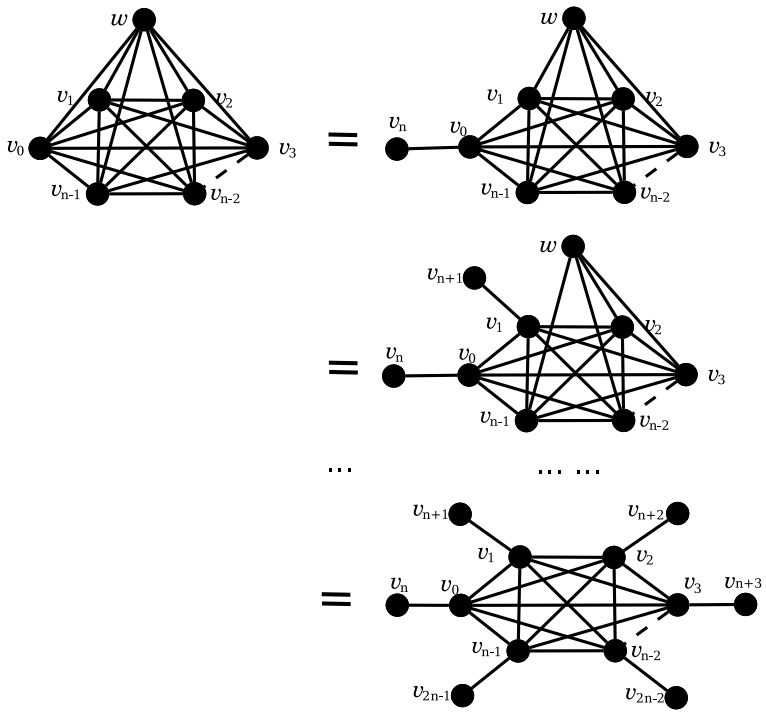

Figure 4: $A(K_{n+1}; p) = A(K_n^*; p)$

5.4 S-factorisations

In this section, we studied the s-factorisations provided by the computational results and found short certificates for two infinite families of s-factorisations.

Theorem 25. *For any $n \geq 2$, we have*

$$A(\Upsilon_{n,2n}; p) = A(P_{n-1}; p) A(\Upsilon_{n+1,n+1}; p)$$

Proof. The following is a certificate of this s-factorisation:

$$\begin{aligned}
A(\Upsilon_{n,2n};p) &= (1-p)A(\Upsilon_{n,2n}\cup P_{n-1};p) + p(1-p)^2A(P_{n-1}\cup P_{n-2};p) && \text{(CS3)}\\
&= (1-p)A(\Upsilon_{n,2n};p)A(P_{n-1};p) + p(1-p)^2A(P_{n-1};p)A(P_{n-2};p) && \text{(CS9)}\times 2\\
&= A(P_{n-1};p)[(1-p)A(\Upsilon_{n,n};p) + p(1-p)^2A(P_{n-2};p)] && \text{(Algebraic)}\\
&= A(P_{n-1};p)A(\Upsilon_{n+1,n+1};p). && \text{(CS4)}
\end{aligned}$$

Figure 5 illustrates this certificate. The length of certificates provided by the proof of Theorem 25 is 5, which is much lower than the exponential upper bound given in Section 5.2. □

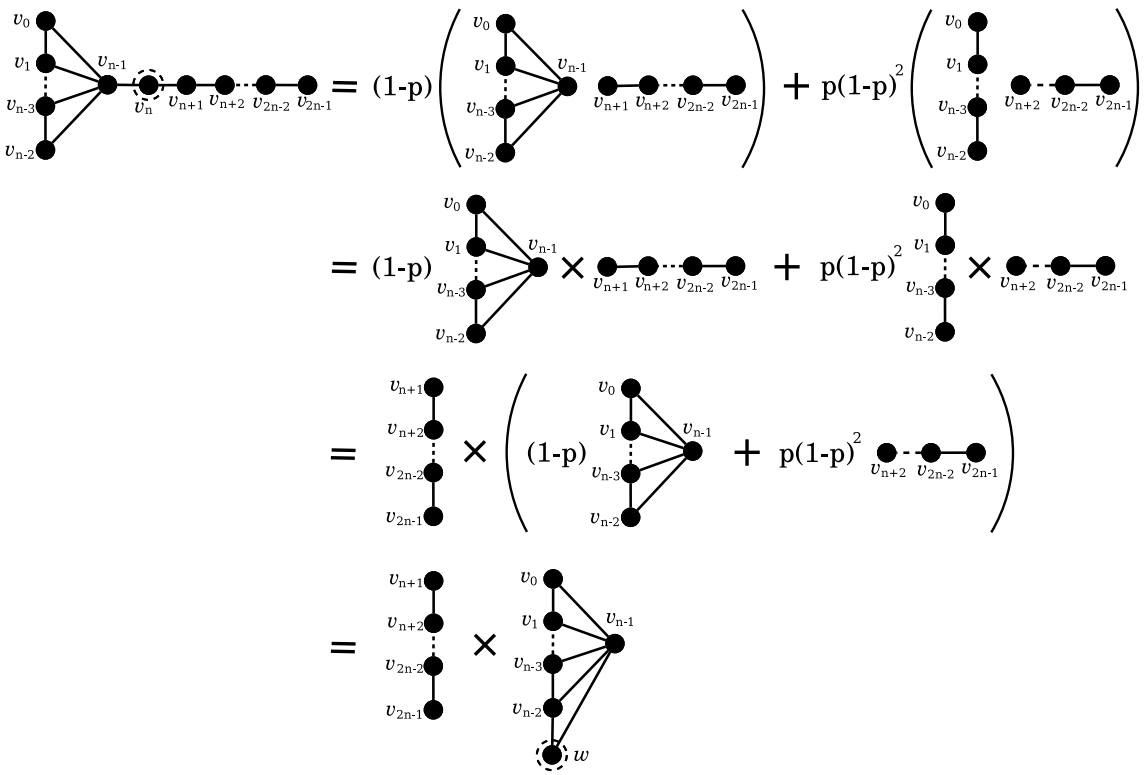

Figure 5: $A(\Upsilon_{n,2n};p) = A(P_{n-1};p)A(\Upsilon_{n+1,n+1};p)$

Theorem 26. *(General symmetry theorem) If $G \cong G', u \in V(G)$ and u' is the image of u in G' under a mapping of the isomorphism,$N(u)$ is the neighbor set of u in G. Let $N_2 \subseteq V(G_2)$, G_3 is a graph with vertex set $V(G_3) = V(G)\cup V(G')\cup V(G_2)$ and edge set $E(G_3) = E(G)\cup E(G')\cup E(G_2)\cup\{uv, u'v : v \in N_2\}\cup\{uu'\}$. Then*

$$A(G_3;p) = A(G-u;p)A(G_4;p)$$

where G_4 is a graph with vertex set $V(G_4) = V(G_2)\cup V(G')\cup\{w\}$ and edge set $E(G_4) = E(G_2)\cup E(G')\cup\{wv, u'v : v \in N(u)\}$

Proof. The following is a certificate of this s-factorisation:

$$
\begin{aligned}
A(G_3;p) =& p(1-p)^{|N(u)\cup N_2\cup\{u'\}|}A((G-u-N(u))\cup(G'-u')\cup(G_2-N_2);p) \\
&+(1-p)A((G-u)\cup(G_3-G);p) && \text{(CS3)} \\
=& p(1-p)^{|N(u)\cup N_2\cup\{u'\}|}A(G-u-N(u);p)A(G'-u';p)A(G_2-N_2;p) \\
&+(1-p)A(G-u;p)A(G_3-G;p) && \text{(CS9)}\times 2 \\
=& p(1-p)^{|N(u)\cup N_2\cup\{u'\}|}A(G-u-N(u);p)A(G_2-N_2;p)] \\
&+A(G-u;p)[(1-p)A(G_3-G;p) && \text{(Algebraic)} \\
=& A(G-u;p)A(G_4;p). && \text{(CS6)}
\end{aligned}
$$

The length of certificates provided by this proof is 5. This independent of the order of the graph and is much lower than the exponential upper bound given in Section 5.2. Figure 6 illustrates this certificate. □

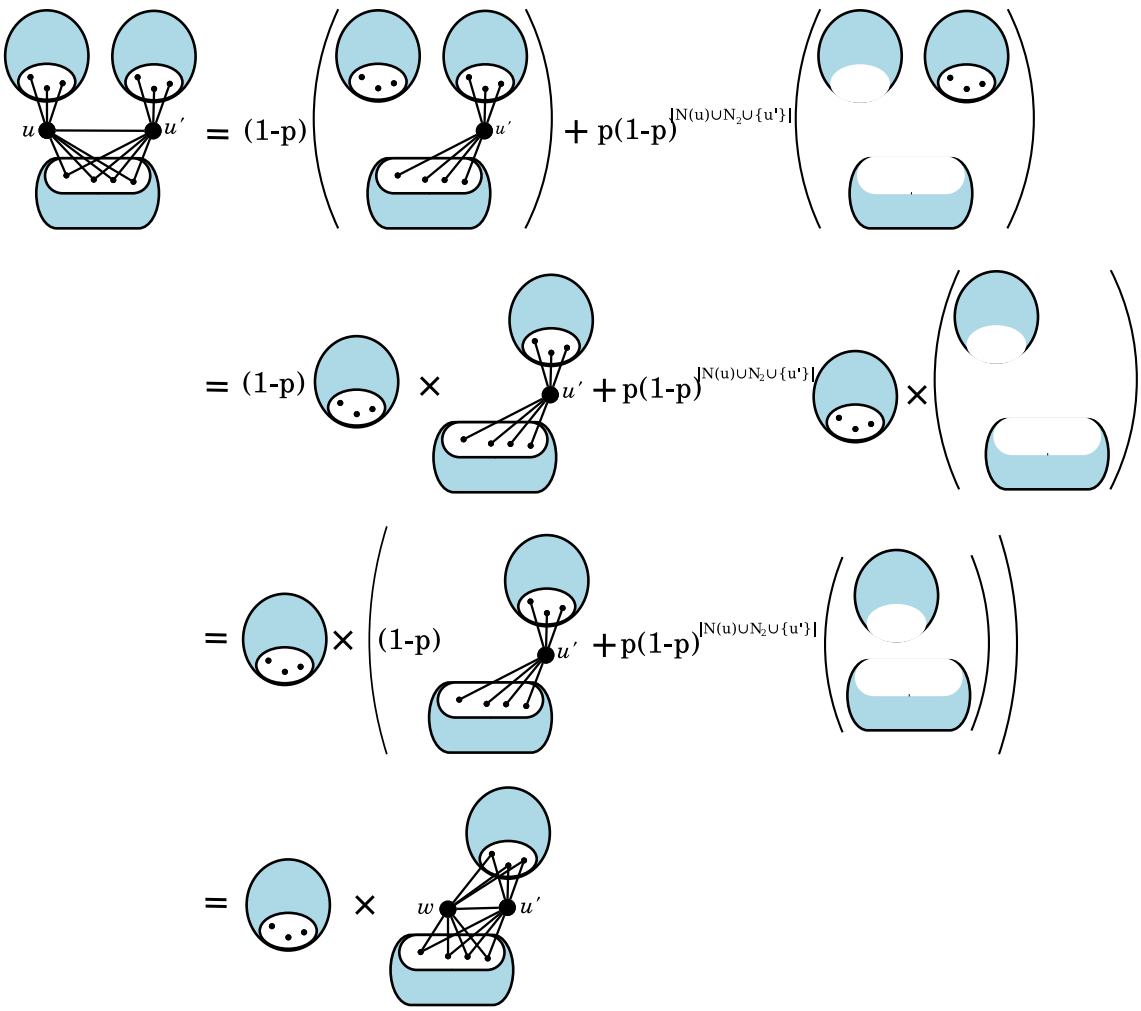

Figure 6: General symmetry theorem

Theorem 26 is a property of stability polynomial which can be used to decompose graphs with isomorphic subgraphs. We applied Theorem 24 on some more specialised cases and found the following interesting corollaries.

Corollary 5. *(Symmetry corollary) If $G \cong G'$, $u \in V(G)$ and u' is the image of u in G' under a mapping of the isomorphism, $N(u)$ is the neighbour set of u in G. Let G_2 be a graph with vertex set $V(G_2) = V(G) \cup V(G')$ and edge set $E(G_2) = E(G) \cup E(G') \cup \{uu'\}$. Then*

$$A(G_2; p) = A(G - u; p)A(G_{u+}; p)$$

where G_{u+} is a graph with vertex set $V(G_{u+}) = V(G) \cup \{w\}$ and edge set $E(G_{u+}) = E(G) \cup \{wu\} \cup \{wv : v \in N(u)\}$

Proof. The following is a certificate of this s-factorisation:

$$\begin{aligned}
A(G_2; p) =& (1-p)A((G-u) \cup G'; p) \\
&+ p(1-p)^d A((G - N(u)) \cup ((G' - u')); p) && \text{(CS3)} \\
=& (1-p)A(G-u; p)A(G') \\
&+ p(1-p)^d A(G - N(u); p)A(G' - u') && \text{(CS9)}\times 2 \\
=& A(G-u; p)[(1-p)G + p(1-p)^{|N(u)\cup\{w\}|}(G - N(u))] && \text{(Algebraic)} \\
=& A(G_{u+}; p). && \text{(CS4)}
\end{aligned}$$

where $d = |N(u) \cup \{u'\}| = |N(u) \cup \{w\}|$. Certificate length of this proof is 5 steps. Figure 7 illustrates this corollary. □

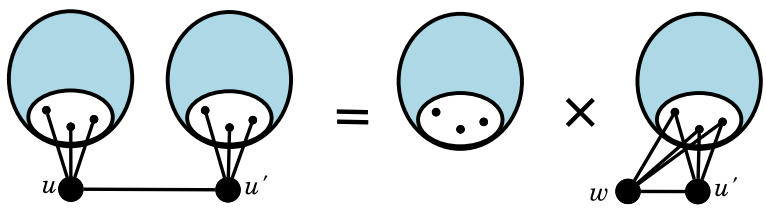

Figure 7: Symmetry corollary

Corollary 5 is a specialisation of Theorem 26. The length of certificates provided by Corollary 5 is 5 steps. It follows from Corollary 5 that

Corollary 6. *For any graph G, there exists a graph H such that H has an s-factorisation with G as a factor.*

This result is analogous to the finding provided by Morgan and Farr's research on chromatic factorisation [18].

Corollary 7. *For any $n \geq 2$, there is*

$$A(K_{n\sim n}; p) = A(K_{n-1}; p)A(K_{n+1}; p)$$

Figure 8 illustrates this corollary.

Corollary 7 is a specialisation of Corollary 5. The length of certificates provided by Corollary 7 is 5.

Corollary 8. *For any $n \geq 2$, there is*

$$A(P_{2n}; p) = A(P_{n-1}; p)A(C_{n+1}; p).$$

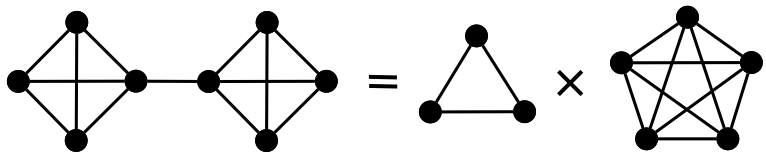

Figure 8: $A(K_{n\sim n};p) = A(K_{n-1};p)A(K_{n+1};p)$ when $n = 4$

Proof. Let G be a P_n, $V(G) = \{v_0, v_1, ..., v_n\}$ and $E(G) = \{v_0v_2\} \cup \{v_iv_{i+1} : 0 \leq i \leq n-2\}$. Let $G' \simeq G$ and v'_i is the image of v_i in G under a mapping of the isomorphism. $G_2 = (G \cup G') + \{v_0v'_0\}$, G_2 is a P_{2n}. G_4 is a Q_n with vertex set $V(G_4) = V(G') \cup \{w\}$ and edge set $E(G_4) = E(G') \cup \{wv, u'v : v \in N_G(v'_0)\}$ By Theorem 26, there is$A(G_2;p) = A(G - v_0;p)A(G_4;p)$.Thus

$$A(P_{2n};p) = A(P_{n-1});p)A(Q_n;p) \tag{14}$$

By Theorem 23 $A(Q_n;p) = A(Y_{n+1};p) = A(C_{n+1};p)$, there is $A(Q_n;p) = A(C_{n+1};p)$, Apply this on Equation 14 and there is

$$A(P_{2n};p) = A(P_{n-1};p)A(C_{n+1};p).$$

The length of certificates provided by the proof of Corollary 8 is $5 + 2 = 7$. Figure 9 illustrates this corollary. □

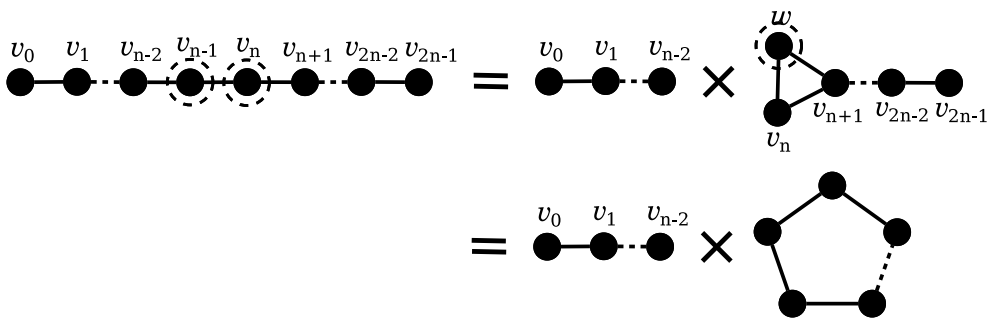

Figure 9: $A(P_{2n};p) = A(P_{n-1};p)A(C_{n+1};p)$

5.5 Certificate schemas of s-factorisations

The certificates for s-factorisations presented in Section 5.4 show that two different families of s-factorisations can be explained by the same sequence of certificate steps. We group these certificates of s-factorisations into two certificate schemas as follows:

$$\begin{aligned}
A(G;p) =& (1-p)A(H_1;p) + p(1-p)^d A(H_2;p) && \text{(CS3)}\\
=& (1-p)A(H_3;p)A(H_4;p) + p(1-p)^d A(H_3;p)A(H_5;p) && \text{(CS9)}\times 2\\
=& A(H_3;p)((1-p)A(H_4;p) + p(1-p)^d A(H_5;p)) && \text{(Algebraic step)}\\
=& A(H_3;p)A(H_6;p) && \text{(CS4)}
\end{aligned}$$

Schema 1

The length of certificates constructed by Schema 1 is 5 steps. This schema can used to build the certificates in Theorems 25 and 26 and Corollaries 5 and 7.

$$
\begin{aligned}
A(G;p) =&(1-p)A(H_1;p) + (1-p)A(H_2;p) - (1-p)^2A(H_3;p) && \text{(CS5)}\\
=&(1-p)A(H_4;p)A(H_5;p) + (1-p)A(H_4;p)A(H_6;p)\\
&- (1-p)^2A(H_4;p)A(H_7;p) && \text{(CS9)}\times 3\\
=&A(H_4;p)((1-p)A(H_5;p) + (1-p)A(H_6;p)\\
&- (1-p)^2A(H_7;p)) && \text{(Algebraic step)}\\
=&A(H_4;p)A(H_8;p) && \text{(CS6)}
\end{aligned}
$$

Schema 2

The length of certificates constructed by Schema 2 is 6. This schema can used to build the certificates of Theorems 25 and 26 and Corollaries 5 and 7

6 Conclusion and further work

This thesis initiates an algebraic study for the stability polynomial. Factorisation is an important algebraic property of polynomials, but previously there had been little work done in the factorisations of the stability polynomial. In this thesis we conducted research on s-factorisations and presented results in five areas: computational results, upper bounds for certificates for stability equivalences and s-factorisations, certificates for stability equivalences and s-factorisations and certificates schemas for s-factorisations. We also present are some open questions for further work.

6.1 Computational results

First we implemented a recursive algorithm to calculate the stability polynomial of all connected graphs of order at most 9 and exhaustively searched for s-factorisations. The computational results showed that 28,576 graphs of order at most 9 are s-factorisable and corresponding to 152 different stability polynomials and 152 different s-factorisations. We found s-factorisable connected graphs have order ≥ 6. The probability that different graphs are stability equivalent appears to increase as the order of graphs increase.

6.2 Upper bounds for certificates for stability equivalences and s-factorisations

We used the notion of a certificate based on Morgan and Farr's research [18] to explain the s-factorisations and stability equivalences given in the computational results. The first question we addressed on certificates is the length. A naive approach was used to construct certificates. This gave a theoretical upper bound of

$$\leq \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_G+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_G+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_H+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_H+1}}{\sqrt{5}} + n_G + n_H + 1$$

on length of certificates for stability equivalence $A(G;p) = A(H_1;p)A(H_2;p)$, where n_G and n_H is the order of graph G and H. This upper bound is approximately $0.724 \times (1.618^{n_G} + 1.618^{n_H}) + n_G + n_H + 1$.

An upper bound on length of certificate of s-factorisation $A(G;p) = A(H_1;p)A(H_2;p)$ was given as

$$\leq \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_G+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_G+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_{H_1}+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_{H_1}+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n_{H_2}+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n_{H_2}+1}}{\sqrt{5}} + n_G + n_{H_1} + n_{H_2} + 1,$$

where n_G, n_{H_1} and n_{H_2} is the order of graph G, H_1 and H_2. This upper bound is approximately $0.724 \times (1.618^{n_G} + 1.618^{n_{H_1}} + 1.618^{n_{H_2}}) + n_G + n_{H_1} + n_{H_2} + 1$.

6.3 Stability equivalences

As a stability polynomial is a algebraic object, we are often interested in when two algebraic objects are equivalent. In this research we found two infinite families of graphs that are stability equivalent and gave short certificates of length at most 5 to explain these stability equivalences. The lengths of these short certificates are much smaller than the theoretical upper bound for certificates for stability equivalences. Furthermore these certificates have constant lengths and are independent of the order of the graphs.

6.4 S-factorisations

Another algebraic property we are interested in is factorisation. The computational results of this researched show that there are 17,461,965 s-factorisations corresponding to 28,576 graphs of order at 9. We found short certificates for two infinite families of s-factorisations. These certificates had of length at most 7. The lengths of these certificates are much shorter than the theoretical upper bound for certificates for s-factorisations and are independent of the size of the graph.

A natural question that arises from the existence of s-factorisations is what graph can be s-factors. We found that for any graph G, there exists a connected graph G' such that G is an s-factor of G'.

6.5 Certificate schemas of s-factorisations

Most of the short certificates for s-factorisations used the same sequence of certificate steps. This allowed us to generalise these certificates to schemas. We found two schemas for the certificates for s-factorisations and the lengths of certificates constructed by these schemas are ≤ 6. Both of this schemas can be used to explain the s-factorisations of our families of graphs.

6.6 Further work

In this research we found short certificates for all s-factorisations of graphs of order 6 and one s-factorisation of graphs of order 7. Finding short certificates for s-factorisations of graphs of order ≥ 7 will be a valuable further work. We may build an automatic certificate searching tool to search for short certificates for these s-factorisations.

An open question arises from this research is what are the shortest certificates for s-factorisations and stability equivalences. We found some short certificates but it is not known if these certificates are the shortest. Finding shortest certificates for infinite families of s-factorisations and stability equivalences will be a challenging work.

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