Monash University • Clayton's School of Information Technology

CSE3313 Computer Graphics

Lecture 13: Mathematical Representations

Reference: Hearn & Baker: Appendix A

In computer graphics, we are interested in 3 major representations of equations:

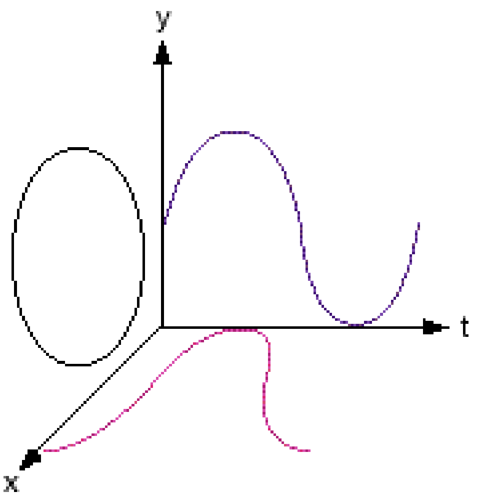
Implicit form: $F(x, y) = 0$

Explicit form: $y = f(x)$

Parametric form:
$$\left.\begin{array}{r}\left.\begin{array}{r}x(t) = f_x(t)\\ y(t) = f_y(t)\end{array}\right\}2\text{D}\\ z(t) = f_z(t)\end{array}\right\}3\text{D}, \quad 0 \le t \le 1$$

Some representations are more convenient than others, depending on our application for them. For example, a circle can be represented:

$$x^2 + y^2 = r^2 \quad \text{implicit form}$$
$$\left.\begin{array}{r}x(t) = r\cos(2\pi t)\\ y(t) = r\sin(2\pi t)\end{array}\right\} \text{parametric form}$$



In the parametric form it is easy do draw a curve with equal spaced samples along t.

Parametric curves:

$$\mathbf{p}(t) = (x(t), y(t))$$

$$\text{surfaces :}$$

$$\mathbf{p}_{\mathbf{3D}}(u,v) = x(u,v)\,\underset{\sim}{\mathbf{i}} + y(u,v)\,\underset{\sim}{\mathbf{j}} + z(u,v)\,\underset{\sim}{\mathbf{k}}$$

For example, a sphere can be represented:

$$f(x,y,z) = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2$$

$$\mathbf{p}(u,v) = r\cos(v)\cos(u)\mathbf{i} + r\sin(v)\mathbf{j} + r\cos(v)\sin(u)\mathbf{k}$$

The implicit form can make certain comparisons easy. Given a point $\mathbf{P}$,

$$\mathbf{P} \text{ is "inside surface" if } f(x,y,z) < 0$$

$$\text{"outside surface" if } f(x,y,z) > 0$$

$$\text{on surface if } f(x,y,z) = 0$$

Surface normal: $$\underline{\mathbf{n}}(u_0, v_0) = \frac{\partial \mathbf{P}}{\partial u} \times \frac{\partial \mathbf{P}}{\partial v}\bigg|_{u=u_0, v=v_0}$$

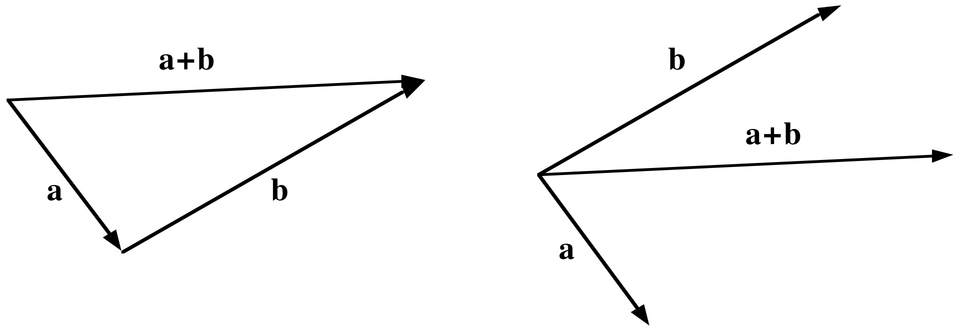
A *parametric curve* can be represented:

$$\mathbf{p}(t) = (x(t), y(t))$$

as a sum of polynomials:

$$p(u) = p_0 + p_1 u + p_2 u^2 + \cdots + p_n u^n = \sum_{i=0}^{n} p_i u^i$$

Sum of Vectors:



dot product (also called *scalar* or *inner* product):

2D:

$$\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2)$$
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

In *n* dimensions:

$$\mathbf{v} = (v_1, v_2, \ldots v_n), \mathbf{w} = (w_1, w_2, \ldots w_n)$$
$$d = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$$

$$\mathbf{a} \cdot \mathbf{b} \equiv \mathbf{b} \cdot \mathbf{a}$$
$$(\mathbf{a} + \mathbf{c}) \cdot \mathbf{b} \equiv \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$$
$$(s\mathbf{a}) \cdot \mathbf{b} \equiv s(\mathbf{a} \cdot \mathbf{b})$$

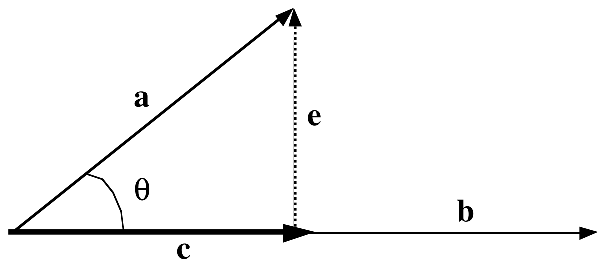
$$\left|\mathbf{b}^2\right| = \mathbf{b} \cdot \mathbf{b} \quad \therefore \left|\mathbf{b}\right| = \sqrt{\mathbf{b} \cdot \mathbf{b}}$$

The angle between 2 vectors:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$
$$\mathbf{u}_a = \frac{\mathbf{a}}{|\mathbf{a}|}, \mathbf{u}_b = \frac{\mathbf{b}}{|\mathbf{b}|}, \cos\theta = \mathbf{u}_\mathbf{a} \cdot \mathbf{u}_\mathbf{b}$$

If:
$$\mathbf{a}\cdot\mathbf{b} > 0 \quad \theta < \frac{\pi}{2}$$
$$\mathbf{a}\cdot\mathbf{b} = 0 \quad \theta = \frac{\pi}{2}$$
$$\mathbf{a}\cdot\mathbf{b} < 0 \quad \theta > \frac{\pi}{2}$$

Projection and Resolution of Vectors



Projection of $\mathbf{a}$ in the direction of $\mathbf{b}$.

$\mathbf{a}$ is resolved into a component $\mathbf{c}$ along $\mathbf{b}$.

we know: $\mathbf{e} = \mathbf{a} - \mathbf{c}$

$$\mathbf{u_c} = \mathbf{u_b} \quad \therefore \text{ need } |\mathbf{c}|$$
$$|\mathbf{c}| = |\mathbf{a}|\cos\theta$$

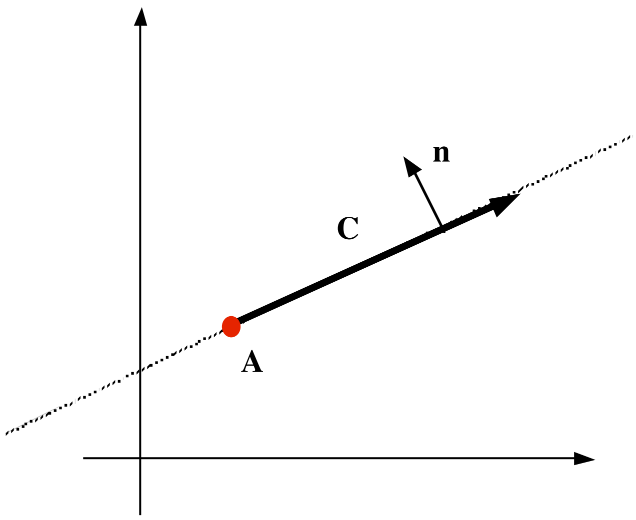
by law of cosines:

$$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$
$$\therefore \quad |\mathbf{c}| = |\mathbf{a}|\frac{(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{a}||\mathbf{b}|}$$
$$= \mathbf{a}\cdot\mathbf{u_b}$$
$$\mathbf{c} = (\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}$$
$$= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

Normal of a line

A line can be represented using a point and a direction vector:

$$\mathbf{L} = \mathbf{A} + \mathbf{C}t$$
$$\mathbf{A} = (A_x, A_y)$$
$$\mathbf{C} = (c_x, c_y) \text{ direction vector}$$
$$\mathbf{n} = (n_x, n_y) \text{ normal}$$



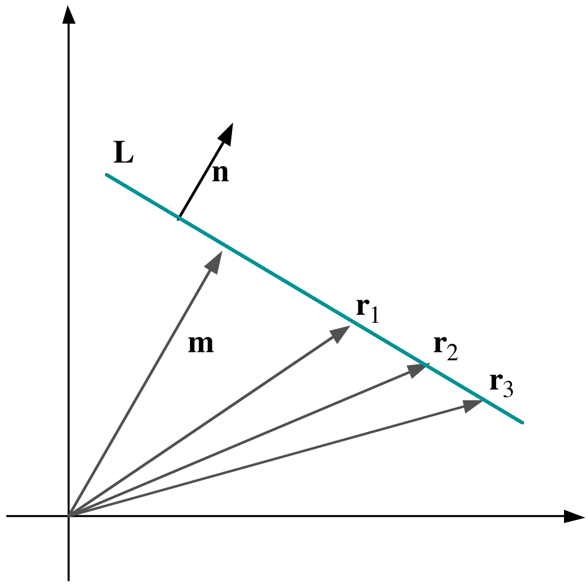
In 2 dimensions:

$$\mathbf{C} \cdot \mathbf{n} = 0 \quad \rightarrow \quad c_x n_x + c_y n_y = 0$$
$$\frac{c_y}{c_x} = \frac{-n_x}{n_y}$$

$\mathbf{n}$ can be any multiple of $(c_y, -c_x)$

Consider an arbitrary point $\mathbf{R} = (x,y)$, on the line, $\mathbf{L}$

$$\mathbf{n} \cdot (\mathbf{R} - \mathbf{A}) = 0$$
$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a} \quad \text{(replace points with vectors)}$$
$$\mathbf{n} \cdot \mathbf{r} = D = n_x A_x + n_y A_y \quad \text{(point normal eqn.)}$$
$$n_x x + n_y y = D$$



all vectors $\mathbf{r}$ have the same projection $\mathbf{m}$ onto $\mathbf{n}$.

$\mathbf{n}.\mathbf{r}$ is proportional to $\mathbf{m}$:

$$\mathbf{n} \cdot \mathbf{r} = D$$
$$|\mathbf{m}| = \mathbf{r} \cdot \mathbf{u}_\mathbf{n}$$
$$= \mathbf{r} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}$$
$$= \frac{D}{|\mathbf{n}|}$$

$\frac{D}{|\mathbf{n}|}$ is the closest distance from the line to the origin.

Likewise for a plane (note this is in 3 dimensions):

$$\mathbf{n} = (n_x, n_y, n_z)$$
$$\mathbf{n} \cdot \mathbf{r} = D$$
$$Ax + By + Cz = D$$
$$n_x x + n_y y + n_z z = D$$

Example:

Consider a plane that passes through the point *(1,2,3)*, with normal *(2,-1,-2)*.

In Point normal form: $(2,-1,-2) . (x,y,z) = D$

$$(2,-1,-2) . (1,2,3) = -6 = D$$

$\therefore$ plane equation is: $2x - y - 2z = 6$

Vector Cross Product:

Given two vectors **v** and **w**, the *cross product* (or *vector product)* is the vector that is orthogonal to both **v** and **w**. There are two possible choices for the direction of the cross product. The correct direction is defined by the *right-hand rule*.

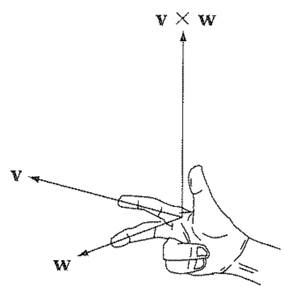


FIGURE 1.15 Cross product direction.

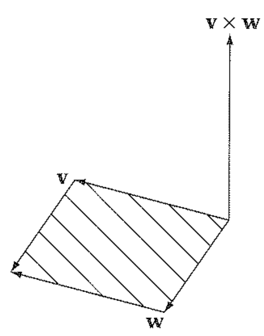


FIGURE 1.16 Cross product length equals area of parallelogram.

The length of the cross product is equal to the area of a parallelogram bordered by the two vectors. This can be computed using the formula:

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$$

where θ is the angle between **v** and **w**. The cross product does not commute, so order is important:

$$\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$$

The formula for the cross product is:

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = (v_y w_z - w_y v_z, \quad v_z w_x - w_z v_x, \quad v_x w_y - w_x v_y)$$

For vectors **u**, **v**, **w**, and scalar *a* the following algebraic rules apply:

$$\begin{aligned}
&\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \\
&\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \\
&(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) \\
&a(\mathbf{v} \times \mathbf{w}) - (a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) \\
&\mathbf{v} \times \mathbf{0} = \mathbf{0} \times \mathbf{v} = \mathbf{0} \\
&\mathbf{v} \times \mathbf{v} = \mathbf{0}
\end{aligned}$$