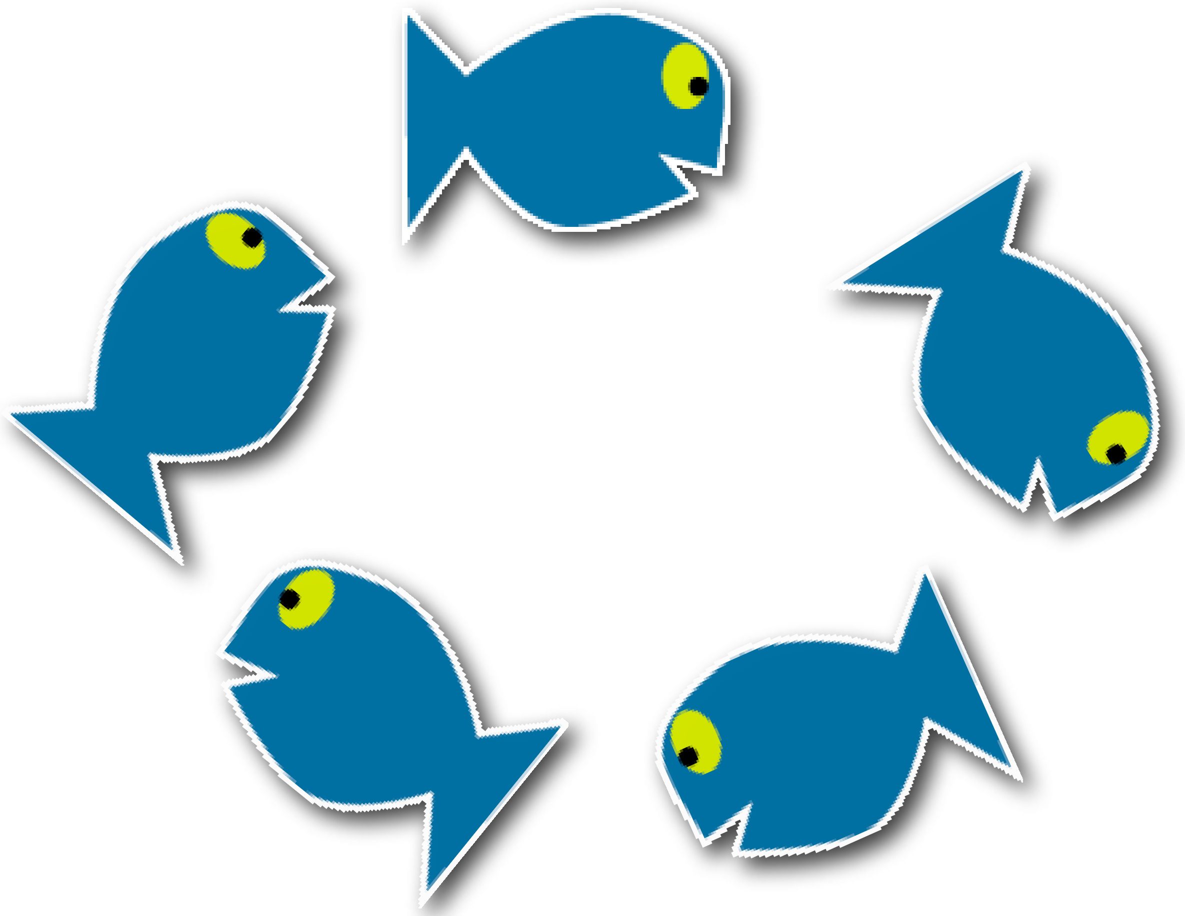


Monash University • Clayton's School of Information Technology

CSE3313 Computer Graphics

Lecture 9: 2D Transformations

Affine Transformations

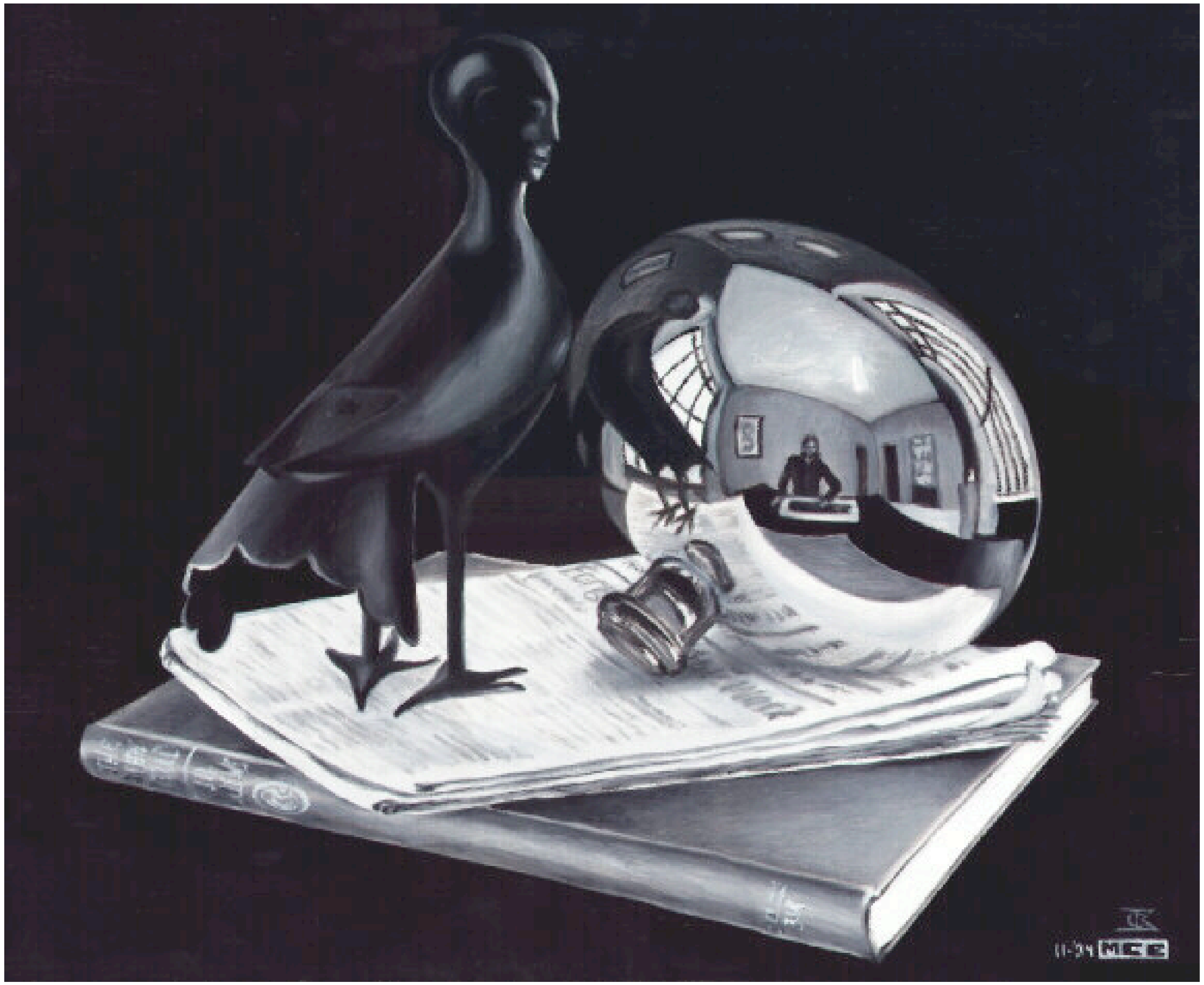
- A line segment is completely determined by its two end points. This is storage efficient.
- If a transformation turns a line into something other than a line, we could represent this either by line segments or by points.
- Transformations which preserve straight lines and parallelism are known as **affine transformations**.
- General 2D affine transformations:

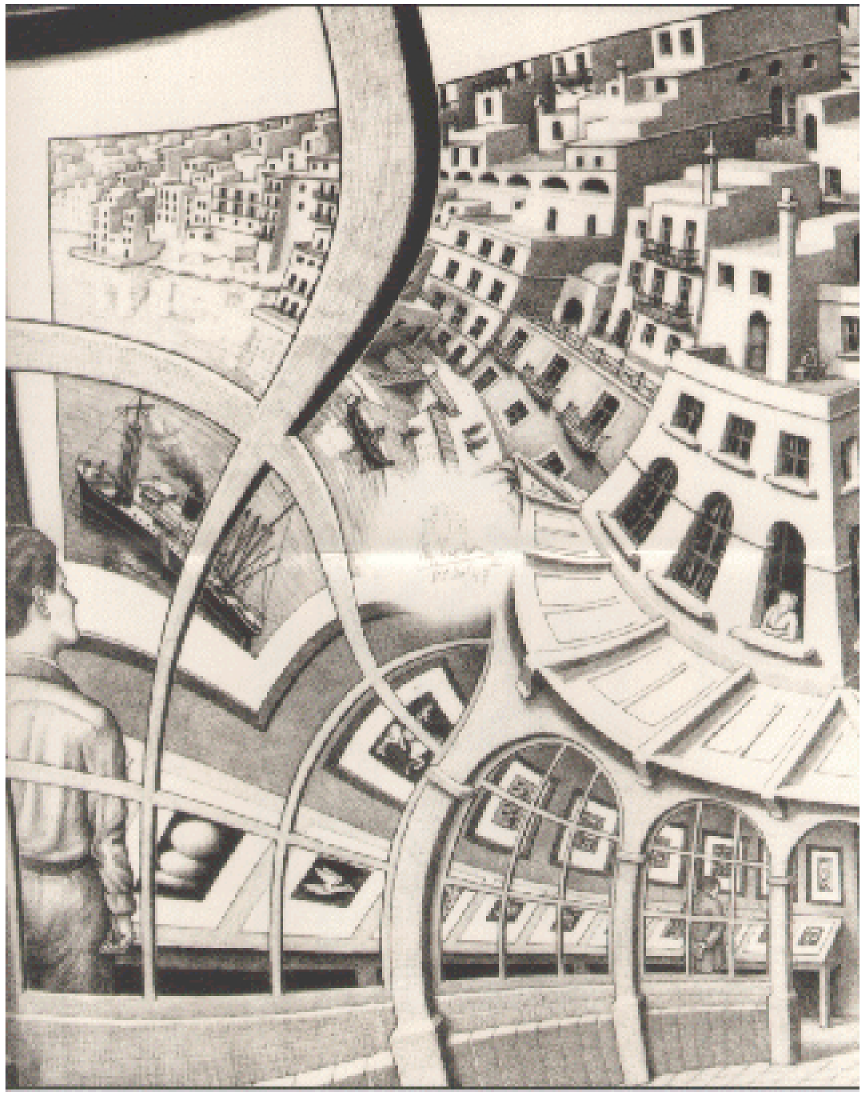
$$x' = a\,x + b\,y + c$$

$$y' = d\,x + e\,y + f$$

or in matrix form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$





Affine Transformations (cont.)

- Since affine transformations transform line segments into line segments, the application of any two successive affine transformations has an equivalent affine transformation.
- We can build complex transformations from a sequence of simple transformations.
- Basic transformations:
  - translation;
  - rotation about the origin;
  - scaling.
- Other transformations:
  - shear
  - skew

Translation

- **Translation** shifts all points by an equal amount.
- Thus the point (x,y) will be translated into:

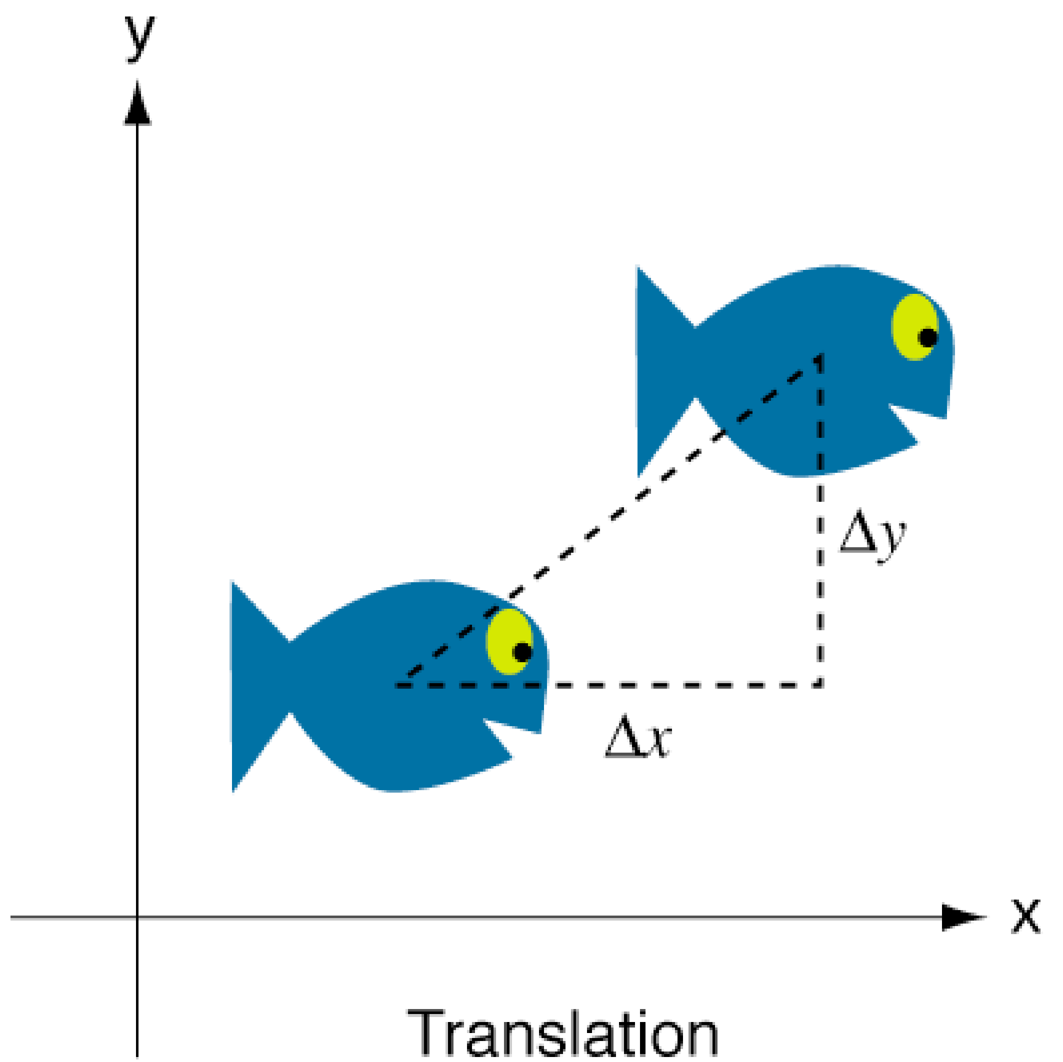
  $x' = x + \Delta x$

  $y' = y + \Delta y$

- In vector notation:

$$\mathbf{p}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

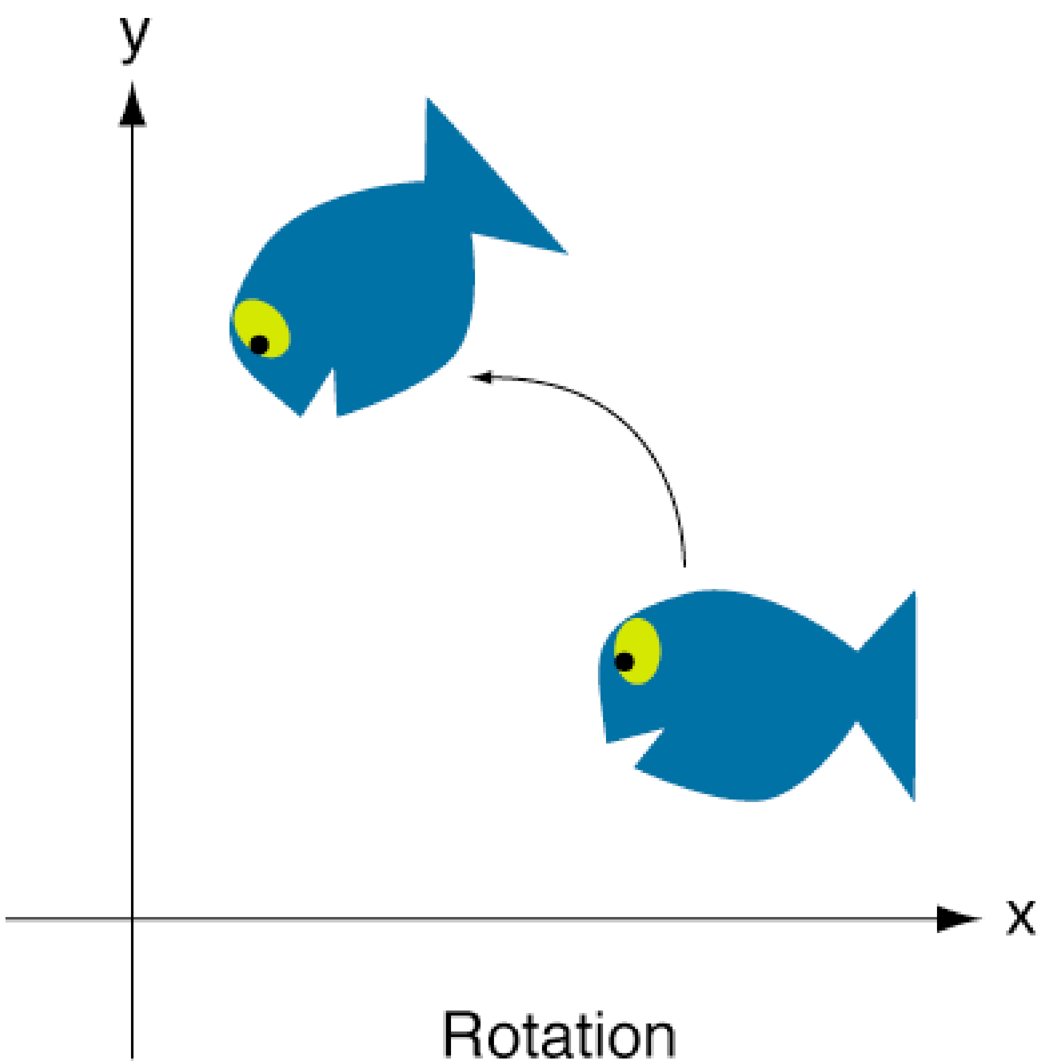
$$\mathbf{p}' = \mathbf{p} + \mathbf{t}$$

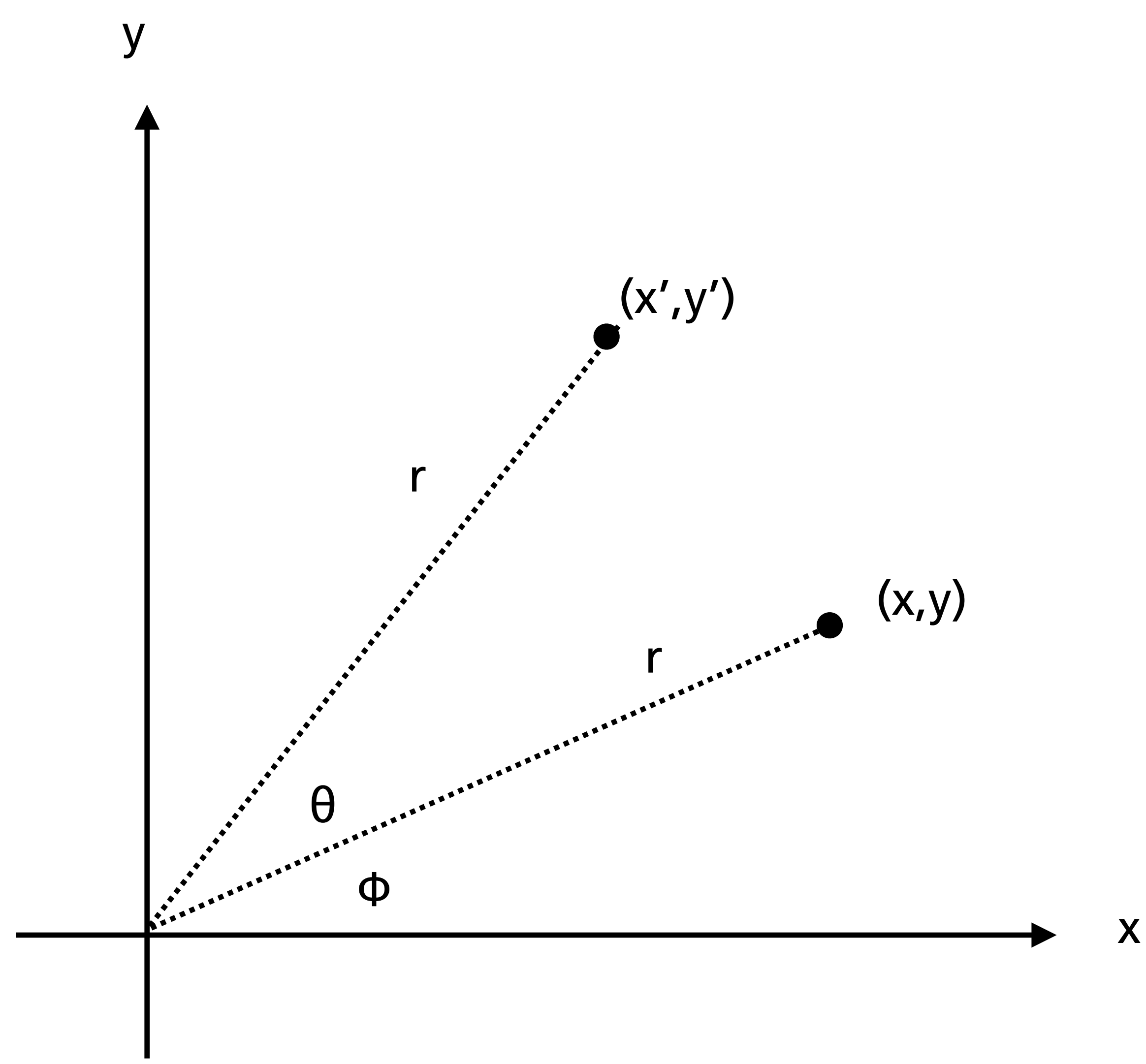


Translation

Rotation

- As we rotate a point by an amount θ about the origin (in an anti-clockwise direction) the point stays a constant distance from the origin.





Rotation (cont.)

- Using polar form we have

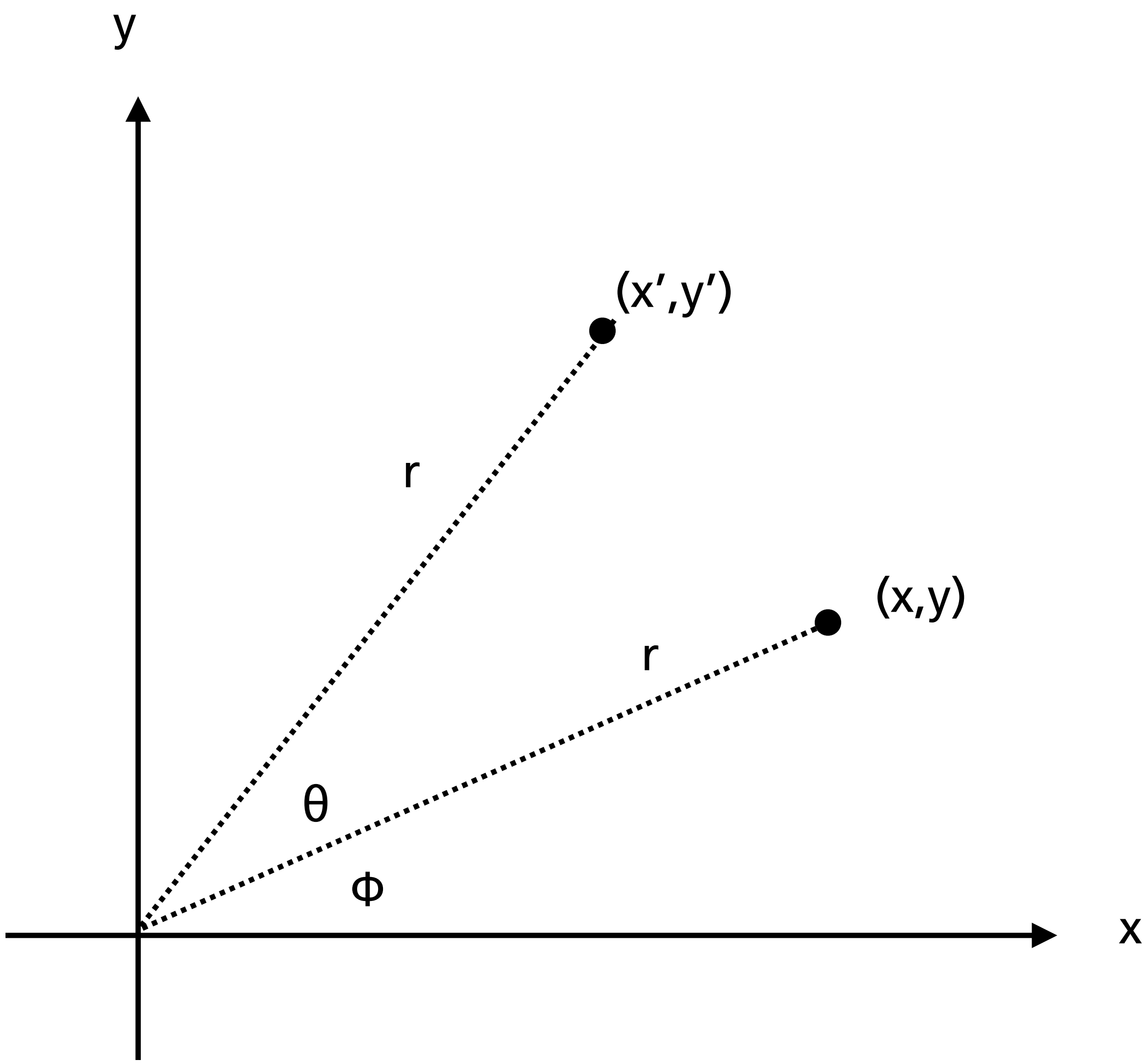
$$x = r\cos\Phi$$

$$y = r\sin\Phi$$

and

$$x' = r\cos(\theta+\Phi)$$

$$y' = r\sin(\theta+\Phi)$$



- Using trigonometric identities for cosine and sine of the sum of two angles:

$$x' = r\cos\theta\ \cos\Phi - r\sin\theta\ \sin\Phi$$

$$= x\cos\theta - y\sin\theta$$

$$y' = r\cos\theta\ \sin\Phi + r\sin\theta\ \cos\Phi$$

$$= x\sin\theta + y\cos\theta$$

- As matrix-vector multiplication:

$$\mathbf{p}' = \mathbf{R}\mathbf{p}$$

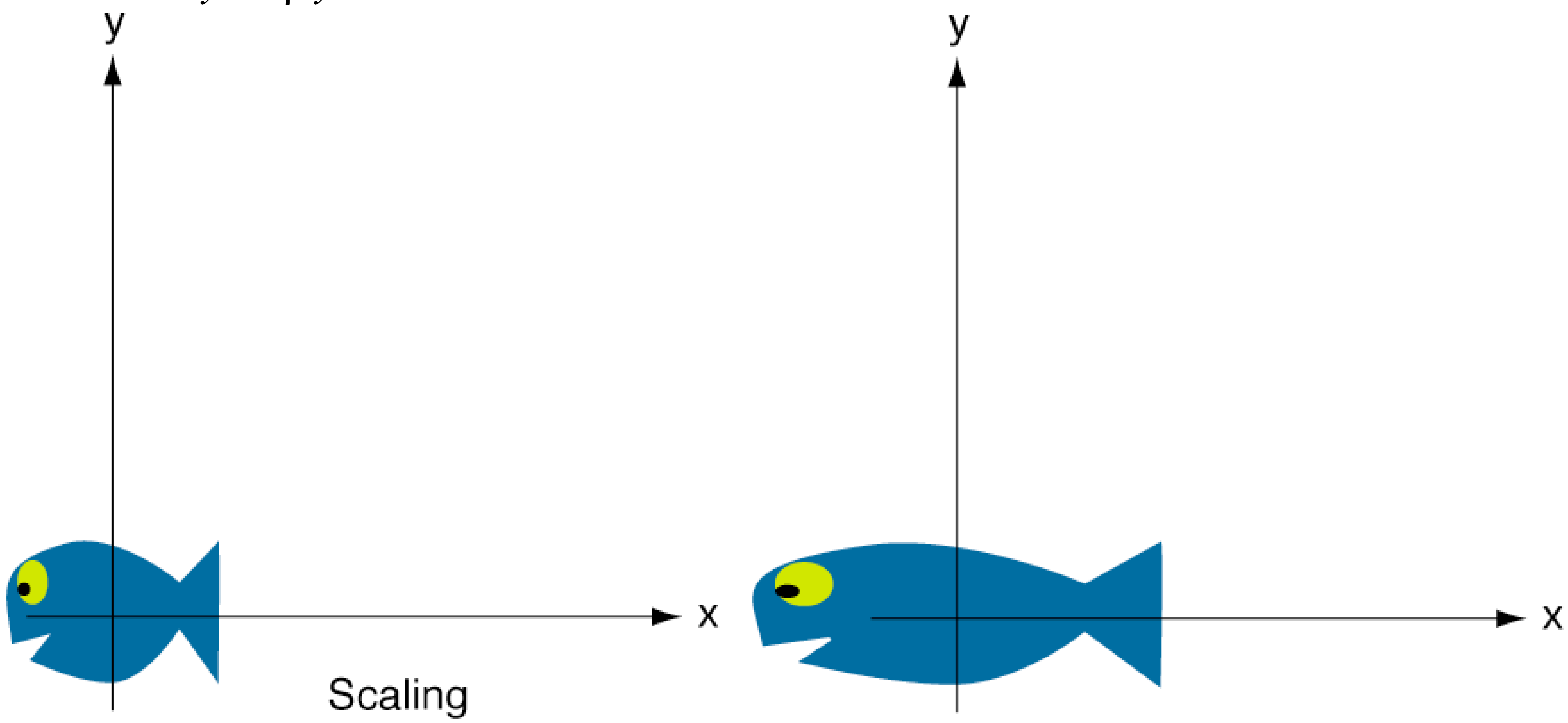
$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Scaling

- We can separate scaling in the x and y axes with constant α and β determining the amount of scaling in each direction.

$$x' = \alpha x$$

$$y' = \beta y$$

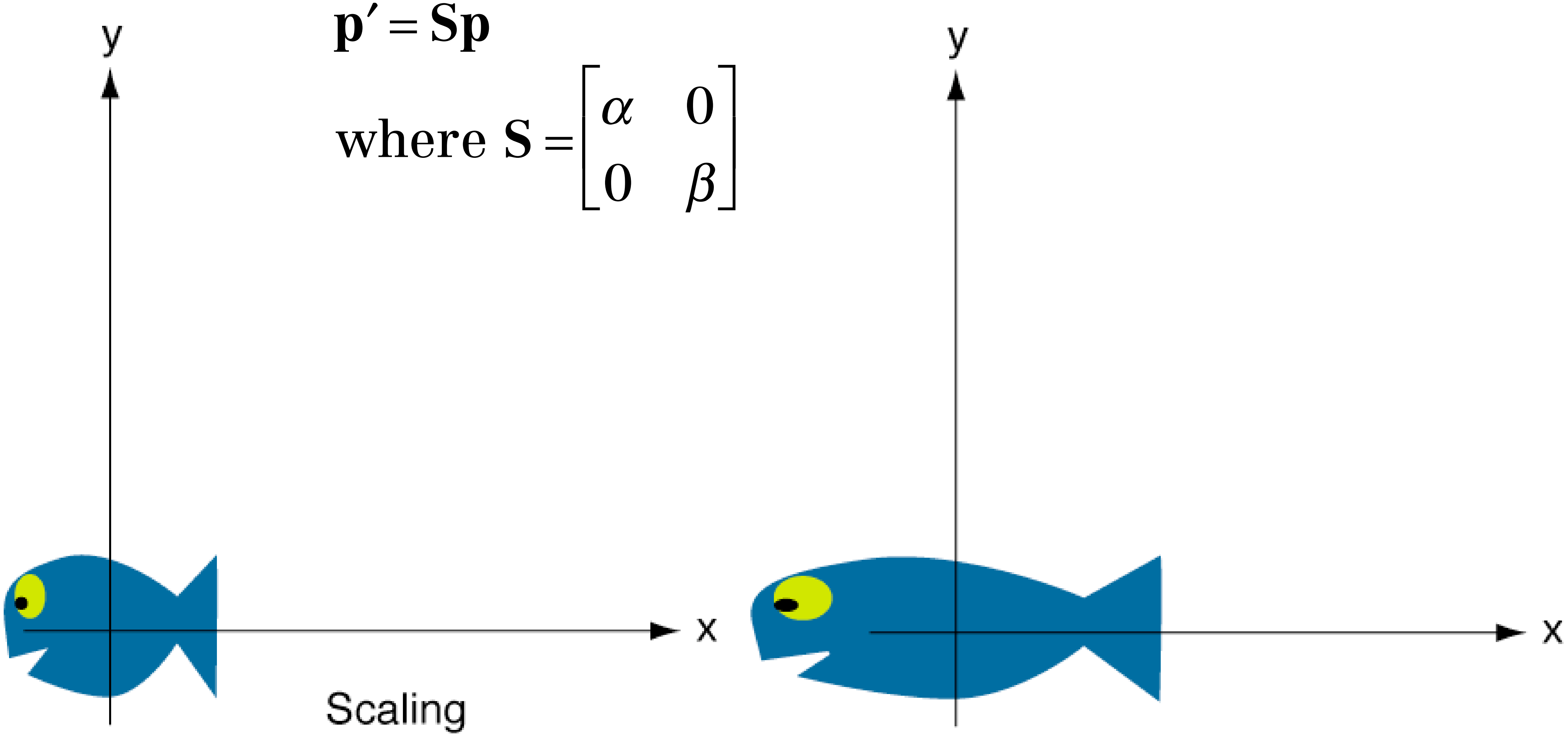


Scaling (cont.)

- A negative scaling factor will cause reflections to occur in that axis.
- Equal scaling in both axes is known as *uniform scaling*.
- Scaling can be expressed as the matrix-vector operation:

$$\mathbf{p}' = \mathbf{S}\mathbf{p}$$

$$\text{where } \mathbf{S} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$



Row and Column Vectors

- Standard mathematical notation expresses vectors in column form:

$$\mathbf{p}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M}\mathbf{p}$$

- It is common in computer graphics to use row vectors

$$\mathbf{p}' = \begin{bmatrix} x' & y' \end{bmatrix} = \mathbf{p}\mathbf{M}^T$$

When using column vectors, transformations accumulate from right to left. With row vectors transformations accumulate from left to right:

$$\text{column}: \quad \mathbf{p}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{p}$$

$$\text{row}: \quad \mathbf{p}' = \begin{bmatrix} x' & y' \end{bmatrix} = \mathbf{p}\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3$$

Combining Transformations

- Two of the basic transformations can be expressed in the form:

  vector ← matrix times vector

- However, translation is expressed by:

  vector ← vector plus constant_vector

- Combining scaling and rotation is straightforward. If $\mathbf{R}$ denotes a rotation matrix and $\mathbf{S}$ a scaling matrix then

  $\mathbf{A} = \mathbf{RS}$ denotes a scaling followed by a rotation;

  $\mathbf{B} = \mathbf{SR}$ denotes a rotation followed by a scaling transformation.

- Matrix multiplication is not commutative, so the order in which calculations are carried out is significant.

- It is not possible to represent a translation as a matrix times vector transformation, if we are restricted to two dimensions.

Homogeneous Coordinates

- However, there is a representation in three dimensions.
- We represent the two dimensional point $[x\ y]$ by the three dimensional point $[wx\ \ wy\ \ w]$ where w is non-zero.
- This is known as a **homogeneous coordinate transformation**.
- In some circumstances w is used as a scaling factor to allow integer arithmetic to simulate rational arithmetic.
- Generally, w is set to 1.
- Suppose the translation maps (x, y) into $(x + \Delta x, y + \Delta y)$, then:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

using column vectors. We may denote the matrix involved by $\mathbf{T}(\Delta x, \Delta y)$.